SOME FLOW PROBLEMS IN MICROPOLAR FLUIDS

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Abstract

In the present paper, we study the flow behaviour of micropolar fluids (whose constitutive equations were formulated by Eringen), in the conventional simple shearing flow, plane Couette flow and Poiseuille flow. We note characteristic features of these fluids, like micro-rotation, unequal shearing stresses and couple stresses in each of the cases. We also compare the flow behaviour of the above fluids with those which occur in the case of non-Newtonian fluids, characterised by other well-known constitutive equations.

1. Introduction

In a series of papers, Eringen[1,2,3] has put forward the theory of simple microfluids. These fluids are characterised by the fact that their properties and behaviour are affected by the local motions of the material particles contained in each of its volume elements. They exhibit certain microscopic effects arising from the local structure and micro-motions of the fluid elements.

In a subsequent paper, Eringen[4] introduced a sub-class of these fluids called the micropolar fluids. They typify the fluids consisting of bar-like elements and dumb-bell molecules and exhibit micro-rotational inertia and other effects.

In the present paper, we study the flow behaviour of these micropolar fluids in the conventional Couette and Poiseuille flows. We evaluate the velocity profile, micro-rotation, shear stress and couple stress in each case and compare them with those which occur in the case of non-Newtonian fluids, characterised by other well known constitutive equations.

2. Basic Equations

The field equations of the micropolar fluids are given by the following partial differential equations:

Continuity Equation :

$$\partial\rho/\partial t + \nabla\cdot(\rho \underset{\sim}{v}) = 0, \qquad [2.1]$$

Momentum Equation:

$$(\lambda_v + 2\mu_v + K_v)\nabla\nabla\cdot\underline{v} - (\mu_v + K_v)\nabla\times\nabla\times\underline{v} + K_v\nabla\times\underline{\nu} - \nabla p + \rho\underline{f}$$
$$= \rho\left[\partial\underline{v}/\partial t - \underline{v}\times(\nabla\times\underline{v}) + \tfrac{1}{2}\nabla(\underline{v}^2)\right], \quad [2.2]$$

First stress moment equation:

$$(\alpha_v + \beta_v + \gamma_v)\nabla\nabla\cdot\underline{\nu} - \gamma_v\nabla\times\nabla\times\underline{\nu} + K_v\nabla\times\underline{v} - 2K_v\underline{\nu} + \rho\underline{l} = \rho j\dot{\underline{\nu}}, \quad [2.3]$$

where $\underline{v}$ and $\underline{\nu}$ are the velocity vector and the micro-rotation vector respectively. ρ is the density of the fluid, λ_v, μ_v, K_v are coefficients of viscosity and α_v, β_v, γ_v are coefficients of gyro-viscosity; $\underline{f}$ and $\underline{l}$ give the body force and body couple respectively; p is the isotropic pressure and the micro-inertial rotation is given by $\dot{\sigma}_k = j\dot{\nu}_k$ where j is a constant on the assumption of micro-isotropy.

3. FLOW THROUGH A STRAIGHT CHANNEL

We consider a micropolar fluid conforming to the above equations to be confined between two infinite parallel plates $y = h$ and $y = -h$ in a Cartesian coordinate system with the axis of x in the direction parallel to the plates and equidistant from them and the axis of y perpendicular to the plates. The two plates are stationary and flow is induced by the introduction of a pressure gradient along the axis of x.

Let u, v, w be the physical components of the velocity vector along the x, y, z axes and ν_x, ν_y, ν_z those of the micro-rotation vector. Then we have

$$u = u(y), \quad v = 0, \quad w = 0, \quad \nu_x = 0, \quad \nu_y = 0, \quad \nu_z = \nu(y) \quad [3.1]$$

The equations of motion are

$$(\mu_v + K_v)\, d^2u/dy^2 + K_v\, d\nu/dy - dp/dx = 0, \quad [3.2]$$

and

$$\gamma_v\, d^2\nu/dy^2 - K_v\, du/dy - 2K_v\nu = 0, \quad [3.3]$$

with boundary conditions

$$u(h) = u(-h) = 0, \quad \nu(h) = \nu(-h) = 0, \quad [3.4]$$

since the fluid sticks to the boundaries. From [3.2] we deduce that $dp/dx =$ constant, say B.

Integrating [3.2] with respect to y and substituting for du/dy in [3.3] we obtain the following equation for ν:

$$d^2\nu/dy^2 - l^2\nu = Py - CK_v/\gamma_1, \quad [3.5]$$

where

$$l^2 = \frac{K_v + 2\mu_v}{K_v + \mu_v} \cdot \frac{K_v}{\gamma_v} > 0, \quad P = \frac{B K_v}{(\mu_v + K_v)\gamma_v},$$

and

$$\gamma_1 = \gamma_v(\mu_v + K_v). \qquad [3.6]$$

On integrating [3.5] we get

$$\nu = A_1 e^{ly} + A_2 e^{-ly} - Py/l^2 + CK_v/\gamma_1 l^2 \qquad [3.7]$$

The corresponding expression for u is

$$u = \frac{1}{(\mu_v + K_v)}\left[\frac{By^2}{2} - Cy - \frac{K_v A_1}{l} e^{ly} + \frac{K_v A_2}{l} e^{-ly} + \frac{PK_v y^2}{2l^2} - \frac{CK_v^2}{\gamma_1 l^2} y + D\right], \qquad [3.8]$$

where A_1, A_2, C, D are arbitrary constants chosen to satisfy the boundary conditions [3.4].

We non-dimensionalise u by $u_0 = -Bh^2/(2\mu_v + K_v)$,

so that

$$u/u_0 = (1 - y'^2) - \frac{K_v}{\mu_v + K_v} \cdot \frac{1}{lh} \cdot \frac{\cosh lh - \cosh lhy'}{\sinh lh}, \qquad [3.9]$$

where $y = hy'$. Introduce two non-dimensional parameters α and β by the relations

$$\alpha = lh, \; \beta = K_v/\mu_v. \qquad [3.10]$$

Then

$$u/u_0 = (1 - y'^2) - \frac{\beta}{\beta + 1} \cdot \frac{1}{\alpha} \cdot \frac{\cosh\alpha - \cosh\alpha y'}{\sinh\alpha}. \qquad [3.11]$$

The first term on the right in [3.11] represents the Newtonian velocity profile. The second term represents the effect of microrotation and gyro-viscosities. Since y' lies between -1 and 1 and α and β are chosen to be positive, this additional term is always negative, so that the velocity profile for micropolar fluids always lies within that for a Newtonian fluid as shown in Fig. 1. We can understand this result by saying that some part of the kinetic energy of the flow is utilised in maintaining the micro-rotations. A similar behaviour is also observed in fluids with couple stresses[5] and for pressure gradient flow in a circular pipe of micropolar fluids[4]. We note below the contribution to the energy equation of the micro-rotational effects and couple stresses in this case :

$$\rho\dot{\epsilon} = -\pi d_{kk} + \lambda_v d_{ll} d_{kk} + (2\mu_v + K_v) d_{kl} d_{lk} + 2K_v(\omega_k - \nu_k)(\omega_k - \nu_k)$$
$$+ \alpha_v \nu_{k,k} \nu_{l,l} + \beta_v \nu_{k,l} \nu_{l,k} + \gamma_v \nu_{l,k} \nu_{l,k} + q_{k,k} + \rho h. \qquad [3.12]$$

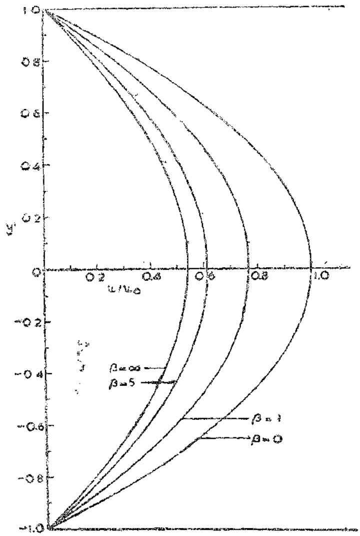


Fig. 1

Velocity Profile $\alpha = 1$.

Contribution of the micro-rotation $= 2\,K_v\,(\omega_k - \nu_k)\,(\omega_k - \nu_k)$

$$= (K_v/2)\left(\frac{h\,(dp/dx)}{\mu_v + K_v}\right)^2 \cdot \frac{\sinh ly}{\sinh lh}. \qquad [3.13]$$

Contribution of the couple stress $= \gamma_v\,\nu_{l,k}\,\nu_{l,k}$

$$= \gamma_v\left(\frac{B}{K_v + 2\,\mu_v}\right)^2 \cdot \left[lh\,\frac{\cosh ly}{\sinh lh} - 1\right]^2. \qquad [3.14]$$

Fig. 2 plots the micro-rotation

$$\nu\,h/u_0 = y' - \sinh \alpha\, y'/\sinh \alpha, \qquad [3.15]$$

against y'. We note that ν is independent of the parameter β and that there is anti-symmetry about the plane $y = 0$, the rotations being in opposite senses in these two regions.

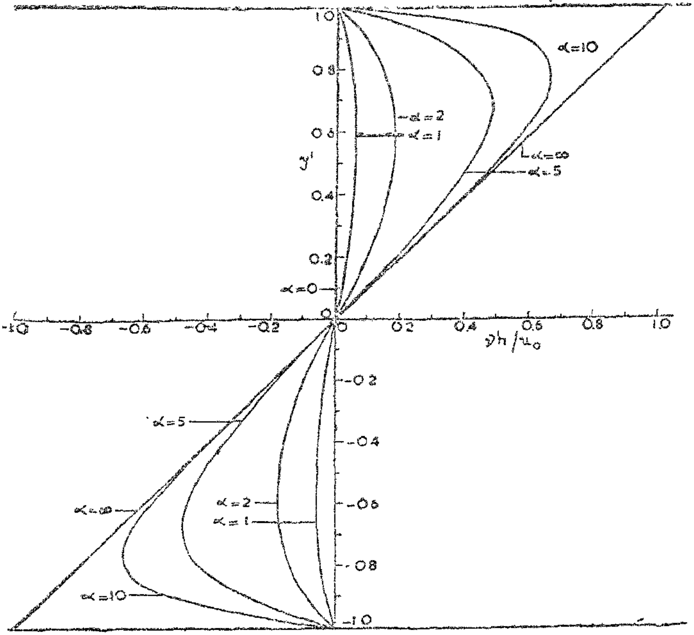


Fig. 2
Micro-Rotation.

The shear-stresses are given by

$$\frac{t_{xy}}{h(dp/dx)} = y' - \frac{\beta}{\beta+1}\cdot\frac{\sinh \alpha y'}{\sinh \alpha} \text{ and } \frac{t_{yx}}{h(dp/dx)} = y'. \qquad [3.16]$$

While t_{yx} is the same as for the Newtonian fluid, the shear-stress t_{xy} is modified by the presence of an additional term. The deviation of the shear stress t_{xy} from that for the Newtonian fluid is shown in Fig. 3. We find that on the boundaries there is a decrease in the magnitude of the shear-stress t_{xy}, when compared with that for a Newtonian fluid. This is in conformity with the experimental observations made in ref. 4.

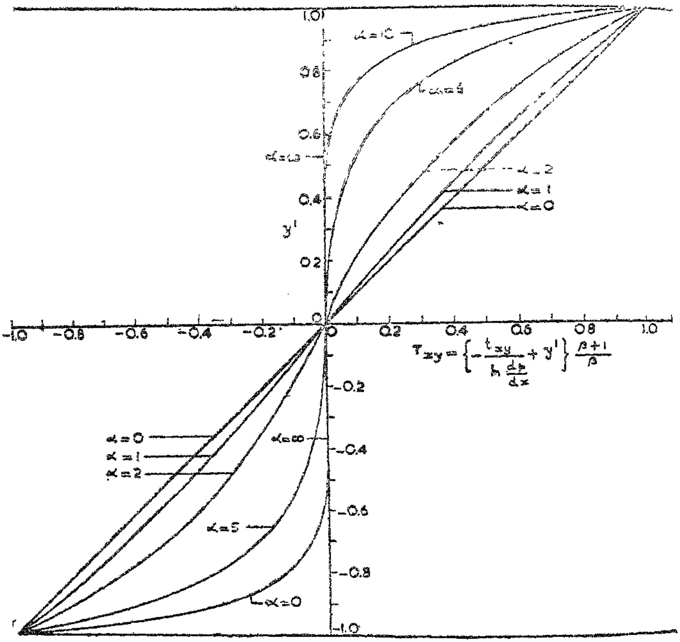


FIG. 3

Shear-stress Difference.

Fig. 4 depicts the couple stress

$$m_{zy} h^2/\beta_v u_0 = 1 - \alpha \cosh \alpha y'/\sinh \alpha = m_{yz} h^2/\gamma_v u_0, \qquad [3.17]$$

which are peculiar to this class of fluids.

In general, non-Newtonian fluids in the above pressure gradient flow display a decrease in apparent viscosity with increasing rate of shear. That is, if $t_{xy} - t_{yx} = \gamma F(\gamma)$ where γ is the rate of shear, then $d[F(\gamma)]/d\gamma < 0$,

Writing the shear-stresses in dimensional form :

$$t_{xy} = 2(2\mu_v + K_v)\gamma - B y, \qquad [3.18]$$

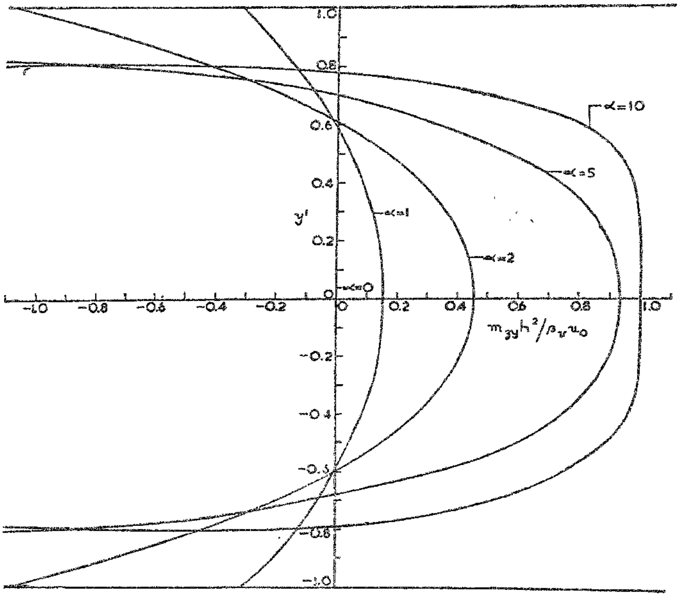


FIG. 4
Couple Stress

and

$$t_{yx} = B\,y. \qquad [3.19]$$

We notice that the symmetric part of the stress tensor, namely,

$$\tfrac{1}{2}\left(t_{xy} + t_{yx}\right) = \left(2\,\mu_v + K_v\right)\gamma, \qquad [3.20]$$

resembles that of a Newtonian fluid of viscosity $(2\,\mu_v + K_v)$.

Again if we write the shear-stresses in the form

$$t_{xy} = \gamma\,F(\gamma), \quad t_{yx} = \gamma\,G(\gamma), \qquad [3.21]$$

we find that $d\,[F(\gamma)]/d\gamma$ and $d\,[G(\gamma)]/d\gamma$ are equal in magnitude but opposite in sign. We cannot, therefore, say anything definitely about the variation of the apparent viscosity with rate of shear.

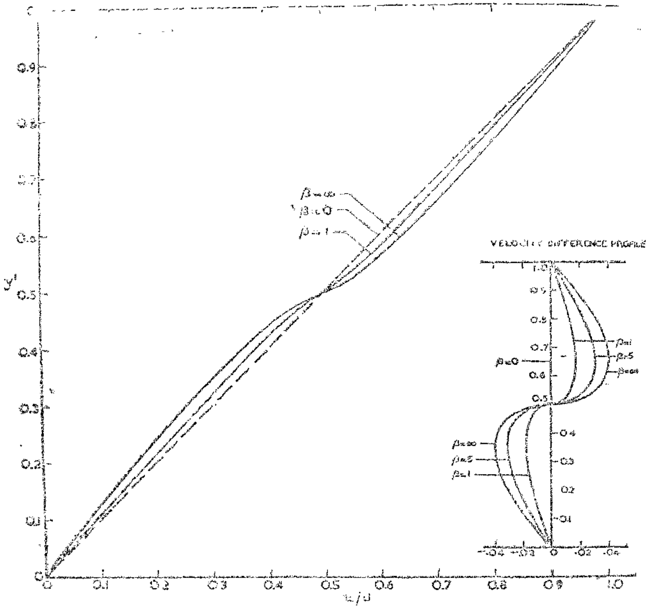


FIG. 5
Velocity Profile α=5.

4. SIMPLE SHEARING FLOW

When the rate of shear γ is a constant we have, using the equation of continuity :

$$u = (du/dy)\, y = 2\,\gamma\, y, \;\; v = 0, \;\; w = 0. \qquad [4.1]$$

To satisfy the balance of first stress moment equation, we require that

$$\nu = -\tfrac{1}{2}\, du/dy = -\gamma. \qquad [4.2]$$

The only non-vanishing vorticity component is

$$\omega_3 = -\,\omega_{12} = -\,\gamma, \qquad [4.3]$$

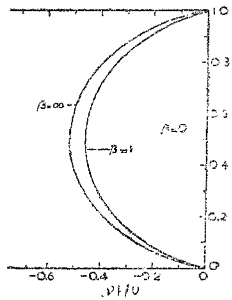


FIG. 6
Micro-Rotation $\alpha = 5$.

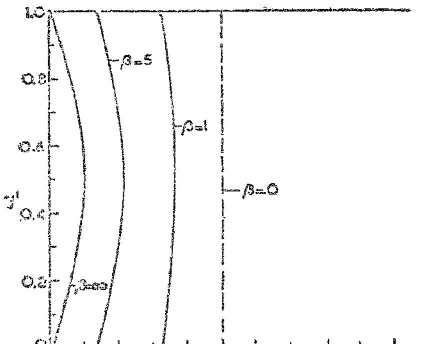
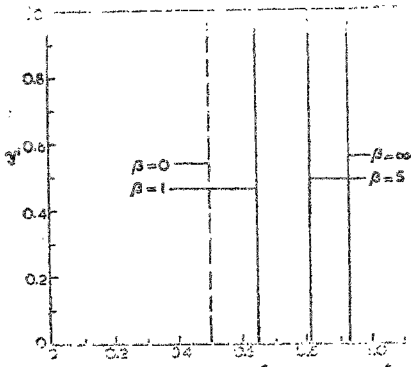


FIG. 7
Shear-Stress $\alpha = 1$.

and $$t_{xy} = (2\,\mu_v + K_v)\,\gamma,\quad t_{yx} = (2\,\mu_v + K_v)\,\gamma. \qquad [44]$$

The apparent viscosity is a constant $= (2\,\mu_v + K_v)$ *i.e.*, no non-Newtonian effects are present and the fluid resembles the Newtonian fluid with viscosity $(2\,\mu_v + K_v)$.



$$T_{yx} = \frac{\beta+1}{\beta} \frac{\alpha}{\alpha - 2 \cdot \frac{e^{\alpha}+e^{-\alpha}-2}{e^{\alpha}-e^{-\alpha}} + \alpha \frac{\beta+2}{\beta}} = \frac{t_{yx} h}{U(2\mu_v + k_v)}$$

FIG. 8

Shear-Stress $\alpha = 1$.

5. PLANE COUETTE FLOW

We now consider the flow of a micropolar fluid between two parallel infinite plates $y = 0$ and $y = h$, when the lower plate is at rest and the upper moves with a constant velocity U, no pressure gradient being present.

In this case $B = 0$, $P = 0$ in [3.7] and [3.8] so that

$$\nu = A_1 e^{ly} + A_2 e^{-ly} + C K_v/\gamma_1 l^2, \qquad [5.1]$$

$$u = [1/(\mu_v + K_v)][-(K_v A_1/l) e^{ly} + (K_v A_2/l) e^{-ly} + D - Cy(1 + K_v^2/\gamma_1 l^2)] \qquad [5.2]$$

Since the fluid sticks to boundaries we have the following boundary conditions:

$$\left.\begin{array}{ll} u = 0 \text{ on } y = 0, & u = U \text{ on } y = h, \\ \nu = 0 \text{ on } y = 0, & \nu = 0 \text{ on } y = h, \end{array}\right\} \qquad [5.3]$$

We have taken ν to be zero at both the boundaries as the fluid sticks to it and no micro-rotation is possible there.

This gives us the following expressions for u, ν:

$$\frac{u}{U} = \frac{\cosh \alpha (1-y') - \cosh \alpha y' - (\cosh \alpha - 1) + \alpha y' \sinh \alpha [1 + (\beta+2)/\beta]}{\alpha [1 + (\beta+2)/\beta] \sinh \alpha - 2(\cosh \alpha - 1)} \qquad [5.4]$$

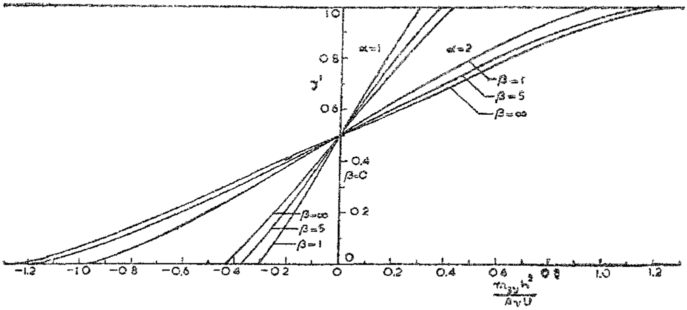


FIG 9
Couple Stress.

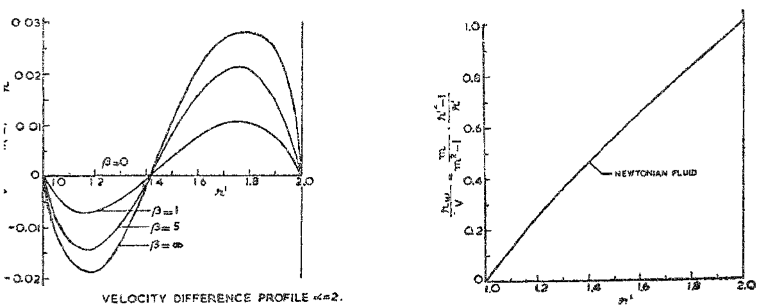


FIG. 10

Velocity Profile.

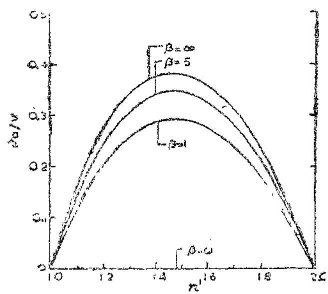


FIG. 11

Micro-Rotation $\alpha = 2$.

and

$$\frac{\nu h}{U} = \frac{\alpha\,(\beta+1)}{\beta}\cdot\frac{\sinh\alpha\,(1-y') + \sinh\alpha\, y' - \sinh\alpha}{\alpha\,[1+(\beta+2)/\beta]\sinh\alpha - 2\,(\cosh\alpha - 1)}, \qquad [5.5]$$

The velocity profile shown in Fig. 5 when $\alpha = 5$, is symmetric about the Newtonian profile given by $u/U = y'$. We notice that the flux across a cross section is the same for all micro-polar fluids irrespective of the values of α and β. (Compare with § 3).

The non-vanishing components of the shear stress, namely.

$$\frac{t_{xy} h}{U\,(2\mu_v + K_v)} = -\alpha\cdot\frac{\sinh\alpha\,(1-y') + \sinh\alpha\, y' - \frac{1}{2}\sinh\alpha\,[1+(\beta+2)/\beta]}{\alpha\,[1+(\beta+2)/\beta]\sinh\alpha - 2\,(\cosh\alpha - 1)} \qquad [5.6]$$

$$\frac{t_{yx} h}{U\,(2\mu_v + K_v)} = \frac{\beta+1}{\beta}\cdot\frac{\alpha\sinh\alpha}{\alpha[1+(\beta+2)/\beta]\sinh\alpha - 2(\cosh\alpha - 1)}, \qquad [5.7]$$

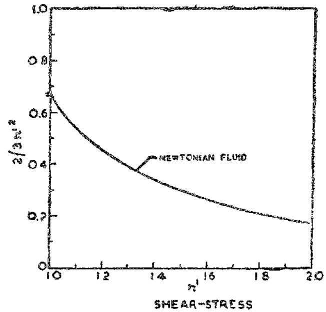
are shown in Figs. 7 and 8, for $\alpha = 1$ and $\beta = 0,\ 1,\ 5,\ \infty$. $\beta = 0$ gives the Newtonian fluid[4] and the shear-stress difference on the boundaries and within the flow is seen in the graphs $t_{yx}h/U(2\mu_v + K_v)$ is a constant, which depends purely on the value of the parameters involved.

Again, we find in dimensional quantities

$$t_{xy} = 2\,(2\,\mu_v + K_v)\,y + C, \quad t_{yx} = -C, \qquad [5.8]$$

where C is an arbitrary constant to be determined from the boundary conditions. Here too, the symmetric part of the stress tensor

$$\tfrac{1}{2}\,(t_{xy} + t_{yx}) = (2\,\mu_v + K_v)\,\gamma\,, \qquad [5.9]$$



Shear-Stress.

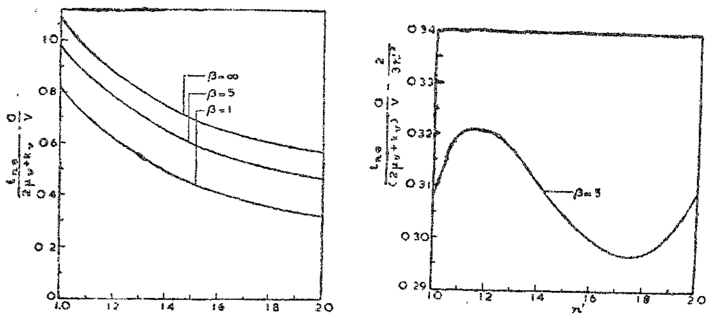


FIG. 12

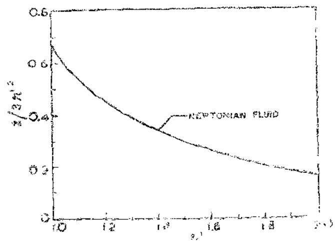
Shear-Stress $\alpha=2$

Shear-Stress Difference $\alpha=2$.

resembles that for a Newtonian fluid of viscosity $(2\,\mu_v + K_v)$.

If we write $$t_{xy} = \gamma\, F(\gamma) \text{ and } t_{yx} = \gamma\, G(\gamma), \qquad [5.10]$$

then $d[F(\gamma)]/d\gamma$ and $d[G(\gamma)]/d\gamma$ are equal in magnitude but opposite in sign, so that we cannot say that there is a decrease in apparent viscosity with increasing rate of shear.



Shear-Stress.

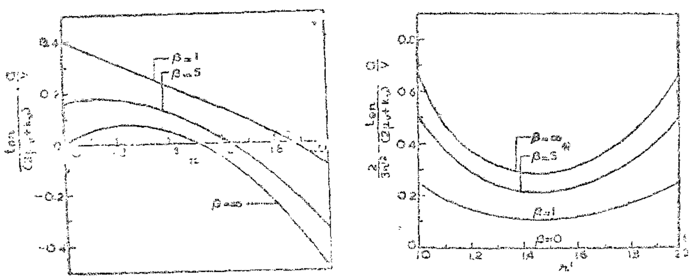


FIG. 13

Shear-Stress $\alpha = 2$.

Shear Stress Difference $\alpha = 2$.

Fig. 9 gives the couple stress

$$\frac{m_{xy}\, h^2}{\beta_0\, U} = \frac{\alpha^2\,(\beta+1)}{\beta}\ \frac{\cosh \alpha\, y' - \cosh \alpha\,(1-y')}{\alpha\,[1+(\beta+2)/\beta]\sinh \alpha - 2(\cosh \alpha - 1)} = \frac{m_{yz}\, h^2}{\gamma_0\, U} \qquad [5.11]$$

According to Eringen[4] the existence of these distributed couples on the fluid surface will produce an effect in a thin layer near the wall, equivalent to reduction of the surface shear and occurrence of boundary layer.

6. FLOW BETWEEN ROTATING VERTICAL COAXIAL CYLINDERS

Next we consider the flow of a micropolar fluid contained between two vertical coaxial cylinders when the outer cylinder rotates with constant velocity. Let the two cylinders be represented by $r = a$ and $r = b$ in a cylindrical polar

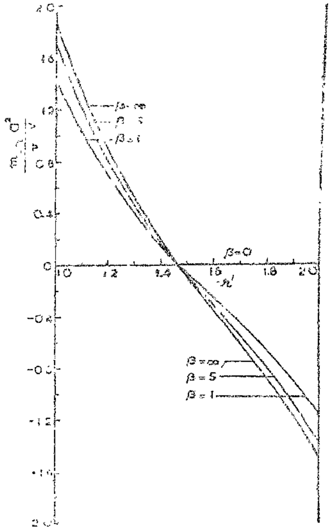


FIG. 14

Couple Stress $\alpha=2$.

coordinate system $(r,\ \theta,\ z)$ and let the outer cylinder $r=b$ rotate with a constant angular velocity V/b. If v_r, v_θ, v_z and ν_r, ν_θ, ν_z are the physical components of the velocity and micro-rotation vectors in the r, θ, z directions, then

$$\left.\begin{aligned} v_r=0,\ \ v_\theta=r\,\omega\,(r),\ \ v_z=0,\\ \nu_r=0,\ \ \nu_\theta=0,\ \ \nu_z=\nu\,(r). \end{aligned}\right\} \qquad [6.1]$$

The balance of momentum and first stress moments give:

$$(\mu_v+K_v)\,\frac{d}{dr}\left\{\frac{1}{r}\ \frac{d}{dr}\,(r^2\,\omega)\right\}-K_v\,\frac{d\nu}{dr}=0, \qquad [6.2]$$

$$-\partial p/\partial r=-\rho\,r\,\omega^2,\ \ \partial p/\partial z=-\rho\,g, \qquad [6.3]$$

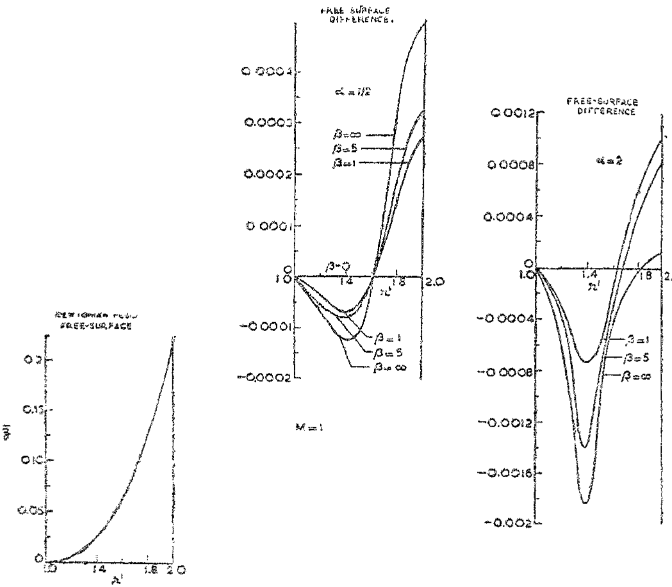


FIG. 15

and

$$\frac{\gamma_r}{r}\frac{d}{dr}\left\{r\frac{dv}{dr}\right\}+\frac{K_v}{r}\frac{d}{dr}\left(r^2\omega\right)-2K_v v=0. \qquad [6.4]$$

The boundary conditions are:

$$\left.\begin{array}{ll} v_\theta=0 \text{ on } r=a, & v_\theta=V \text{ on } r=b,\\ v=0 \text{ on } r=a, & v=0 \text{ on } r=b.\end{array}\right\} \qquad [6.5]$$

Integrating [6.2] with respect to r and substituting for $d(r^2\omega)/dr$ in [6.4] we have the following equation determining v:

$$d^2v/dr^2+(1/r)\,dv/dr-l^2v=-CK_v/\gamma_1, \qquad [6.6]$$

where C is an arbitrary constant and l^2, γ_1 are defined in [3.6].

Solving for ν we get

$$\nu = A\, I_0(lr) + B\, K_0(lr) + C\, K_v/\gamma_1\, l^2, \qquad [6.7]$$

where I_0, K_0 are modified Bessel functions of the first and second kinds of order zero. The corresponding expression for v_θ is:

$$r\,\omega(r) = \frac{K_v}{l(\mu_v + K_v)}\,[A\, I_1(lr) - B\, K_1(lr)] + \frac{Cr}{2(\mu_v + K_v)}\left[1 + \frac{K_v^2}{\gamma_1\, l^2}\right] + D, \qquad [6.8]$$

where A, B, C, D are arbitrary constants to be determined by the boundary conditions [6.5]

$$\begin{aligned} r\omega/V = (1/F)\,\langle I_1(\alpha r')\,\{K_0(\alpha\, m) - K_0(\alpha)\} - K_1(\alpha\, r')\,\{(I_0(\alpha) - I_0(\alpha\, m)\} \\ - \alpha[(1 - r'^2)/2r']\,[(\beta + 2)/\beta + 1]\,\{I_0(\alpha m)\, K_0(\alpha) - I_0(\alpha)\, K_0(\alpha\, m)\} \\ + (1/\alpha\, r')[1 - \alpha\,\{K_0(\alpha\, m)\, I_0(\alpha) + I_0(\alpha\, m)\, K_0(\alpha)\}]\rangle, \qquad [6.9] \end{aligned}$$

where $r = ar'$ and $b = am$ and

$$\begin{aligned} F = (\alpha/m)\langle 2/\alpha^2 - (m/\alpha)\{K_0(\alpha)\, I_1(\alpha m) + I_0(\alpha)\, K_1(\alpha m)\} - (1/\alpha)\,\{K_0(\alpha m)\, I_1(\alpha) \\ + I_0(\alpha m)\, K_1(\alpha)\} + \tfrac{1}{2}\,[(\beta + 2)/\beta + 1]\,(m^2 - 1)\,\{I_0(\alpha\, m)\, K_0(\alpha) \\ - K_0(\alpha\, m)\, I_0(\alpha)\}\rangle. \qquad [6.10] \end{aligned}$$

For $\alpha = 2$, the velocity for micropolar fluids differs only very slightly from the Newtonian profile $r\omega/V = [m/(m^2 - 1)]\cdot[(r'^2 - 1)/r']$, so that we have drawn in Fig 10, the Newtonian velocity profile together with the velocity-difference profile for micropolar fluids, with $\alpha = 2$ and $\beta = 1, 5, \infty$. We notice that the flux across a meridian section is greater for micropolar fluids than for Newtonian fluids. Fig. 11 shows the micro-rotation when $\alpha = 2$ and $\beta = 1, 5, \infty$. It is zero on the two boundaries and attains a maximum close to $r' = 1.5$.

$$\begin{aligned} \nu a/V = [\alpha(\beta + 1)/\beta F]\,[\{K_0(\alpha\, m)\, K_0(\alpha)\}\, I_0(\alpha\, r') + \{I_0(\alpha) - I_0(\alpha m)\}\, K_0(\alpha\, r') \\ + I_0(\alpha\, m)\, K_0(\alpha) - K_0(\alpha\, m)\, I_0(\alpha)]. \qquad [6.11] \end{aligned}$$

The non-vanishing components of shearing stress are:

$$\begin{aligned} &[t_{r\theta}/(2\,\mu_v + K_v)]\,(a/V) \\ &= -(1/r'\,F)\langle I_1(\alpha\, r')\,[K_0(\alpha\, m) - K_0(\alpha)] - K_1(\alpha\, r')\,[I_0(\alpha) - I_0(\alpha\, m)] \\ &\quad - (\alpha/r')\cdot[(\beta + 1)/\beta]\,[I_0(\alpha\, m)\, K_0(\alpha) - K_0(\alpha\, m)\, I_0(\alpha)] \\ &\quad + (1/\alpha\, r')\,[1 - \alpha\,\{K_0(\alpha\, m)\, I_1(\alpha) + I_0(\alpha\, m)\, K_1(\alpha)\}]\rangle, \qquad [6.12] \end{aligned}$$

and

$$\frac{t_{\theta r}}{2\,\mu_v + K_v} \cdot \frac{a}{V} = \frac{\alpha}{F}\,\{[I_0(\alpha\,r') - I_1(\alpha\,r')/\alpha\,r']\,[K_0(\alpha\,m) - K_0(\alpha)] + [K_0(\alpha\,r') + K_1(\alpha\,r')/\alpha\,r']\,[I_0(\alpha) - I_0(\alpha\,m)] + (1/r'^2)\,[I_0(\alpha\,m)\,K_0(\alpha) - K_0(\alpha m)\,I_0(\alpha)]\,(\beta+1)/\beta - (1/\alpha^2 r'^2)\,[1 - \alpha\,\langle K_0(\alpha m)\,I_1(\alpha) + I_0(\alpha\,m)\,K_1(\alpha)\rangle]\}. \qquad [6.13]$$

Figs. 12 and 13 give the shear stresses for micropolar fluids with $\alpha = 2$ and $\beta = 1, 5, \infty$. We have also drawn the corresponding shear-stress for a Newtonian fluid $t_{\theta r}\,a/\mu\,V = t_{r\theta}\,a/\mu\,V = 2/(3\,r'^2)$ The shear-stress difference on the two boundaries is the same for a given α and β and while $t_{\theta r}$ exhibits a decrease on the boundary in comparison with the Newtonian shear stress, $t_{r\theta}$ shows an increase.

We notice again in dimensional form

$$t_{r\theta} = (\mu_v + K_v)\,r\,d\omega/dr + K_v\,(\omega - \nu), \qquad [6.14]$$

and

$$t_{\theta r} = \mu_v\,r\,d\omega/dr - K_v\,(\omega - \nu), \qquad [6.15]$$

so that the symmetric part of the stress tensor

$$\tfrac{1}{2}\,(t_{r\theta} + t_{\theta r}) = \tfrac{1}{2}\,(2\,\mu_v + K_v)\,r\,d\omega/dr = (2\,\mu_v + K_v)\,\gamma, \qquad [6.16]$$

which corresponds to a Newtonian fluid with viscosity $(2\,\mu_v + K_v)$.

The couple stress is shown in Fig. 14.

$$m_{zr}\,a^2/(\gamma_v\,V) = (\alpha^2/F)\cdot[(\beta+1)/\beta]\,\{[K_0(\alpha m) - K_0(\alpha)]\,I_1(\alpha r') - [I_0(\alpha) - I_0(\alpha m)]\,K_1(\alpha r')\} = m_{rz}\,a^2/(\gamma_v\,V). \qquad [6.17]$$

From [6.3], we have

$$p = -\rho\,g\,z + \rho\int^{r} r\,\omega^2\,dr + k_0, \qquad [6.18]$$

where k_0 is a constant, so that the normal stress

$$t_{rr} = t_{\theta\theta} = t_{zz} = -p = \rho\,g\,z - \rho\int^{r} r\,\omega^2\,dr + k_0. \qquad [6.19]$$

Unlike the usual non-Newtonian fluids, visco-inelastic and visco-elastic, we do not find any extra normal stresses in this case. In general for non-Newtonian fluids[6, 7, 8]

$$t_{zz} = \rho\,gz - \int^{r} \rho\,r\,\omega^2\,dr - p_1(r) + \text{ constant}, \qquad [6.20]$$

where $p_1(r)$ represents an additional radial variation of vertical normal pressure over and above that arising from the weight of the liquid and its centripetal acceleration. In sufficiently viscous liquids, the dominant part of the vertical normal pressure required to keep the upper surface of the liquid horizontal is $p_1(r)$. When the upper surface is not held horizontal, but is free, the form of the calculated function $p_1(r)$ will determine, in general, the form of the free surface. The absence of $p_1(r)$ in these fluids is worthy of note.

The equation to the free surface following Bhatnagar and Rathna[9] is given by

$$\bar{z} = z' + p_0/(\rho g a) + k_0 - (1/M) \int^{r'} r' (c\omega/V)^2 \, dr', \quad [6.21]$$

where $M = ga/V^2$ and p_0 is the value of the non-dimensional pressure on the free surface. We note that

$$d\bar{z}/dr' = 1/M \cdot r' \cdot (c\omega/V)^2 \quad [6.22]$$

is positive throughout the annulus and thus no Weissenberg effect is present. $d\bar{z}/dr'$ is zero on the inner boundary and equal to a positive constant on the outer, and $\bar{z}$ is a monotonic increasing function of r' for all $1 < r' < m$. While the positive Weissenberg effect has been predicted by the usual constitutive equations of non-Newtonian fluids such is not the case for micropolar fluids.

To facilitate working we have plotted (Fig. 15)

$$\bar{z} = (1/M) \int_1^{r'} r' (c\omega/V)^2 \, dr', \quad [6.23]$$

thus absorbing in $\bar{z}$ a constant, which depends on the non-Newtonian parameters involved. We have

$$\bar{z} = \frac{1}{F^2 M}\left[A \int_1^{r'} \frac{I_1^2(\alpha r')}{r'} dr' + B \int_1^{r'} \frac{K_1^2(\alpha r')}{r'} dr' - 2AB \int_1^{r'} \frac{I_1(\alpha r')\, K_1(\alpha r')}{r'} dr' \right.$$

$$+ \frac{C^2}{4}\left(\frac{r'^2}{2} - 2 \ln r' - \frac{1}{2r'^2}\right) + \frac{D^2}{2}\left(1 - \frac{1}{r'^2}\right) + (2AD - AC) \int_1^{r'} \frac{I_1(\alpha r')}{r'^2} dr'$$

$$- (2BD - BC) \int_1^{r'} \frac{K_1(\alpha r')}{r'^2} dr' + \frac{AC}{\alpha}[I_0(\alpha r') - I_0(\alpha)]$$

$$\left. + \frac{BC}{\alpha}[K_0(\alpha r') - K_0(\alpha)] + CD\left(\frac{1}{2r'^2} + \ln r' - \frac{1}{2}\right)\right] \quad [6.24]$$

where

$$A = K_0(\alpha m) - K_0(\alpha),$$

$$B = I_0(\alpha) - I_0(\alpha m)$$

$$C = \alpha\,[(\beta+2)/\beta+1]\,[I_0(\alpha m) K_0(\alpha) - K_0(\alpha m) I_0(\alpha)],$$

$$D = (1/\alpha)\ [1 - \alpha\{K_0(\alpha m) I_1(\alpha) + I_0(\alpha m) K_1(\alpha)\}],$$

$$\int_1^{r'} [I_1^2(\alpha r')/r']\,dr' = \tfrac{1}{2}\,[I_0^2(\alpha r') - I_1^2(\alpha r') - I_0^2(\alpha) + I_1^2(\alpha)],$$

$$\int_1^{r'} [K_1^2(\alpha r')/r']\,dr' = \tfrac{1}{2}\,[K_0^2(\alpha r') - K_1^2(\alpha r') - K_0^2(\alpha) + K_1^2(\alpha)],$$

$$\int_1^{r'} [I_1(\alpha r') K_1(\alpha r')/r']\,dr' = -\tfrac{1}{2}\,[I_1(\alpha r') K_1(\alpha r') + I_0(\alpha r') K_0(\alpha r') - I_1(\alpha) K_1(\alpha) - I_0(\alpha) K_0(\alpha)],$$

$$\int_1^{r'} \frac{I_1(\alpha r')}{r'^2}\,dr' = \frac{\alpha}{2}\left[\int_0^{\alpha r'} \frac{I_0(t)-1}{t}\,dt - \int_0^{\alpha} \frac{I_0(t)-1}{t}\,dt + \ln r'\right] - \frac{I_1(\alpha r')}{2r'} + \frac{I_1(\alpha)}{2},$$

$$\int_1^{r'} \frac{K_1(\alpha r')}{r'^2}\,dr' = -\frac{K_1(\alpha r')}{2r'} + \frac{K_1(\alpha)}{2} + \frac{\alpha}{2}\left[\int_{\alpha r'}^{\infty} \frac{K_0(t)}{t}\,dt - \int_{\alpha}^{\infty} \frac{K_0(t)}{t}\,dt\right] \qquad [6\ 25]$$

The free-surface for Newtonian Fluids and the corresponding free surface difference for micropolar fluids are shown in Fig. 15.

7. CONCLUSIONS

We note that while the theory of micropolar fluids takes into account various phenomena such as micro-rotation and micro-rotational inertia and they are able to support couple stress and body couples, they do not predict certain observed phenomena, such as decrease in apparent viscosity with increasing rate of shear in §3, §4 and §5 and the Weissenberg effect in §6. This may perhaps, be due to the linearisation involved[3] in the constitutive equations. We note that the normal stresses for shearing flow for these fluids resembles those for a Newtonian fluid with coefficient of viscosity $(2\mu_v + k_v)$ *i.e.* μ [*Ref.* 4, (3.4), (4.15)] and a modification of the constitutive equations is necessary before we can find a resemblance between these micropolar fluids and general non-Newtonian fluids.

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