Branching Particle Systems and Superprocess

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Abstract | In this paper, we describe a system of particles that perform independent random motions in space and at the end of their lifetimes give birth to a random number of offspring. We show that the system in the large density, small mass, rapid branching or long time scale limit converges to a measure-valued diffusion called the superprocess.

1. Introduction

In this paper, we describe a system of branching-diffusing particle system and a high density measure-valued diffusion limit for such systems. The limiting measure-valued process is called the Dawson-Watanabe process or the superprocess. Branching-diffusing particle systems and their high density limits have been studied intensively in the last 30 years of so. Interest in these processes stems from their application in population genetics, or from the fact that they arise as limits of the rescaled voter model and critical oriented percolation. The particle as well as the limiting superprocess are also of interest since their log-Laplace functionals are associated with a class of non-linear partial differential equations. These processes satisfy the Markov property and so the rich mathematical machinery developed for the study of such processes become available. Though these processes are built from individual particles performing Markovian or diffusion motion, the limiting process exhibits significantly different path behaviour as compared to the individual particles. For example, if individual particle motions are Brownian, then they "hit" points only in dimension 1, whereas the limiting superprocess does so for dimensions 3 and below. In dimensions larger than 2, even if the initial measure has a density with respect to the Lebesgue measure, the process instantaneously becomes singular.

For a detailed description of these processes, we refer the reader to Ref. [4] or the expository article by Perkins[14]. A highly readable introduction to superprocesses can be found in Ref. [9]. Dynkin (1993)[8] and Le Gall[12] explore connections between superprocesses and partial differential equations. There are other constructions of a superprocess, for example, the construction using the Browian snake[11], via convergence of rescaled lattice trees to a variant of the super Brownian motion called the Integrated super Brownian motion[7], the rescaled voter model[2], or as limits of the critical oriented percolation[10]. Superprocesses in random environment[3,17], catalytic superprocesses[5] and superprocesses over stochastic flows[1], superprocesses with interactions are examples of some of the generalizations that have been explored. Many properties such as propagation and dimension of the support, continuity, persistence and local extinction, genealogy of particles and their connections to random trees have been studied. We restrict ourself in this article to two approaches to construction of a simple superprocess.

2. Branching-Diffusing Particle Systems and their High Density Limits

In this section, we will describe a simple branching-diffusing particle system and the conditions under which we can obtain a high density measure-valued diffusion limit for these systems. In order to describe the processes of interest, we need the following notations: For any function ϕ and measure μ

defined on some measure space, we denote by $\langle \phi, \mu \rangle$ the integral of ϕ w.r.t. μ. Let $\mathcal{M}_F(\Re^d)$ be the space of finite measures on $\Re^d$. Let $\mathcal{M}_a(\Re^d)$ be the subset of $\mathcal{M}_F(\Re^d)$ of purely atomic measures on $\Re^d$. Let $f(a,x) \in C_b^2(\Re^d)$ be the space of positive bounded continuous functions that are twice differentiable in $x \in \Re^d$ respectively. $C_b^+(\Re^d)$ be positive bounded continuous functions on $\Re^d$.

Consider a system of particles that start at time $t = 0$ distributed according to a Poisson random measure π_λ with intensity measure $\lambda \in \mathcal{M}_F(\Re^d)$. Note that the Poisson random measure specifies both the number and location of the initial particles. These particles then start performing independent random motions according to a diffusion with generator **A** with Feller semigroup T_t. For example if individual particles perform independent Brownian motion, then the generator is the Laplacian with the associated semigroup being the heat semigroup.

Each particle lives an independent lifetime with exponential distribution with parameter γ. At the end of its lifetime, each particle independent of everything else, gives birth to a certain number of offspring according to a distribution $\{p_k, k \geq 0\}$, with generating function

$$F(s) = \sum_{k=0}^{\infty} p_k s^k.$$

The offspring distribution is assumed to be critical, that is the mean of the offspring distribution, $m := \sum_{k=1}^{\infty} k p_k = 1$. A simple example of a critical generating function is

$$F(s) = s + c(1-s)^2, \qquad 0 < c \leq \frac{1}{2}. \qquad (2.1)$$

The offspring immediately upon birth, start moving independently from the location of death of their parents. The newborn particles evolve in the same way as the parent particles and this process continues till the system becomes extinct. Our interest is in the measure-valued stochastic process X_t, where $X_t(A)$ is the total number of particles in set A at time t. Define the Laplace transform of the particle system by

$$L_t(\phi, \nu) = E\left[e^{-\langle \phi, X_t \rangle} | X_0 = \nu\right], \qquad \phi \in C_b(\Re^d),\ \nu \in M_a(\Re^d). \qquad (2.2)$$

We also use the notation $L_t\phi(x) = L_t(\phi, \delta_x)$. The following property is referred to as the branching property:

$$L_t(\phi, \nu_1 + \nu_2) = L_t(\phi, \nu_1) L_t(\phi, \nu_2).$$

From this it follows that

$$L_t(\phi, \nu) = e^{\langle \log L_t(\phi), \nu \rangle}, \qquad \nu \in M_a(\Re^d). \qquad (2.3)$$

We now describe the Dawson-Watanabe measure-valued process that is obtained as the high density limit of the above particle system. To do this we consider for any $n \geq 1$, the above branching-diffusing particle system with the following parameters:

1. The initial distribution of the particles is Poisson with intensity measure $n\lambda$. We will denote it by $\pi_{n\lambda}$. The mean number of particles generated will be $n\lambda(\Re^d)$. The Laplace transform of the distribution of particles under such a Poisson measure is given by
$$E_{\pi_\lambda}\left[e^{-\langle \phi, X_0 \rangle}\right] = \int_{M_a(\Re^d)} e^{-\langle \phi, \nu \rangle} \pi_\lambda(d\nu) = e^{-\langle 1 - e^{-\phi}, \lambda \rangle}. \qquad (2.4)$$
2. We will consider such a high density—small mass system at time nt. Thus we need to slow down the diffusion in order to obtain a meaningful limit. The diffusion semigroup of the particle motion for the nth system is given by $T_t^n = T_{t/n}$.
3. The lifetime distribution G is exponential with parameter γ.
4. The generating function of the offspring distribution F_n satisfies
$$\lim_{n \to \infty} \sup_{u \leq N} \| n^2(F(1 - u/n) - (1 - u/n)) - cu^2 \| \to 0, \qquad (2.5)$$
for all $N > 0$. Note that (2.1) satisfies the above condition.
5. Each particle has mass n^{-1}. We will study the mass process,
$$X_t^n(A) := \frac{1}{n} \sum_{\alpha \in \mathcal{I}_{nt}} 1_A(X_{nt}^\alpha), \qquad (2.6)$$
where $\mathcal{I}_t$ is the index set of all the particles that are alive at time t.

Theorem 2.1 (Roelly-Coppoletta, 1986[15]). *The process X^n converges weakly on the Skorokhod space $D(\Re_+, \mathcal{M}_F(\Re^d)$ to a measure-valued Markov process Y which is the unique solution of the following martingale problem: For each $\phi \in \mathcal{D}(\mathcal{A})$, the process*

$$Z_t(\phi) = \langle \phi, Z_t \rangle := \langle \phi, Y_t \rangle - \langle \phi, \lambda \rangle - \int_0^t \langle \mathcal{A}\phi, Y_s \rangle ds, \qquad (2.7)$$

is a martingale with increasing process

$$\langle Z_t(\phi)\rangle_t = c\gamma \int_0^t \langle \phi^2, Y_s\rangle. \quad (2.8)$$

The Laplace functional of the process Y is given by

$$E_\lambda\left[e^{-\langle\phi, Y_t\rangle}\right] = e^{-\langle u_t,\lambda\rangle}, \qquad \lambda \in \mathcal{M}_F(\Re^d), \quad (2.9)$$

where u_t is the unique strong solution of the initial value problem

$$\frac{\partial}{\partial t} u_t(x) = \mathcal{A}u(x) - c\gamma u_t^2; \qquad u_0(x) = \phi(x),$$
$$\phi \in C_b^+(\Re^d). \quad (2.10)$$

3. Sketch of Proof of Theorem 2.1

We first show convergence via the Laplace functional approach.

Lemma 3.1. *For any $t \geq 0$, and $x \in \Re^d$ the Laplace function $L_t\phi(x) = E_{\delta_x}[\exp(-\langle\phi, X_t\rangle)]$ satisfies the integral equation*

$$L_t\phi = T_t e^{\phi} + \gamma \int_0^t T_s(F(L_{t-s}\phi) - L_{t-s}\phi)\, ds \quad (3.1)$$

Proof. Condition on the disjoint but covering events

$$\{T^d > t\}, \qquad \{T^d = s, \text{ for some } 0 \leq s \leq t\},$$

where T^d is the time of death of the ancestor, which we recall is an exponential random variable with parameter γ. We obtain, by the branching property,

$$L_t\phi(x) = e^{-\gamma t} T_t e^{-\phi}(x) + \int_0^t \gamma e^{-\gamma s}\, T_s \times [F(L_{t-s}\phi)](x)\, ds$$

Using the identity $\exp(-\gamma t) = 1 - \int_0^t \gamma \exp(-\gamma(t-r))dr$, we get,

$$L_t\phi(x) = T_t e^{-\phi}(x) + \gamma \int_0^t T_s \times [F(L_{t-s}\phi)](x)\, ds - A_t(x),$$

where

$$A_t(x) = \int_0^t \gamma e^{-\gamma(t-s)}\, ds T_t e^{-\phi}(x) + \gamma^2 \int_0^t \int_0^s e^{-\gamma(s-r)}\, dr\, T_s \times [F(L_{t-s}\phi)](x)\, ds.$$

Interchange the order of integration in the last term above. The second term now becomes

$$\gamma^2 \int_0^t \left(\int_r^t e^{-\gamma(s-r)}\, T_s[F(L_{t-s}\phi)](x)\, ds\right) dr$$

Shift s to $s-r$:

$$\gamma^2 \int_0^t \left(\int_0^{t-r} e^{-\gamma s}\, T_{s+r}[F(L_{t-(s+r)}\phi)](x)\, ds\right) dr$$

Notationally switch s and r to obtain,

$$\begin{aligned} A_t(x) &= \int_0^t \gamma e^{-\gamma(t-s)}\, ds T_t e^{-\phi}(x) \\ &\quad + \gamma^2 \int_0^t \left(\int_0^{t-s} e^{-\gamma r}\, T_{r+s} \times [F(L_{t-s-r})\phi](x)\, dr\right) ds \\ &= \gamma \int_0^t T_s\Big(e^{-\gamma(t-s)} T_{t-s} e^{-\phi}(x) \\ &\quad + \gamma \int_0^{t-s} e^{-\gamma r} T_r[F(L_{t-s-r}\phi)]\, dr\Big)(x)\, ds \\ &= \int_0^t T_s(\gamma L_{t-s}\phi)(x)\, ds \end{aligned}$$

To complete the proof, substitute the above expression in the equation for $L_t\phi(x)$

Lemma 3.2. *The Laplace transform of the branching particle system X_t described above under the Poisson initial measure π_λ is given by*

$$E_{\pi_\lambda}\left[e^{-\langle\phi, X_t\rangle}\right] = e^{-\langle v_t(\phi),\lambda\rangle}, \quad (3.2)$$

where $v_t(\phi) = 1 - L_t(\phi)$, and L_t is as defined in (2.2).

Proof. Proof follows from (2.3) and (2.4).

$$\begin{aligned} E_{\pi_\lambda}\left[e^{-\langle\phi, X_t\rangle}\right] &= \int_{M_a(\Re^d)} E_\nu\left[e^{-\langle\phi, X_t\rangle}\right]\pi_\lambda(d\nu) \\ &= \int_{M_a(\Re^d)} e^{-\langle v_t(\phi),\nu\rangle}\pi_\lambda(d\nu) \\ &= e^{-\langle e^{1-\log L_t(\phi)},\lambda\rangle} \\ &= e^{-\langle 1-L_t(\phi),\lambda\rangle} \end{aligned}$$

Let $v_t(\phi) = 1 - L_t(\phi)$. Then, following (3.1) we can write an evolution equation for v_t as

$$v_t(\phi)(x) = T_t(1-e^{-\phi})(x) - \gamma \int_0^t T_s(F(1-v_{t-s}\phi) - 1 + v_{t-s}\phi)(x)\, ds \quad (3.3)$$

We now write the Laplace transform for the approximating high density—small mass particle system X^n:

$$E_{\pi_{n\lambda}}\left[e^{-\langle\phi,X_t^n\rangle}\right]=e^{-\langle v_t^n(\phi),\lambda\rangle}, \tag{3.4}$$

where v^n satisfies the evolution equation,

$$\begin{aligned}v_t^n(\phi)(x)&=T_{nt}^n n(1-e^{-\phi})(x)\\&\quad-n\gamma\int_0^{nt}T_s^n\left(F\left(1-\frac{1}{n}v_{t-s/n}^n\phi\right)-1\right.\\&\quad\left.+\frac{1}{n}v_{t-s/n}^n\phi\right)(x)\,ds\\&=T_t n(1-e^{-\phi})(x)\\&\quad-n^2\gamma\int_0^{t}T_s\left(F\left(1-\frac{1}{n}v_{t-s}^n\phi\right)-1\right.\\&\quad\left.+\frac{1}{n}v_{t-s}^n\phi\right)(x)\,ds\end{aligned} \tag{3.5}$$

(2.10) now follows from the assumptions on the offspring distribution function, the contraction property of T_t and an application of the Gronwall's inequality.

This proves the convergence of the marginal disributions of the process X^n to those of the limiting superprocess Y. Convergence of the finite dimensional distributions can be shown using the Markov property and the convergence in distribution of $\sum_{i=1}^k\langle\phi_i,X_{t_i}^n\rangle$ to $\sum_{i=1}^k\langle\phi_i,Y_{t_i}\rangle$ (see Dynkin, 1993[8] or Dawson, Gorostiza and Li, 2002[6] for details on the construction of a more general superprocess).

To complete the proof of weak convergence, we need to show that the sequence of approximating particle processes X^n are tight in $D(\Re_+,\mathcal{M}_F(\Re^d)$ for which we refer the reader to Ref. [9], Proposition 1.19.

Worms Eye View

Finally we give a microscopic description of the particle system, leading to the stochastic evolution equation for the limiting superprocess given in (2.7). Thus the microscopic birth, evolution, branching and death of the individual particles can be assimilated and represented in a neat form in terms of smooth stochastic differential equations. This representation enables one to extend the superprocess directly by adding terms corresponding to interaction etc. without reference to the particle system.

We start with a particular labeling system for the particles. Define the family of multi indices,

$$\begin{aligned}\mathcal{A}:=\{\alpha=(\alpha_0,\alpha_1,\ldots,\alpha_N):\alpha_0\in\{1,2,..\},\\ \alpha_i=1,2,i\geq 1,N\in\{0,1,2,..\}\}.\end{aligned}$$

For example, the label $(1,2,2)$ will stand for the second child of the second child of the first particle, etc. Let $|\alpha|=N$ denote the length of the string $(\alpha_0,\alpha_1,\ldots,\alpha_N)$. Further, let $\alpha|_i:=(\alpha_0,\ldots,\alpha_i)$ and $\alpha-i=(\alpha_0,\ldots,\alpha_{|\alpha|-i}),i=1,\ldots,|\alpha|-1$, be the predecessors of α. We provide $\mathcal{A}$ with a partial ordering

$$\beta\prec\alpha\Longleftrightarrow\beta=\alpha|_i \text{ for some } i\leq|\alpha|.$$

We simplify matters by assuming that the lifetimes are deterministic and equal $1/n$. For any $t>0$ write $\alpha\sim t$, if and only if

$$\frac{|\alpha|}{n}\leq t<\frac{1+|\alpha|}{n}.$$

Let $\{\xi_t^\alpha,\alpha\in\mathcal{A}\}$ be a family of $\Re^d$-valued independent diffusions with generator $\mathbf{A}$ on the interval $[0,1/n)$ with $\xi_0^\alpha=0$. Let $\{N^\alpha,\alpha\in\mathcal{A}\}$ be a family of i.i.d. random variables taking values 0 and 2 with equal probabilities. ξ_t^α describes the spatial motion of the particle, while N^α gives the number of offsprings of the particle α.

At time t = 0, we have K, particles distributed independently in $\Re^d$ according to the Poisson measure with intensity measure $n\lambda$. Label these particles $1,2,\ldots,K$, and let their locations be $y_1,\ldots,y_K$ respectively. Define processes $Y^1,\ldots,Y^K$ on the interval $[0,1/n)$ by

$$Y^i(t)=y_i+\xi^i(t).$$

The families $\{\xi^\alpha\}$, $\{N^\alpha\}$ and $\{Y_0^i\}$ are independent. We now define recursively a binary tree of processes, for each α, by

$$Y^\alpha(t)=\begin{cases}\lim_{s\to\frac{k}{n}^-}Y^{\alpha-1}(s)+\xi^\alpha\left(t-\frac{k}{n}\right) & \text{if } \alpha\sim t,k=|\alpha|\\ \Lambda & \text{otherwise.}\end{cases}$$

Here Λ, called the cemetery, is a point disjoint from $\Re^d$, and, as usual, the convention is that any function on $\Re^d$ is extended to $\Re^d\cup\Lambda$ by $f(\Lambda)=0$. Thus the process $Y^\alpha(t)$ is alive only if $\alpha\sim t$. We still need to take into account the fact that the particle α is alive in the interval $[|\alpha|/n,(|\alpha|+1)/n)$ provided each of its predecessor $\alpha-i,i=1,\ldots,|\alpha|-1$, gave birth to two offsprings. To this end we define, for each α, the killing times

$$\tau^\alpha=\begin{cases}0 & \text{if } \alpha_0>K\\ \min\left\{\frac{i+1}{n}:N^{\alpha|_i}=0,i\leq|\alpha|\right\} & \text{if this set is non-empty and } \alpha_0\leq K\\ \frac{1+|\alpha|}{n} & \text{otherwise.}\end{cases}$$

Note that $Y^\alpha(t)$ gives the position of the particle α, if the particle is alive at time $t \geq 0$. Therefore we can define the path of any particle $\alpha \in \mathcal{A}$ as

$$X_t^\alpha = \begin{cases} Y^\alpha(t) & \text{if } t < \tau^\alpha \\ \Lambda & \text{if } t \geq \tau^\alpha. \end{cases} \quad (3.6)$$

Finally, we define the measure valued process for the above particle system as

$$X_t^\mu(A) = \frac{\#\{X_t^\alpha \in A : \alpha \sim t\}}{n} = \frac{1}{n}\sum_{\alpha\sim t} 1_A(X_t^\alpha). \quad (3.7)$$

3.1. *Martingale Measures*

Definition 1. *Let $\{\mathcal{F}_t\}$ be a right continuous filtration. A process $\{M_t(A), \mathcal{F}_t, t \geq 0, A \in \mathcal{B}(\Re^d)\}$ is a martingale measure if*

(i) $M_0(A) = 0$;
(ii) For each $t \geq 0$, M_t is a σ-finite $\mathcal{L}^2$-valued measure;
(iii) $\{M_t(A), \mathcal{F}_t, t \geq 0\}$ is a martingale.

Definition 2. *A martingale measure M is orthogonal if, for any two disjoint sets A and B in $\mathcal{B}(\Re^d)$, the martingales $\{M_t(A), t \geq 0\}$ and $\{M_t(B), t \geq 0\}$ are orthogonal.*

Equivalently, M is orthogonal if $M_t(A)M_t(B)$ is a martingale for any two disjoint sets A and B. This in turn is equivalent to having $< M(A), M(B) >_t$, the predictable process of bounded variation, vanish.

3.2. *An Integral Equation for* X_t^μ

Let $\{B^\alpha : \alpha \in \mathcal{A}\}$ be a family of d-dimensional $\Re^d$-valued Brownian motions on the same probability space such that

$$\xi_t^\alpha = \int_0^t \mathbf{b}(\xi_s^\alpha)ds + \int_0^t \boldsymbol{\sigma}(\xi_s^\alpha)\cdot dB_s^\alpha, \quad (3.8)$$

where $\boldsymbol{\sigma} = (\sigma_{ij})_{1\leq i,j\leq d}$ is such that $a_{ik}(x) = \frac{1}{2}\sum_{j=1}^d \sigma_{ij}\sigma_{jk}$. a_{ij} and b_j satisfy the following assumptions.

A1. The coefficients a_{ij}, b_i $(i, j = 1, \ldots d)$ are bounded and satisfy a Hölder condition on $\Re^d$.

A2. There exists a constant $C > 0$ such that for all $x \in \Re^d$ and arbitrary real numbers $\lambda_1, \ldots, \lambda_d$

$$\sum_{i,j=1}^d a_{ij}(x)\lambda_i\lambda_j \geq C\sum_{i=1}^d \lambda_i^2.$$

Define two martingale measures W^n and Z^n as follows:

$$W_t^n(A) = \frac{1}{\sqrt{n}}\sum_\alpha \int_0^t 1_A(X_s^\alpha)\boldsymbol{\sigma}(\xi_s^\alpha)\cdot dB_s^\alpha \quad (3.9)$$

$$\begin{aligned} Z_t^n(A) &= \frac{1}{n}\sum_\alpha \big(1_A(X_{\beta(\alpha)}^\alpha)1(\beta(\alpha)\leq t) \\ &\quad -1_A \times (X_{\zeta(\alpha)-}^\alpha)1(\zeta(\alpha)\leq t)\big) \\ &= \frac{1}{n}\sum_\alpha (N^\alpha - 1)\big(1_A(X_{\zeta(\alpha)-}^\alpha) \\ &\quad \times 1(\zeta(\alpha)\leq t)\big), \end{aligned} \quad (3.10)$$

where $\beta(\alpha)$ is the birth time and $\zeta(\alpha)$ is the death time of the particle α. The measure W^n keeps track of the diffusion and ignores branching, while Z^n does just the opposite. It measures the births minus the deaths and ignores the diffusion. Following the procedure in Walsh (1986) (applying the Itô formula for each particle using (3.8) and then suitably summing over all α) X_t^μ can be shown to satisfy the following evolution equation:

$$\begin{aligned} X_t^\mu(f) &= X_0^n(f) + \int_0^t X_s^n(\mathbf{A}f)ds \\ &\quad + \int_0^t \int_{\Re^d} f(x)Z^n(dx, ds) \\ &\quad + \frac{1}{\sqrt{n}}\int_0^t \int_{\Re^d} \nabla f(x)\cdot W^n(dx, ds), \\ &\quad f \in C_b^2(\Re^d). \end{aligned} \quad (3.11)$$

Proposition 3.3. *W^n and Z^n are orthogonal martingale measures. W^n is continuous and $\Re^d$-valued while Z^n is purely discontinuous and real valued. Moreover,*

(i) $< W^n(f) >_t = \frac{1}{n}\sum_\alpha \int_0^t f^2(X_s^\alpha)\boldsymbol{\sigma}(\xi_s^\alpha)\boldsymbol{\sigma}'(\xi_s^\alpha)ds$
(ii) $< \frac{1}{\sqrt{n}}\int_0^t \int_{\Re^d} \nabla f(x)\cdot W^n(dx, ds) >$

$$= \frac{1}{n}\sum_\alpha \int_0^t \nabla f(X_s^\alpha)\boldsymbol{\sigma}(\xi_s^\alpha)\boldsymbol{\sigma}'(\xi_s^\alpha)(\nabla f(X_s^\alpha))' ds$$

(iii) $< Z^n(f) >_t = \sum_{k<[tn]} \eta_{\frac{k}{n}}(f^2)\frac{1}{n} = \int_0^t \eta_s(f^2)ds + O(\frac{1}{n})$

Proof. The orthogonality of W^n and (i), (ii) follow immediately from the definition of W^n and the fact that the $\{B^\alpha\}$ are independent (see Walsh (1986), Prop. 8.1). The computations for Z^n are given in Ref. [14] (see proof of Theorem 2.13).

Remark 1. *The martingale measures W^n and Z^n are mutually orthogonal since W^n is continuous and Z^n is purely discontinuous. The diffusion and the branching are conditionally independent given η_0.*

Remark 2. *Let* $\boldsymbol{\sigma}^{\star} = max_{1\le k\le d} \parallel \sum_{j=i}^{d} \sigma_{kj}^2 \parallel_\infty$. *The following inequalities are a straightforward consequence of Proposition 3.3.*

$$E\left\{\left(\int_0^t \int_{\Re^d} \nabla f(x)\cdot W^n(dx,ds)\right)^2\right\} \le \boldsymbol{\sigma}^{\star} E\left\{\int_0^t X_s^n(|\nabla f(x)|^2)ds\right\} \quad (3.12)$$

$$E\left\{\left(\int_0^t \int_{\Re^d} f(x) Z^n(dx,ds)\right)^2\right\} = E\left\{\int_0^t X_s^n(f^2)ds\right\} + O\left(\frac{1}{n}\right) \quad (3.13)$$

Equation (3.11) along with the above moment formula gives an idea as to why the superprocess (which is the limit of the above particle process, as $n \to \infty$) is a solution of the martingale problem of Theorem 2.1. The measure W^n converges to a Gaussian measure. However, because of the additional normalization by $\frac{1}{\sqrt{n}}$, the term in W^n vanishes in the limit. On the other hand Z^n converges to a martingale measure, thus yielding the superprocess as a solution of the desired martingale problem (see Walsh (1986) for the details).

Any solution to the martingale problem can be shown via the Ito's formula to satisfy the log-Laplace functional equation given by (2.9), (2.10). Further, using the Markov property, one can extend this representation to the finite-dimensional distributions of the process. Since (2.10) and its finite-dimensional extension has a unique solution, it follows that the solution to the martingale problem is unique. For more general superprocesses, a nice representation in terms of the log-Laplace equation does not exist. In such cases, the uniqueness problem is more difficult.

Uniqueness of the solution to the martingale problem is guaranteed by the convergence of the log-Laplace functionals shown earlier.

Several extensions of the above superprocesses have been studied, for example, multi-type superprocesses, superprocesses in random environment, superprocesses with catalytic branching, superprocesses with age dependent branching and diffusion, multi-level superprocess where each particle itself is a superprocess, competing species superprocess etc. Interesting questions in the area include the uniqueness of solutions to martingale problems characterizing these processes, problems of local extinction or persistence, long term behaviour of the associated branching particle systems etc. Superprocesses can be a powerful tool in the study of the certain elliptic boundary value problems.

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