

# AN EQUIVALENT LINEARISATION FOR THE SOLUTION OF CERTAIN CLASS OF NON-LINEAR VIBRATION PROBLEMS

BY V. ACHYUTA RAO BAPAT & P. SRINIVASAN

*(Department of Mechanical Engineering, Indian Institute of Science, Bangalore-12, India)*

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## ABSTRACT

A piecewise linear method based on the principles of minimum mean square error between the non-linear and an equivalent linear characteristic has been presented. This enables the non-linear differential equation to be reduced to a sequence of linear differential equations which are solved by usual methods. The method is illustrated by examples.

## INTRODUCTION

Several approximate methods are available for the solution of vibration problems governed by non-linear differential equations. Most of these methods are applicable to the so called quasilinear systems, where the system executes small motions. Some of the approximate methods like the Kryloff and Bogoliuboff method, perturbation method, etc., assume the presence of a small parameter in association with the non-linear terms of the differential equation, which renders their contribution quite small as compared to the linear terms. In the perturbation method, the solution is then developed as a convergent power series in the small parameter, with coefficients which are determined by solving a sequence of linear equations. In the Kryloff and Bogoliuboff method, the generating solution of the quasi-linear system is assumed to be harmonic. It is then assumed that the amplitude and phase of this harmonic motion vary slowly with time and the original second order equation is reduced to two first order equations, giving the variation of amplitude and phase with time. If however the contribution of the non-linear terms becomes comparable to that of the linear terms, either because of large initial values or large values of the parameter, the above methods will be inaccurate.

Graphical methods like the phase-plane method for second order systems and the general phase-space methods for higher order systems are available for the solution of autonomous, non-linear differential equations. For non-autonomous systems, Jacobson's<sup>1</sup> phase plane-delta method can be employed.

These methods however are applicable only on an one-off basis for solving individual problems and lack flexibility as they cannot be programmed on a digital computer.

Ergin<sup>2</sup> has proposed an approximate method for the transient response of a non-linear spring mass system. In this method, the non-linear characteristic is approximated by bilinear segments whose slopes are chosen according to the principle of minimum mean square error, between the actual and the approximate characteristics. The method seems to work well for second order conservative systems as is illustrated by three examples<sup>2</sup>. Although calculations made on this basis for first order system (example 1 of this paper) gave good agreement, the results for a second order non-linear dissipative system were not satisfactory. (example 5 of this paper).

#### EQUIVALENT PIECEWISE LINEARISATION METHOD

A method is presented here in which the non-linear characteristic is replaced by equivalent, linear segments over finite intervals instead of two linear segments over the entire characteristic as in Ergin's method. The slopes of these segments are chosen according to the principle of minimum mean square error between the non-linear and the equivalent characteristics. This modification of the bilinear method gives good agreement even in the case of the non-linear dissipative system as shown in (example 5) this paper. The principle of the method is explained below.

Referring to Figure 1 let  $f(\xi)$  be the non-linear characteristic of the system. Consider the range  $\xi_1 \leq \xi \leq \xi_2$ . Let the slope of the equivalent linear segment over this range be  $K_{eq}$ . Then the mean square error in the differential equation is

$$\bar{E}^2 = [1/(\xi_2 - \xi_1)] \int_{\xi_1}^{\xi_2} [f(\xi) - K_{eq} \cdot \xi]^2 d\xi \quad [1]$$

and for a minimum of (1),

$$\partial \bar{E}^2 / \partial K_{eq} = 0$$

$$\therefore \int_{\xi_1}^{\xi_2} [f(\xi) - K_{eq} \cdot \xi] d\xi = 0$$

or

$$K_{eq} = \left[ \int_{\xi_1}^{\xi_2} f(\xi) d\xi \right] / \left[ \int_{\xi_1}^{\xi_2} \xi \cdot d\xi \right] \quad [2]$$

Using this value of  $K_{eq}$  the associated linear differential equation (obtained by replacing  $f(\xi)$  by  $K_{eq} \xi$ ) is solved for the interval  $\xi_1 \leq \xi \leq \xi_2$ . The process is repeated for subsequent segments.

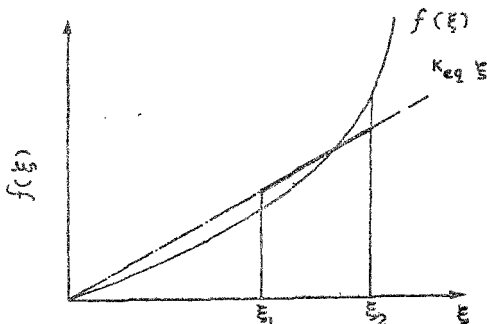


FIG. 1

The accuracy of the method increases as the interval  $(\xi_2 - \xi_1)$  diminishes. It has been found that if the segments are chosen so as to coincide with the original characteristic approximately then the error in the solution of the differential equation is negligible.

#### PHYSICAL INTERPRETATION

The method has an elegant physical interpretation at least in case of certain types of non-linearities. For example (a) in the case of a nonlinear spring if  $f(x)$  is the spring characteristic then

$$\int_{x_1}^{x_2} f(x) dx$$

represents the energy stored in (or released by) the spring over the interval

$$x_1 \leq x \leq x_2.$$

$$\int_{x_1}^{x_2} K_{eq} \cdot x dx,$$

represents the energy stored in (or released by) the spring in the same interval  $x_1 \leq x \leq x_2$ . Thus by equating these two integrals, the energy level of the nonlinear system and the equivalent linear system is maintained the same over the interval. If the interval is quite small it is then reasonable to expect the behaviour of the two systems to be almost identical.

(b) In the case of a non-linear dissipating device whose characteristic is  $\phi(\dot{x})$ , the equivalent viscous damping coefficient  $C_{eq}$  over the range

$$\dot{x}_1 \leq \dot{x} \leq \dot{x}_2$$

is defined by

$$C_{eq} = \left[ \int_{\dot{x}_2}^{\dot{x}_1} \phi(\dot{x}) d\dot{x} \right] / \left( \int_{\dot{x}_1}^{\dot{x}_2} \dot{x} d\dot{x} \right).$$

While

$$\int_{\dot{x}_1}^{\dot{x}_2} \phi(\dot{x}) d\dot{x},$$

gives the total power dissipated by the nonlinear device

$$\int_{\dot{x}_1}^{\dot{x}_2} C_{eq} \dot{x} d\dot{x},$$

gives the power dissipated by the equivalent viscous damper over the same range  $\dot{x}_1 \leq \dot{x} \leq \dot{x}_2$ . For sufficiently small range  $(\dot{x}_2 - \dot{x}_1)$  an almost identical behaviour can be expected.

#### EXTENSION TO THE CASE OF MIXED NONLINEARITIES

Superficially the method appears to be restricted to cases where the nonlinearity is a function of a single argument such as  $x$ ,  $\dot{x}$  etc. However in certain vibrational problems mixed types of nonlinearities which are functions of both  $x$  and  $\dot{x}$  occur. Some engineering examples are cited below in the author's review. In most of the cases a suitable change of variable reduces the nonlinearity either to the form  $f(x)$  or  $\phi(\dot{x})$ . For example:

(1) van der Pol's equation

$$\dot{x} - \epsilon(1 - x^2)\dot{x} + x = 0$$

reduces to the Rayleigh's form

$$\dot{y} - \epsilon(1 - \frac{1}{3}y^2)\dot{y} + y = 0$$

by the transformation  $x = y$ .

So  $f(x, \dot{x}) = -\epsilon(1 - x^2)\dot{x}$ ,  $\phi(y) = -\epsilon(1 - \frac{1}{3}y^2)\dot{y}$

(2) pulling into step of a salient pole synchronous motor:

$$\dot{y} + (K - b \cos 2y)\dot{y} + a \sin 2y = (b \sin y) H(t) = d$$

Here the non-linearity is of the type:

$$f(y, \dot{y}) = -(b \cos 2y)\dot{y} = -\frac{b}{2} \frac{d}{dt} (\sin 2y) \approx -(b/2)(K_{eq} \dot{y})$$

where

$$K_{eq} = \left( \int_{y_1}^{y_2} \sin 2y dy \right) / \left( \int_{y_1}^{y_2} y dy \right)$$

(3) oscillations in a surgetank :

$$\dot{y}' + K |\dot{y}| + \{K_1/\sqrt{(b+y)} + K_2 \sqrt{(b+y)}\} \dot{y} + ay = 0$$

The mixed nonlinearity is

$$\begin{aligned} f(y, \dot{y}) &= \{K_1/\sqrt{(b+y)} + K_2 \sqrt{(b+y)}\} \dot{y} \\ &= \frac{d}{dt} \{2K_1 \sqrt{(b+y)} + \frac{2}{3} K_2 (b+y)^{3/2}\} \\ &\simeq \frac{d}{dt} \{K_{eq} y\} = K_{eq} \dot{y} \end{aligned}$$

$$\text{where, } K_{eq} = \frac{\int_{y_1}^{y_2} \{2K_1 \sqrt{(b+y)} + \frac{2}{3} K_2 (b+y)^{3/2}\} dy}{\int_{y_1}^{y_2} y dy}$$

(4) In some cases such a transformation may have to be applied repeatedly to bring the nonlinearity to a form suitable for applying this method. For example under certain conditions the oscillations in a surge tank are represented by :

$$\dot{y}' = (K/y) |\dot{y}| \dot{y} + ay (y-b) = 0$$

Here the mixed non-linearity is

$$f(y, \dot{y}) = (K/y) |\dot{y}| \dot{y}$$

$$\text{Let, } \phi_1(y, \dot{y}) = (K/y) \dot{y} = \frac{d}{dt} [K \ln y] \simeq K_{1 eq} \dot{y},$$

$$\text{where } K_{1 eq} = \left( \int_{y_1}^{y_2} \ln y dy \right) / \left( \int_{y_1}^{y_2} y dy \right)$$

$$\begin{aligned} \therefore f(y, \dot{y}) &= \phi_1(y, \dot{y}) |\dot{y}| \\ &\simeq K_{1 eq} \dot{y} |\dot{y}| \\ &\simeq (C_{eq} \dot{y}) (\text{Sgn. } \dot{y}) \end{aligned}$$

$$\text{where, } C_{eq} \simeq \left( \int_{y_1}^{y_2} K_{1 eq} \dot{y}^2 dy \right) / \left( \int_{y_1}^{y_2} \dot{y} dy \right)$$

Therefore equivalent linear equation will be

$$\ddot{y} + C_{eq} \dot{y} + a_{eq} y - ab.y = 0$$

where

$$a_{eq} = \left( \int_{y_1}^{y_2} ay^2 dy \right) / \int_{y_1}^{y_2} y dy$$

Although the physical interpretation is not straightforward in cases (1) to (4), it becomes clear after effecting the transformation, *i.e.* the energy level of the non-linear system and the equivalent linear system is maintained same over the interval and also the total power dissipated by the equivalent viscous damper over the same range  $\dot{x}_1 \leq \dot{x} \leq \dot{x}_2$  is equal to the total power dissipated by the non-linear device.

*Example 1:* (Fig. 2)

Consider the first order non-linear differential equation governing the capacitor discharge through a diode.<sup>4</sup>

$$dx/dt + Ax + Bx^2 = 0; \quad x(0) = x_0$$

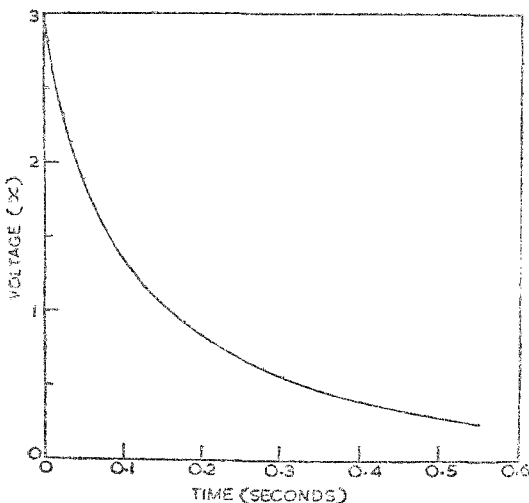


Fig. 2

The exact solution is

$$x = \frac{x_0 \exp. (-A t)}{1 + (B x_0 / A) [1 - \exp. (-A t)]}$$

For illustrating the equivalent linearisation method let  $x_0 = 3, A = 2, B = 3$ . Then the exact solution is

$$x = \frac{3 e^{-2t}}{(5.5) - (4.5 e^{-2t})} = \frac{1}{(1.833 e^{2t}) - (1.5)}$$

For the equivalent linearisation method the characteristic of  $(3x^2)$  Vs.  $x$  is drawn. The characteristic is divided into convenient linear segments. The equivalent linear coefficient  $K_{eq}$  for each segment is calculated from

$$K_{eq} = \left( \int_{x_1}^{x_2} 3x^2 dx \right) / \left( \int_{x_1}^{x_2} x dx \right) = 2 (x_2^2 + x_1 x_2 + x_1^2) / (x_1 + x_2)$$

The solution of the corresponding linear equation will be

$$\tilde{x} = x_1 e^{-At}, \quad x_1 \leq x \leq x_2.$$

As a check the approximate and exact values of  $x$  are compared and tabulated.

TABLE I

Item	1	2	3	4	5	6	7
$x_1$	3.0	2.50	2.25	2.00	1.50	1.00	0.5
$x_2$	2.5	2.25	2.00	1.50	1.00	1.50	0.25
$K_{eq}$	8.27	7.14	6.375	5.28	3.80	2.33	1.1667
$\tilde{x}$	2.50	2.25	2.00	1.50	1.00	0.50	0.25
$\tilde{f}_{sec}$	0.01772	0.02914	0.04323	0.0827	0.1534	0.3134	0.5534
$x_{Exact} = \tilde{x}$	2.50	2.25	2.00	1.50	1.00	0.500	0.25
$\tilde{f}_{Ergin}$	0.018		0.0434	0.0797	0.1478		
$f_{Exact}$	0.01772	0.02914	0.0432	0.0826	0.1530	0.313	0.550
$x_{Ergin}$	2.481	2.225	1.970	1.518	1.1955		

It is seen from the Table I that the approximate solution differs very little from the exact solution.

Example 2: (Fig. 3)

Response of a nonlinear spring mass system to a Rectangular pulse:

The system considered<sup>2</sup> is described by

$$\ddot{x} + x + x^3 = F$$

with

$$x(0) = 0 = \dot{x}(0) \text{ and } F = 1.$$

The approximate solution obtained by equivalent linearisation method is compared with Ergin's solution (2). The spring constant of the equivalent linear equation is computed as

$$K_{eq} = \left( \int_{x_1}^{x_2} x^3 dx \right) / \left( \int_{x_1}^{x_2} x dx \right) \\ = (x_1^2 + x_2^2) / 2$$

The solution of equivalent linear equation will be

$$x = x_1 [\cos \sqrt{(1 + K_{eq})} t + [\dot{x}_1 / \sqrt{(1 + K_{eq})}] [\sin \sqrt{(1 + K_{eq})} t] \\ + [1 / (1 + K_{eq})] \{1 - [\cos \sqrt{(1 + K_{eq})} t]\}$$

The results are tabulated below:

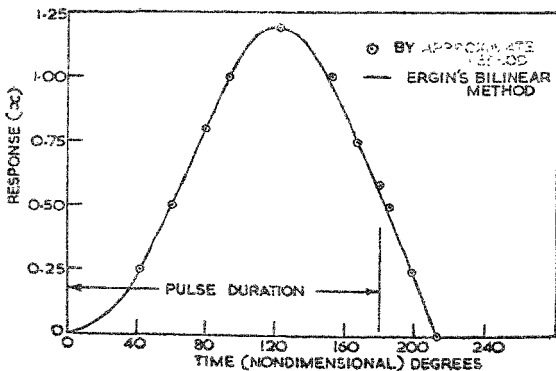


FIG. 3



TABLE 2

Sl. No. (of step)	$x_1$	$x_2$	$K_{cr}$	$\sim$	$\tilde{t}$ deg.	$x$
1	0	0.25	0.03125	0.25	41.4	0.660
2	0.25	0.50	0.15625	0.50	60.25	0.846
3	0.50	0.80	0.4450	0.80	80.03	0.868
4	0.80	1.00	0.8200	1.00	94.45	0.706
5	1.00	1.10	1.1050	1.101	123.95	0.000
6	1.191	1.00	1.210	1.00	152.20	-0.734
7	1.00	0.80	0.820	0.82	166.18	-0.890
8 (a)	0.80	0.50	0.445	0.50	185.88	-0.938
8 (b)	0.80	0.50	0.445	0.585	180.00	-0.889
Pulse duration ends						
8 (c)	0.585	0.50	0.295	0.500	185.26	-0.952
9	0.5	0.25	0.15625	0.25	199.44	-1.056
10	0.25	0	0.03125	0	212.71	-1.089

Example 3: (Fig. 4)

Response of a non-linear spring mass system to a sinusoidal pulse

The differential equation motion is given by (Ref. 2, pp. 640)

$$\dot{x} + x + 5x^3 = \begin{cases} \sin 2t, & 0 \leq t \leq \pi/2 \text{ radians} \\ 0, & t > \pi/2 \text{ radians} \end{cases}$$

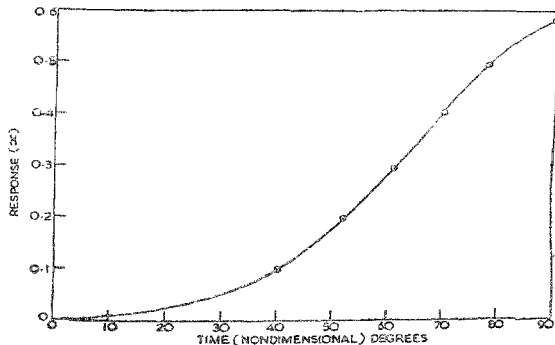


FIG. 4

The equivalent spring constant over the linear stretch from  $x_1$  to  $x_2$  is,

$$K_{eq} = \frac{5}{2} (x_1^2 + x_2^2).$$

The results are tabulated below along with Ergin's bilinear solution. According to Ergin's method

$$x = \frac{1}{3} (2 \sin t - \sin 2t) \text{ for } t \leq 5.5^\circ$$

and

$$x = 1.98 \sin [\sqrt{(3.5)} t - 1^\circ] + 0.172 - 2 \sin 2t \text{ for } t > 55.5^\circ$$

TABLE 3

Sl. No. (of step)	$x_1$	$x_2$	$K_{eq}$	$\tilde{x}$	$t$	$\tilde{x}$	$x$ Ergin	$\tilde{x}$ Ergin
1	0	0.1	0.025	0.1	40	0.3963	0.1	0.396
2	0.1	0.2	0.125	0.2	52	0.57	0.201	0.571
3	0.2	0.3	0.325	0.298	61	0.661	0.296	0.669
4	0.3	0.4	0.622	0.405	70	0.687	0.406	0.684
5	0.4	0.5	1.035	0.498	78	0.622	0.494	0.624
6	0.5	0.6	2.020	0.582	90	0.382	0.581	0.38

Note: Since there is a term on the RHS explicitly involving time, the appropriate time lag (time elapsed upto the commencement of the step) should be introduced in each step. For example for step 2, the equation of motion should read  

$$x + 1.125x = \sin 2(t + 40^\circ)$$
 and so on.)

Example 4: (Fig. 5)

Consider the van der Pol's oscillator (Ref. 5 pp. 63-64)<sup>3</sup> represented by the differential equation in the Rayleigh's form viz.,

$$\ddot{\xi} - \left[ \dot{\xi} - \frac{1}{3} \dot{\xi}^3 \right] + \xi = 0$$

Also assume the initial conditions

$$\xi(0) = -0.05, \dot{\xi}(0) = 0.$$

The equivalent viscous damping coefficient for each linear stretch is obtained from

$$C_{eq} = \left[ \int_{\xi_1}^{\xi_2} (\dot{\xi}^3/3) d\xi \right] / \int_{\xi_1}^{\xi_2} \dot{\xi} d\xi = (\xi_1^2 + \xi_2^2)/6$$

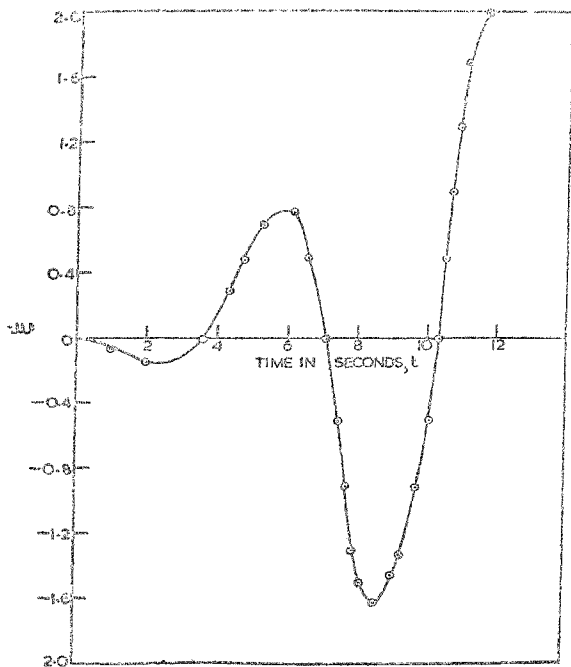


FIG. 5

The results show very good agreement with  $\xi$  vs.  $t$  plot obtained by integration of the phase trajectory. [Ref. (5)]

The values are tabulated in Table 4

TABLE 4

Sl No. of steps	$\xi_1$	$\xi_2$	$C_{ex}$	$\xi$	$\xi$	$\bar{r}$ seconds
1	0	0.3	0.015	-0.191	0.3013	4.365
2	0.3	0.5	0.057	-0.028	0.501	4.775
3	0.5	0.7	0.123	+0.242	0.700	5.225
4	0.7	0.9	0.217	+0.9272	0.784*	6.145
5	0.7	0.5	0.123	0.8602	0.4958	6.545
6	0.5	0	0.048	+1.2050*	0	7.105
7	0	-0.5	0.041	1.150	-0.5	7.463
8	-0.5	-0.9	0.177	1.022	-0.9	7.683
9	-0.9	-1.3	0.417	0.742	-1.3	7.943
10	-1.3	-1.5	0.657	0.429	-1.5	8.163
11(a)	-1.5	-1.7	0.857	-0.237*	-1.638*	-1.58
(b)	-1.5	-1.7	0.857	-0.995	-1.450	9.063
12	-1.5	-1.3	0.657	-1.380	-1.300	9.333
13	-1.3	-0.9	0.417	-1.840	-0.910	9.743
14	-0.9	-0.5	0.177	-1.920	-0.502	10.041
15	-0.5	0	0.041	-2.000*	0	10.321
16	0	0.5	0.041	-1.972	+0.497	10.546
17	0.5	0.9	0.177	-1.880	+0.900	10.686
18	0.9	1.3	0.417	-1.720	+1.300	10.838
19	1.0	1.7	0.687	-1.500	+1.700	11.042
20	1.7	2.0	1.071		2.08*	11.662

\* (indicate relative extreme values)

Example 5: (Fig. 6)

Relaxation oscillations of the van der Pol Oscillator<sup>3</sup>:

Consider the van der Pol equation,  $\dot{x} - \epsilon(1-x^2)\dot{x} + x = 0$  with  $\epsilon$  having a large value. Here  $\epsilon$  is set, equal to 5. The solution of such a system is characterised by Jerky, nonsinusoidal oscillations. The results available from a phase plane plot of the above equation are compared with the approximate solution obtained by piecewise linearisation.

The differential equation is:

$$\dot{y} - 5(1-y^2/3)\dot{y} + y = 0$$

where

$$y = \int x \, dt$$

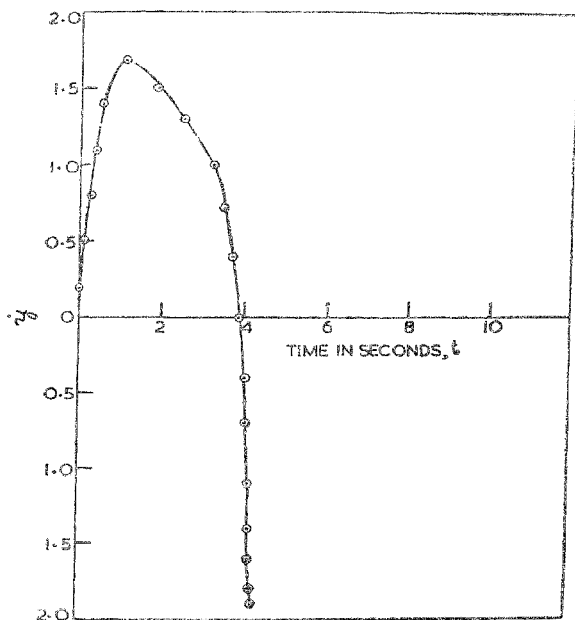


FIG. 6

The equivalent viscous damping coefficient is calculated as

$$C_{eq} = \left( \int_{y_1}^{y_2} y^3 dy \right) / \int_{y_1}^{y_2} y dy = \frac{4}{6} (y_1^2 + y_2^2)$$

The equivalent linear equation then becomes

$$\ddot{y} - (5 - C_{eq}) \dot{y} + y = 0$$

The results are tabulated in Table 5 :

TABLE 5

$t$	$\tilde{y}$	$\tilde{y}$
0.1980	0.0637	0.5071
0.3090	0.1355	0.8000
0.4080	0.2290	1.1000
0.5195	0.3680	1.4000
0.5705	0.4380	1.5000
1.0735	1.2250	1.6820
1.8235	2.3050	1.5000
2.5285	3.3800	1.3000
3.2685	4.2940	1.0000
3.5285	4.5248	0.7070
3.7785	4.6180	0.4000
3.9460	4.6800	0.0000
4.0114	4.6460	-0.4000
4.0544	4.5980	-0.7000
4.1054	4.5150	-1.1030
4.1474	4.4600	-1.4000
4.1784	4.4000	-1.6000
4.2238	4.2900	-1.8000
4.2338	4.2200	-1.9000

Note: Ergin's method gives an unbounded solution for the second bilinear segment)

#### *Suitability for Digital programming :*

On a digital computer, the method can be easily programmed to yield very accurate results. The method is also adaptable for long hand calculations as it only involves the solution of a sequence of ordinary equivalent linear differential equations.

#### *Accuracy of the method :*

Here accuracy is meant to indicate the closeness between the equivalent linear differential equation and the original nonlinear differential equation rather than the closeness of the solutions. If the nonlinear characteristic is

$f(\xi)$  and  $K_{eq}$ .  $\xi$  is its linear equivalent, then the error is proportional to  $[f(\xi) - K_{eq} \xi]$  and by choosing small intervals in  $\xi$ , the absolute error can be maintained small in addition to maintaining the mean square error a minimum. For long hand computation, it is sufficient to draw the characteristic and divide it into approximately linear segments and calculate  $K_{eq}$  over each segment.

#### CONCLUSIONS

Several approximate methods are available for the solution of vibration problems governed by nonlinear differential equations. None of these methods are general and they are applicable in certain cases only. Here a method has been presented that is fairly general and covers both the autonomous and non-autonomous cases. No restriction is laid on the size of the non-linearity.

Some examples of non-linearities normally occurring in engineering applications have been solved by this method.

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