STUDY OF MECHANICAL SYSTEMS SUBJECTED TO COMBINED DETERMINISTIC AND RANDOM EXCITATION

BY B. V. DASARATHY AND P. SRINIVASAN

(*Department of Mechanical Engineering, Indian Institute of Science, Bangalore-12, India*)

ABSTRACT

This paper describes the study of a mechanical system idealised by a single degree of freedom system, when subjected to combined random and deterministic excitation. The results reveal the similar features existing between the case under consideration and the case of a similar system subjected to pure sinusoidal excitation, which fact is explained by the assumption that the random part of the combined excitation is a narrow band process with a central frequency equal to the frequency of the deterministic part of the excitation. The analysis has been extended to the case of a two degree of freedom system.

INTRODUCTION

It is well known that mechanical systems are not always subjected to deterministic forces such as a harmonic force of fixed frequency and amplitude. Quite frequently one comes across systems with inputs which are random in nature. Under such circumstances, the instantaneous values of the amplitude and phase of the response have little meaning and so recourse is taken to statistical analysis. Systems with random inputs have been extensively studied among others by Thomson and Burton[1], Crandall[2], Robson[3], Caughey and Stumpf[4]. It is possible that in some cases, the forces acting on the systems, may, instead of being purely deterministic or purely random in nature, be of a combined nature, *i.e.* partly deterministic and partly stochastic in character.

The object of this paper is to study the response of linear mechanical systems subjected to combined deterministic (sinusoidal or periodic type) and random (narrow band, Gaussian) excitation. Use has been made of the studies made by Rice[5], Middleton[6], Davenport and Root[7] and others, concerning narrow band processes etc., in arriving at the response of the system and its statistical properties. The mechanical system, for purposes of analysis, is idealised by a linear single degree of freedom system with viscous damping. The analysis has then been extended to systems with two degrees of freedom.

Analysis

Consider a linear damped single degree of freedom system subjected to combined deterministic and random excitation. A typical example for this, would be a system with random base motion, subjected to deterministic forces. When the base motion is a direct consequence of regarding the base as a lightly damped system subjected to broad band random excitation, the random part of the combined excitation on the main system can be construed to be a narrow band process.

The governing equation of motion for such a system can be written as

$$\ddot{x} + 2 \beta \dot{x} + p_0^2 x = f(t) + F_0 \cos(p_c t - \theta) = F(t) \qquad [1]$$

with
$$p_0^2 = K/M; \quad \beta = C/2M; \qquad [2]$$

where M, C and K are the system parameters,

- x is the displacement or response of the system
- $f(t)$ is a stationary narrow band Gaussian random variable
- F_0 is the constant amplitude of the deterministic part of the excitation.
- p_c is the frequency of the deterministic part of the excitation
- θ is an arbitrary phase angle in the interval: $(0 - 2\pi)$

It is possible [6,7] to separate $f(t)$ into cosine and sine functions in the form

$$f(t) = f_c(t) \cos p_c t + f_s(t) \sin p_c t \qquad [3]$$

Here the random coefficients $f_c(t)$ and $f_s(t)$ are normal variables with mean zero. These variables will be statistically independent provided $f(t)$ has a narrow band spectrum symmetrical about p_c, the frequency of the deterministic excitation. Writing,

$$F(t) = F_c(t) \cos p_c t + F_s(t) \sin p_c t \qquad [4]$$

gives
$$F_c(t) = F_0 \cos \theta + f_c(t) \qquad [5]$$

and
$$F_s(t) = F_0 \sin \theta + f_s(t) \qquad [6]$$

Hence the sum process $F(t)$ can be put as

$$F(t) = F_E(t) \cos [p_c t - \phi(t)] \qquad [7]$$

where
$$F_c(t) = F_E(t) \cos \phi(t) \qquad [8]$$

$$F_s(t) = F_E(t) \sin \phi(t) \qquad [9]$$

With these relations, it is possible to arrive at the probability density function for the envelope of the sum process, $F_E(t)$, and this is shown to be approximately Gaussian under certain conditions. (Refer appendix.)

Expanding $f(t)$ as an infinite series in Cosine and Sine functions,

$$f(t) = \sum_{n=0}^{\infty} (a_n \cos p_n t + b_n \sin p_n t) \tag{10}$$

with $\quad\quad a \le t \le b\,; \; p_n = (2\pi n/T) + p_c\,; \; T = b - a$

where a_n and b_n are normally distributed random coefficients with mean zero. Setting $p_n = p'_n + p_c$ in equation [10] and using the relation [3]

$$f_c(t) = \sum_{n=0}^{\infty} (a_n \cos p'_n t + b_n \sin p'_n t) \tag{11}$$

$$f_s(t) = \sum_{n=0}^{\infty} (-a_n \sin p'_n t + b_n \cos p'_n t) \tag{12}$$

$$a \le t \le b\,; \; p'_n = 2\pi n/T\,; \; T = b - a$$

Here $\quad \langle a_m c_n \rangle = \langle b_m b_n \rangle\,; \; \langle a_m b_m \rangle = \langle a_n b_m \rangle = 0\,; \; \langle a_m \rangle = \langle b_n \rangle = 0 \tag{13}$

Hence, $\quad\quad \langle f_c(t) \rangle = 0 = \langle f_s(t) \rangle \tag{14}$

Now, $\quad\quad \langle f_c(t_1)\, f_c(t_2) \rangle = \langle f_s(t_1)\, f_s(t_2) \rangle$
$$= \sum_{m,n} a_m a_n \cos(p'_n t_2 - p'_m t_1) \tag{15}$$

and $\quad\quad -\langle f_c(t_1)\, f_s(t_2) \rangle = \langle f_s(t_1)\, f_c(t_2) \rangle$
$$= \sum_{m,n} a_m a_n \sin(p'_n t_2 - p'_m t_1) \tag{16}$$

Allowing 'a' and 'b' to tend to $-\infty$ and $+\infty$ respectively and noting the fact, $f_c(t)$ and $f_s(t)$ are stationary

$$\underset{\text{as } T \to \infty}{Lt} \langle a_m a_n \rangle = \delta_{m,n} S_f(p_m)\, dp/2\pi \tag{17}$$

Hence $\quad \langle f_c(t_1)\, f_c(t_2) \rangle = \underset{T \to \infty}{Lt} \sum_m 1/T.\quad S_f(p_m) \cos p'_m (t_2 - t_1)$
$$= (1/2\pi). \int_0^{\infty} S_f(p) \cos(p - p_c)\, t\, dp = \sigma_f^2 \rho_c(t) \tag{18}$$

Similarly $\left\langle f_c\left(t_1\right) f_s\left(t_2\right) \right\rangle = -\left(1/2\pi\right) \int_0^{\infty} S_f\left(p\right) \sin\left(p - p_c\right) t\; dp = \sigma_f^2\, \mu_c\left(t\right)$

as $dp' = p - p_c ; \quad t = t_2 - t_1 ; \quad 1/T = \triangle p' \rightarrow dp'$ in the limit.   [19]

Therefore, $\left\langle f_c^2 \right\rangle = \left\langle f_s^2 \right\rangle = \left(1/2\pi\right) \int_0^{\infty} S_f\left(p\right) dp = \sigma_f^2$ [20]

Also, $\rho_c\left(0\right) = 1 ; \quad \mu_c\left(0\right) = 0 ; \quad \rho_c\left(t\right) = \rho_c\left(-t\right) ; \quad \mu_c\left(t\right) = -\mu_c\left(-t\right) ;$

Hence, $R_{f_c}\left(t\right) = R_{f_s}\left(t\right) = \sigma_f^2 f_c\left(t\right)$ [21]

and $R_{f_c f_s}\left(t\right) = -R_{f_s f_c}\left(t\right) = \sigma_f^2 \mu_c\left(t\right)$ [22]

[21] and [22] give the correlation properties of f_c and f_s. When $f(t)$ is a narrowband process with its spectrum symmetrical about the central frequency p_c, $f_c\left(t\right)$ and $f_s\left(t\right)$ become statistically independent normal processes and expression [22] will vanish.

Now, considering the excitation as the sum of a series of impulses and assuming the initial conditions to be $x\left(0\right) = 0 = \dot{x}\left(0\right)$; the response of the system to this combined excitation is obtained in the form of a convolution integral

$$x\left(t\right) = \int_{-\infty}^{t} h\left(t - t_1\right) F\left(t_1\right) dt_1 \tag{23}$$

Changing the variable $t_1 = t - \tau$

$$x\left(t\right) = \int_0^{\infty} h\left(\tau\right) F\left(t - \tau\right) d\tau \tag{24}$$

where $h(\tau) =$ the impulse response of the given system

$$= \left(e^{-\beta \tau}/p_1\right) \sin p_1 \tau \tag{25}$$

with $p_1^2 = p_0^2 - \beta^2$

Assuming stationary type of excitation with zero mean,

Expectation or mean of $x\left(t\right)$: $\left\langle x\left(t\right) \right\rangle = 0$ [26]

Therefore $\sigma_x^2 = \left\langle x^2\left(t\right) \right\rangle = R_x\left(0\right)$ [27]

Now,

$$R_x\left(\tau\right) = \left\langle x\left(t\right) x\left(t + \tau\right) \right\rangle$$

$$= \int_0^{\infty} h\left(\tau_1\right) \int_0^{\infty} h\left(\tau_2\right) F\left(t - \tau_1\right) F\left(t + \tau - \tau_2\right) d\tau_1 d\tau_2 \tag{28}$$

The terms in the angular brackets $= R_F (\tau - \tau_1 - \tau_2)$

Now, from the relations previously established,

$$R_F(\tau) = \left\langle F(t) \; F(t + \tau) \right\rangle = \tfrac{1}{2} \left(F_0^2 + 2 \, \sigma_f^2 \right) \cos p_c \, \tau \qquad [29]$$

Using [29] in [28] and integrating,

$$R_x(\tau) = \tfrac{1}{2} \left(F_0^2 + 2 \, \sigma_f^2 \right) \cos p_c \, \tau / [(p_0^2 - p_c^2)^2 + 4 \, p_c^2 \; \beta^2] \qquad [30]$$

Mean Square Response and Variance :

$$\sigma_x^2 = \left\langle x^2 (t) \right\rangle = A / [(1 - p^2)^2 + 4 \, p^2 \, q^2] \qquad [31]$$

where $p =$ ratio of the forcing frequency p_c to the undamped natural frequency p_0 of the system $= p_c / p_0$

$q = \beta / p_0$

$$A = \tfrac{1}{2} \left(F_0^2 + 2 \, \sigma_f^2 \right) / p_0^4$$

Statistical properties of the velocity $\dot{x}(t)$ and acceleration $\ddot{x}(t)$: using [26],

$$\left\langle \dot{x}(t) \right\rangle = 0 = \left\langle \ddot{x}(t) \right\rangle \qquad [32]$$

Also,

$$R_{\dot{x}}(\tau) = - \left(d^2 / d \tau^2 \right) R_x(\tau) = p_c^2 \cdot R_x(\tau) \qquad [33]$$

Therefore, mean square velocity $= \left\langle \dot{x}^2 (t) \right\rangle = A_1 \, p^2 / [(1 - p^2)^2 + 4 \, p^2 \, q^2] \qquad [34]$

where

$$A_1 = \tfrac{1}{2} \left(F_0^2 + 2 \, \sigma_f^2 \right) / p_0^2$$

Similarly mean square acceleration $= \left\langle \ddot{x}^2 (t) \right\rangle$

$$= A_2 \, p^4 / [(1 - p^2)^2 + 4 \, p^2 \, q^2] \qquad [35]$$

where

$$A_2 = \tfrac{1}{2} \left(F_0^2 + 2 \, \sigma_f^2 \right).$$

Extending the analysis to the case of a two degree of freedom system, the equations of motion can be written as

$$m_1 \, \ddot{x}_1 + k_1 \, x_1 + k_2 \, (x_1 - x_2) = F(t) \qquad [36]$$

$$m_2 \, \ddot{x}_2 + k_2 \, (x_2 - x_1) = 0 \qquad [37]$$

where m_1, m_2, k_1 and k_2 are the system parameters and $F(t)$ is, as in the previous case, the combination of periodic and random excitation.

Using the method of orthogonal normal modes,

with

$$x_1 = x_{11} + x_{12} ; \quad x_{21} = c_1 \, x_{11}$$

$$x_2 = x_{21} + x_{22} ; \quad x_{22} = c_2 \, x_{12} \qquad [38]$$

gives the responses in the form,

$$x_1(t) = \int_0^t \left[\frac{F(\tau)}{M_1 p_1} \sin p_1(t-\tau) + \frac{F(\tau)}{M_2 p_2} \sin p_2(t-\tau) \right] \, d\tau \quad (t \geqslant \tau) \qquad [39]$$

$$x_2(t) = \int_0^t \left[\frac{c_1 F(\tau)}{M_1 p_1} \sin p_1(t-\tau) + \frac{c_2 F(\tau)}{M_2 p_2} \sin p_2(t-\tau) \right] d\tau \quad (t \geqslant \tau) \qquad [40]$$

Here, $c_n = k_2/(m_2 \lambda_n^2 + k_2); \quad \lambda_n^2 = -p_n^2 \qquad (n = 1, 2)$

$$\lambda_1^2 + \lambda_2^2 = -\frac{k_2}{m_2} + \frac{k_1 + k_2}{m_1}; \quad \lambda_1^2 \lambda_2^2 = k_1 k_2/m_1 m_2$$

and $M_n = m_1 + c_n^2 m_2 \ (n = 1, 2)$ \qquad [41]

By transformation of variables [39] and [40] can be rewritten as,

$$x_1(t) = \int_0^t [h_1(\tau) + h_2(\tau)] \, F(t-\tau) \, d\tau \qquad [42]$$

$$x_2(t) = \int_0^t [c_1 h_1(\tau) + c_2 h_2(\tau)] \, F(t-\tau) \, d\tau \qquad [43]$$

where $h_n(\tau) = \sin p_n \tau/M_n p_n \ (n = 1, 2)$ \qquad [44]

As before, assuming a combined deterministic and random excitation on the system, the statistical properties are determined.

Mean or Expectation of x_1 and x_2:

As in the previous case, $<x_1> = 0 = <x_2>$ \qquad [45]

Using [29], the autocorrelation functions are determined as

$$R_{x_1}(\tau) = \tfrac{1}{2}(A + B + C)(F_0^2 + 2\sigma_f^2) \cos p_c \tau \qquad [46]$$

$$R_{x_2}(\tau) = \tfrac{1}{2}(c_1^2 A + c_2^2 B + 2c_1 c_2 C)(F_0^2 + 2\sigma_f^2) \cos p_c \tau \qquad [47]$$

Here $A = f(m, n), \ m = n = 1$;
$$B = f(m, n), \quad m = n = 2 ; \qquad [48]$$
$$C = f(m, n), \quad m = 1 ; \ n = 2 ;$$

where $f(m, n)$

$$= \left\{ p_c^2 \cdot \sin p_m t \sin p_n t - (p_m \sin p_n t + p_n \sin p_m t) \cdot p_c \sin p_c t \right.$$
$$+ p_m p_n \left[1 + \cos p_m t \cos p_n t - \cos p_c t \cdot (\cos p_m t + \cos P_n t) \right] \right\}$$
$$\div \left\{ M_m M_n p_m p_n (p_m^2 - p_c^2)(p_n^2 - p_c^2) \right\}$$

Mean square Values of x_1 and x_2:

$$\sigma_{x_1}^2(t) = \langle x_1^2(t) \rangle = \tfrac{1}{2}(A + B + C)(F_0^2 + 2\sigma_f^2) \qquad [49]$$

$$\sigma_{x_2}^2(t) = \langle x_2^2(t) \rangle = \tfrac{1}{2}(c_1^2 A + c_2^2 B + 2c_1 c_2 C)(F_0^2 + 2\sigma_f^2) \qquad [50]$$

The mean square values of the velocities and accelerations bear the same ratios to the mean square responses as in the previous case.

CONCLUSIONS

Comparing the results obtained above for the case of a linear damped single degree of freedom system subjected to combined deterministic and random excitation, with the wellknown results for a similar system subjected to pure sinusoidal excitation, it is seen that the mean square values of the three quantities, *viz*, displacement, velocity and acceleration, bear the same ratio $[p_c^2]$ in both the cases. This is because of the fact, that in the case of combined excitation, the stochastic part of the excitation is a narrow band process symmetrical about the central frequency p_c which happens to be the frequency of the deterministic part of the excitation.

In view of the fact a lightly damped oscillator acts as a narrow band filter when subjected to broad band random excitation, the results obtained here correspond to the case of systems which are excited through an auxiliary lightly damped system subjected to ordinary broad band excitation.

Figures 1, 2 and 2 give the plot of the mean square values of the response, its velocity and acceleration, respectively *Vs*. the frequency ratio p, which is defined as the ratio of the forcing frequency p_c to the undamped natural frequency p_o of the system. These have been plotted for different damping ratios and as is to be expected, the increase in damping ratios corresponds to decrease in the mean square values. From Fig. 1. it is seen that the effect of the frequency ratio is very similar to the case of system subjected to sinusoidal excitation the maximum value of the mean square response being in the neighbourhood of the resonace (*i e. p* = unity) depending on the extent of damping. The mean square velocity plot is similar to the mean square response plot, but for the difference in the value at zero frequency. While the mean square acceleration plot shows that as the frequency ratio increases the value tends to unity. in case of mean square response and velocity, their value tends to zero as frequency ratio increases beyond the peak values.

APPENDIX

The probability density function of the envelope $F_E(t)$ can be arrived at by first evaluating the joint probability density function $F_c(t)$, $F_s(t)$ and θ, which gives the joint p.d.f. of $F_0(t)$, ϕ and θ, which is then integrated with respect to ϕ and θ.
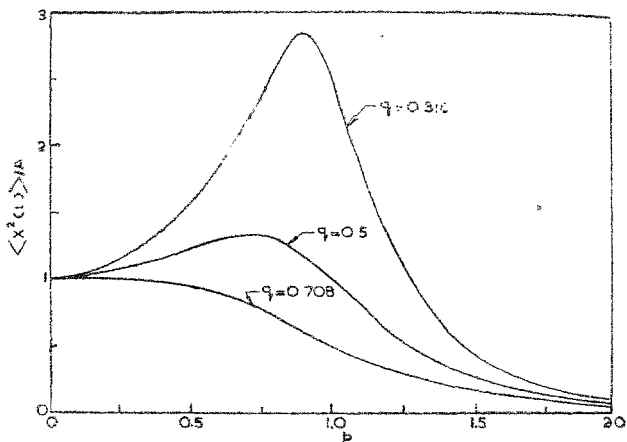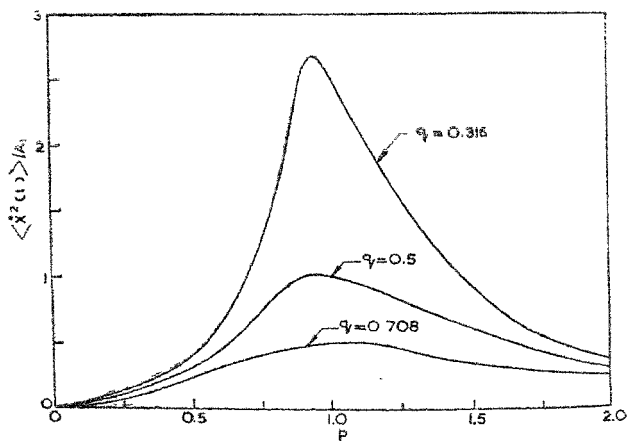
FIG. 1
Mean square response vs Frequency Ratio

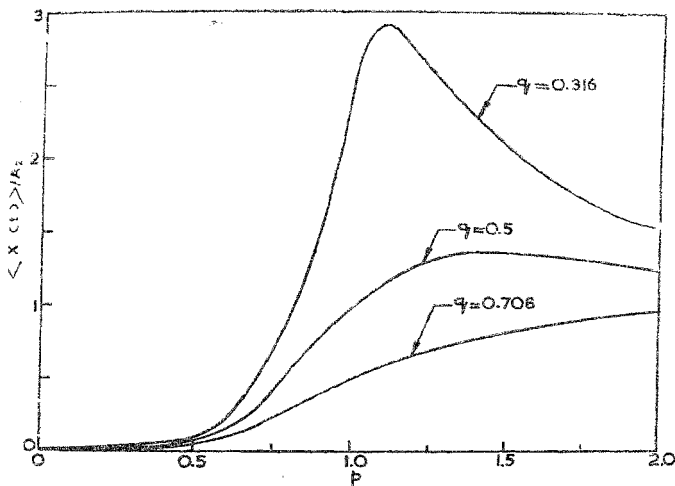FIG. 2
Mean square velocity vs Frequency Ratio

FIG 3
Mean Square Acceleration *Vs.* Frequency Ratio

$$p\left(F_E\right) = \left(F_E/\sigma_f^2\right) \exp\left[-\left(F_E^2 + F_0^2\right)/2\,\sigma_f^2\right] I_0\left(F_0\,E_E/\sigma_f^2\right) \text{ for } F_E > 0$$
$$= 0 \text{ otherwise.}$$

For large values of $F_0\,F_E/\sigma_f^2 = Z$

$$I_0\left(Z\right) = \frac{e^Z}{\left(2\pi\,Z\right)^{1/2}} \left(1 + \frac{1^2}{1\left(8Z\right)} + \frac{1^2\,3^2}{2\left(8\,Z\right)^2} \cdots\right)$$

for values of $F_0\,F_E >> \sigma_f^2$

$$p\left(F_E\right) = \frac{1}{\sigma_f} \left[\frac{F_E}{2\pi\,F_0}\right]^{1/2} \exp-\left[\frac{\left(F_E - F_0\right)^2}{2\,\sigma_f^2}\right]$$

hen F_0 is large compared to σ_f and F_E is nearly equal to F_0, the p.d.f. of the avelope F_E of the sum process is approximately Gaussian.

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