ON A MIXED BOUNDARY VALUE PROBLEM OF A CYLINDRICAL HOLE IN AN ELASTIC MATRIX

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ABSTRACT

A formal solution is developed in this note for an axi-symmetric localized indentation problem of cylindrical hole in an elastic matrix.

INTRODUCTION

Many boundary value problems connected with a cylindrical hole in an elastic infinite and semi-infinite space have been solved using classical techniques Tranter[1] has analysed the distortion of a cylindrical hole in an infinite elastic space by a localized hydrostatic pressure on the cylindrical boundary adopting the Fourier integral approach. Bowie[2] has investigated the elastic stresses due to a semi-infinite band of hydrostatic pressure acting over a cylindrical hole in an infinite solid All these problems are of the first boundary value type with the boundary conditions given only in stresses. Blenkarn and Wilhoit[3] have evaluated the stress condition generated by a band of normal stress at the entrance of a circular hole in an elastic half space. This again, is a first boundary value problem. Muki[4] has formulated a general integral-solution for the asymmetric problem of the elastic half space The particular mixed boundary value problem then considered is again the case for which the half space is free of surface shears. Recently Westmann[5] has solved the asymmetric mixed boundary value problems of the elastic half space using the integral formulation techniques of Muki. In the present note, using Love function, a solution is given for the localised piston-type indentation problem of a cylindrical hole in an infinite elastic space. The problem is reduced to the solution of a pair of dual integral equations which have been solved through series method adopting Legendre orthogonal polynomials.

THE PROBLEM

The co-ordinate system employed is shown in Fig. 1. The stresses and displacements in an axisymmetric problem can be derived from the Love function ϕ satisfying

$$\nabla^2 \nabla^2 \phi = 0 \qquad [1]$$

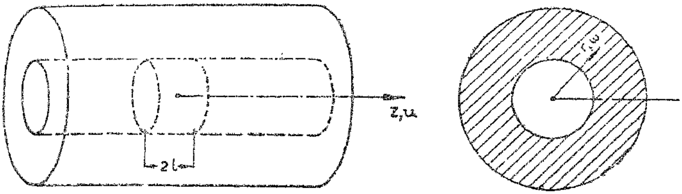

FIG. 1
Co-ordinate Axes and Geometry

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \qquad [2]$$

The stress and displacement field is[6]

$$\sigma_r = \frac{\partial}{\partial z}\left[\mu\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\phi$$

$$\tau_{rz} = \frac{\partial}{\partial r}\left[(1-\mu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\phi$$

$$\sigma_\theta = \frac{\partial}{\partial z}\left[\mu\nabla^2 - \frac{1}{r}\,\frac{\partial}{\partial r}\right]\phi$$

$$\sigma_z = \frac{\partial}{\partial z}\left[(2-\mu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\phi$$

$$u = \frac{(1+\mu)}{E}\left[2(1-\mu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\phi$$

$$\omega = -\frac{(1+\mu)}{E}\,\frac{\partial^2\phi}{\partial r\,\partial z} \qquad [3]$$

where u and w are the axial and radial displacements and E the Young's Modulus and Poisson's ratio respectively. The boundary conditions are, on the inner surface of the hole, $r = a$, throughout its length, it is assumed that

$$\tau_{rz} = 0 \qquad [4]$$

$$w = \text{constant} = \delta \text{ for } |Z| \leq l \quad \text{and} \quad \sigma_r = 0 \text{ for } |Z| > l \qquad [5]$$

i.e., the cylindrical boundary has a punch type axisymmetric displacement δ for a length $2\,l$.

THE SOLUTION

The expression for ϕ is taken as follows:

$$\phi = \int_0^\infty (1/\alpha^3) \left[A(\alpha) K_0(\alpha r) + B(\alpha) \alpha r K_1(\alpha r)\right] \sin \alpha z \, d\alpha \qquad [6]$$

where $A(\alpha)$ and $B(\alpha)$ are function of α, $K_0(\alpha r)$ and $K_1(\alpha r)$ are modified Bessel functions of second kind order zero and one respectively. The expressions for stresses and displacements using Eqs. [6] and [3] are:

$$\sigma_r = \int_0^\infty \left[\{(1-2\mu) B(\alpha) - A(\alpha)\} K_0(\alpha r) - \{A(\alpha)/\alpha r + B(\alpha) \alpha r\} K_1(\alpha r)\right] \cos \alpha z \, d\alpha.$$

$$\tau_{rz} = \int_0^\infty \left[\{2(1-\mu) B(\alpha) - A(\alpha)\} K_1(\alpha r) - B(\alpha) \alpha r K_0(\alpha r)\right] \sin \alpha z \, d\alpha.$$

$$\sigma_z = \int_0^\infty \left[\{A(\alpha) - 2(2-\mu) B(\alpha)\} K_0(\alpha) + B(\alpha) \alpha r K_1(\alpha r)\right] \cos \alpha z \, d\alpha.$$

$$\sigma_\theta = \int_0^\infty \left[A(\alpha) K_1(\alpha r)/\alpha r + (1-\mu) B(\alpha) K_0(\alpha r)\right] \cos \alpha z \, d\alpha. \qquad [7]$$

and

$$u = [(1+\mu)/E] \int_0^\infty (1/\alpha) \left[A(\alpha) K_0(\alpha r) + \{\alpha r K_1(\alpha r) - 4(1-\mu) K_0(\alpha r)\} B(\alpha)\right] \sin \alpha z \, d\alpha.$$

$$\omega = [(1+\mu)/E] \int_0^\infty 1/\alpha \left[A(\alpha) \cdot K_0(\alpha r) + B(\alpha) \cdot \alpha r K_1(\alpha r)\right] \cos \alpha z \, d\alpha. \qquad [8]$$

Substituting in boundary condition (4) that shearing stress is zero on the the inner curved rim, we get

$$B(\alpha) = A(\alpha) K_1(\alpha a)/[2(1-\mu) K_1(\alpha a) - \alpha a K_0(\alpha a)] \qquad [9]$$

Putting in the boundary condition that $\sigma_r \Big|_{r=a} = 0$ for $|z| > l$;

we get
$$\int_0^\infty \zeta(\alpha) \cos \alpha z \, d\alpha = 0 \text{ for } |z| \leq l. \qquad [10]$$

where

$$\zeta(\alpha) = A(\alpha) \left[-\{K_0(\alpha a) + K_1(\alpha a)/\alpha a\} + \frac{\{(1-2\mu) K_0(\alpha a) - \alpha a K_1(\alpha a)\}}{\{2(1-\mu) - \alpha a [K_0(\alpha a)/K_1(\alpha a)]\}} \right]$$

The remaining condition to be satisfied is,

$$w\Big]_{r=a} = \delta \text{ for } |z| \leq l$$

This gives,

$$\int_0^\infty \lambda(\alpha)\, \zeta(\alpha) \cos \alpha z \, d\alpha = \delta \text{ for } |z| \leq l \qquad [11]$$

where

$$\lambda(\alpha) = \frac{(H\mu)}{\alpha E}\left[K_0'(\alpha a) + \frac{\alpha a\, K_1(\alpha a)}{2(1-\mu) - \alpha a\,[K_0(\alpha a)/K_1(\alpha a)]}\right] \div \Big[-\{K_0(\alpha a)$$
$$+ K_1(\alpha a)/\alpha a\} + \{(1-2\mu)\, K_0(\alpha a) - \alpha a\, K_1(\alpha a)\}/\{2(1-\mu)$$
$$- \alpha a\, K_0(\alpha a)/K_1(\alpha)\}\Big]$$

Hence the problem reduces to the solution of the pair of dual integral equations [10] and [11].

Solution of the Dual Integral Equations

For simplicity, a and l are taken to be unity. In the equations [10] and [11], the unknown is $\zeta(\alpha)$ or $A(\alpha)$. Since the surface traction $\sigma_r|_{r=1}$ and not $\zeta(\alpha)$ is of direct interest, the former is expanded in series of Legendre polynomials[7] as follows[8]:

$$\sigma_r|_{r=1} = \sum_{n=0}^{\infty} a_n\, P_n(1-2z^2) \qquad \text{for } |z| \leq 1$$
$$= 0 \qquad \text{for } |z| > 1 \qquad [12]$$

The problem is considered solved when the constants a_n are determined. Equation [12] identically satisfies the first integral equation [10]. Then it follows[9] that

$$\zeta(\alpha) = \sum_{n=0}^{\infty} a_n (-1)^n\, J_{(n+\frac{1}{2})}(\alpha/2)\, J_{-(n+\frac{1}{2})}(\alpha/2) \qquad [13]$$

The second integral equation then takes the form

$$\delta = \int_0^\infty \lambda(\alpha) \sum_{n=0}^{\infty} a_n (-1)^n\, J_{(n+\frac{1}{2})}(\alpha/2)\, J_{-(n+\frac{1}{2})}(\alpha/2) \cos \alpha z\, d\alpha \quad \text{for } |z| \leq 1 \quad [14]$$

Putting

$$\lambda_n(z) = \int_0^\infty \lambda(\alpha)\,(-1)^n\, J_{(n+\frac{1}{2})}(\alpha/2)\, J_{-(n+\frac{1}{2})}(\alpha/2) \cos \alpha z\, d\alpha.$$

Equation [14] becomes

$$\sum_{n=0}^{\infty} a_n \lambda_n(z) = \delta. \qquad [15]$$

The dual integral equations [10] and [11] have now been reduced to equation [15] in which the a_n's are the unknowns. Eqn. [15] is identical in form with the series solution of a Fredholm integral equation of the first kind-Schmidt method[10] can be used for its solution.

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