SOME INTEGRALS INVOLVING THE EULER AND BERNOULLI'S NUMBERS

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ABSTRACT

This paper gives analytical expressions for the integrals of the type

$$\int_0^a \frac{\substack{\sinh \\ \cosh}(\alpha t)}{\substack{\cosh \\ \sinh}(t)} e^{\beta t} t^{\gamma} dt.$$

Values of some other important integrals have been deduced from these.

1. The aim of the present note is to obtain analytical expressions for the following integrals

$$\text{(i)} \quad \int_0^a \frac{\substack{\sinh \\ \cosh}(\alpha t)}{\cosh(t)} e^{\beta t} t^{\alpha} dt, \qquad \text{(ii)} \quad \int_0^a \frac{\substack{\sinh \\ \cosh}(\alpha t)}{\sinh(t)} e^{\beta t} t^{\gamma} dt, \qquad [1.1]$$

where α, β, γ, a are constants.

In essence we first obtain expressions for

$\frac{\substack{\sinh \\ \cosh}(\alpha t)}{\cosh t}$ in terms of Euler's Numbers and

$\frac{\substack{\sinh \\ \cosh}(\alpha t)}{\sinh t}$ in terms of Bernoulli's Numbers

starting from their generating functions We then multiply these expressions by appropriate functions and integrate between given limits provided the integrals are convergent.

We can by simple operations deduce analytical expressions for integrals of the following type:

$$\text{(iii)} \quad \int_0^a \frac{\substack{\sin \\ \cos}(\alpha t)}{\cos t} e^{\beta t} t^{\gamma} dt, \qquad \text{(iv)} \quad \int_0^a \frac{\substack{\sin \\ \cos}(\alpha t)}{\sin t} e^{\beta t} t^{\gamma} dt,$$

and (v) $\int_0^a \frac{e^{\beta t}\, t^{\gamma}}{f(t)}\, dt$, where $f(t) = \cosh t$, $\sinh t$, $\cos t$ or $\sin t$. [1.2]

It is also clear that we can deduce a chain of simpler integrals from the above integrals. In passing we mention that such integrals occur in many physical problems such as gravitational instability of polytropic sheets in uniform rotation[1] and oscillatory problems associated with spherical geometries[2].

2. Substituting $\pm 2t$ for t in the following relation

$$\frac{2\, e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x)\, \frac{t^n}{n!}, \quad |t| < \pi, \tag{2.1}$$

and performing elementary algebraic operations, we get

$$\frac{\cosh(\alpha t)}{\cosh t} = \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!}\, E_{2n}\left(\frac{1+\alpha}{2}\right) t^{2n}, \tag{2.2}$$

and

$$\frac{\sinh(\alpha t)}{\cosh t} = \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!}\, E_{2n+1}\left(\frac{1+\alpha}{2}\right) t^{2n+1} \tag{2.3}$$

provided $|t| < \pi/2$, where the following gives the expression for the Euler's polynomials $E_n[(1+\alpha)/2]$ in terms of Euler's numbers E_n:

$$E_n\left(\frac{1+\alpha}{2}\right) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^n} \alpha^{n-k}. \tag{2.4}$$

Similarly, starting from the generating function

$$\frac{t\, e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\, \frac{t^n}{n!}, \quad |t| < 2\pi \tag{2.5}$$

we can easily show that

$$\frac{\cosh(\alpha t)}{\sinh t} = \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!}\, B_{2n}\left(\frac{1+\alpha}{2}\right) t^{2n-1}, \tag{2.6}$$

and

$$\frac{\sinh(\alpha t)}{\sinh t} = \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!}\, B_{2n+1}\left(\frac{1+\alpha}{2}\right) t^{2n}, \tag{2.7}$$

provided $|t| < \pi$, where Bernoulli's polynomials $B_n\,[(1+\alpha)/2]$ in terms of Bernoulli's numbers B_n are given by

$$B_n\left(\frac{1+\alpha}{2}\right) = -\sum_{k=0}^{n} \binom{n}{k} B_k\,\alpha^{n-k} \left\{\frac{1}{2^{n-k}} - \frac{1}{2^{n-1}}\right\}. \qquad [2.8]$$

3. Multiplying [2.2], [2 3], [2.6] and [2.7] by $e^{\beta t}\,t^{\gamma}$ and integrating with respect to t from 0 to a we get

$$\int_0^a \frac{\cosh(\alpha t)}{\cosh t}\, e^{\beta t}\, t^{\gamma}\, dt = \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} E_{2n}\left(\frac{1+\alpha}{2}\right) I\,(a\,;\,\beta\,;\,2n+\gamma), \qquad [3.1]$$

$$\int_0^a \frac{\sinh(\alpha t)}{\cosh t}\, e^{\beta t}\, t^{\gamma}\, dt = \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} E_{n+1}\left(\frac{1+\alpha}{2}\right) I\,(a\,;\,\beta\,;\,2n+1+\gamma) \qquad [3.2]$$

$$\int_0^a \frac{\cosh(\alpha t)}{\sinh t}\, e^{\beta t}\, t^{\gamma} dt = \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!}\, B_{2n}\left(\frac{1+\alpha}{2}\right) I\,(a\,;\,\beta\,;\,2n-1+\gamma),\ \gamma > 0 \qquad [3.3]$$

and

$$\int_0^a \frac{\sinh(\alpha t)}{\sinh t}\, e^{\beta t}\, t^{\gamma} dt = \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!}\, B_{2n+1}\left(\frac{1+\alpha}{2}\right) I\,(a\,;\,\beta\,;\,2n+\gamma) \qquad [3.4]$$

where

$$I(a\,;\,\beta\,;\,n) = \int_0^a e^{\beta t}\, t^n\, dt. \qquad [3.5]$$

when $n > -1$ for convergence.

We can evaluate this integral when β is negative and equal to $-\beta_1\,(\beta_1 > 0)$ in terms of incomplete Gamma functions :

$$I\,(a\,;\,\beta\,;\,n) = \frac{1}{\beta_1^{\,n+1}}\,\gamma\,(n+1\,;\,a\beta_1), \qquad [3.6]$$

where $$\gamma\,(a\,;\,x) = \int_0^x e^{-u}\, u^{a-1}\, du, \qquad [3.7]$$

when β is positive, we have

$$I\,(a\,;\,\beta\,;\,n) = \frac{1}{\beta^{n+1}} \int_0^{a\beta} e^{u}\, u^{n}\, du, \qquad [3.8]$$

where the integral on the right hand side is of the form

$$\int_0^{a\beta} \frac{e^u}{u^f}\, du, \qquad \text{where } 0 < f < 1 \tag{3.9}$$

or of the form

$$\int_0^{a\beta} e^u\, u^f\, du, \qquad \text{where } 0 < f < 1 \tag{3.10}$$

or when $n > 1$, can be reduced to the form [3.10] by using the reduction formula

$$\int_0^{a\beta} e^u\, u^n\, du = e^{a\beta}\,(a\beta)^n - n \int_0^{a\beta} e^u\, u^{n-1}\, du, \tag{3.11}$$

In the special case where n is a positive integer, the integral [3.5] can be evaluated completely :

$$I(a\,;\,\beta\,;\,n) = \frac{(-1)^n\, n\,!}{\beta^{n+1}} \left[e^{\beta a} \sum_{r=0}^{n} \frac{(-1)^r\,(a\beta)^r}{r\,!} - 1 \right]. \tag{3.12}$$

4. On putting $i\,t$ for t in [2 2], [2.3], [2.6] and [2.7] and integrating with respect to t from 0 to a after multiplying by $e^{\beta t}\, t^{\gamma}$, we get

$$\int_0^a \frac{\cos(\alpha\, t)}{\cos t}\, e^{\beta t}\, t^{\gamma}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n}}{(2\,n)\,!}\, E_{2n} \left(\frac{1+\alpha}{2}\right) I(a\,;\,\beta\,;\,2\,n+\gamma), \tag{4.1}$$

$$\int_0^a \frac{\sin(\alpha\, t)}{\cos t}\, e^{\beta t}\, t^{\gamma}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n+1}}{(2\,n+1)\,!}\, E_{2n+1} \left(\frac{1+\alpha}{2}\right) I(a\,;\,\beta\,;\,2\,n+1+\gamma), \tag{4.2}$$

$$\int_0^a \frac{\cos(\alpha t)}{\sin t}\, e^{\beta t}\, t^{\gamma}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n}}{(2\,n)\,!}\, B_{2n} \left(\frac{1+\alpha}{2}\right) I(a\,;\,\beta\,;\,2n-1+\gamma),\ \gamma>0 \tag{4.3}$$

and

$$\int_0^a \frac{\sin(\alpha\, t)}{\sin t}\, e^{\beta t}\, t^{\gamma}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n+1}}{(2\,n+1)\,!}\, B_{2n+1} \left(\frac{1+\alpha}{2}\right) I(a\,;\,\beta\,;\,2\,n+\gamma). \tag{4.4}$$

5. Dividing [3.2], [3.4], [4.2] and [4.4] by α and proceeding to the limit as $\alpha \to 0$, we get

$$\int_0^a \frac{e^{\beta t}\, t^{\gamma+1}}{\begin{smallmatrix}\cosh\\ \cos\end{smallmatrix}(t)}\, dt = \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!}\, I(a\,;\,\beta\,;\,2n+1+\gamma)\, X_{2n+1} \begin{bmatrix} 1 \\ (-1)^n \end{bmatrix}, \quad [5.1]$$

and

$$\int_0^a \frac{e^{\beta t}\, t^{\gamma+1}}{\begin{smallmatrix}\sinh\\ \sin\end{smallmatrix}(t)}\, dt = \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!}\, I(a\,;\,\beta\,;\,2n+\gamma)\, Y_{2n+1} \begin{bmatrix} 1 \\ (-1)^n \end{bmatrix}, \quad [5.2]$$

where

$$X_{2n+1} = \frac{(-1)^n \cdot 2 \cdot (2n+1)!}{\pi^{2n+1}}\, \beta(2n+1), \quad [5.3]$$

where

$$\beta(n) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^n}, \quad [5.4]$$

and

$$Y_{2n+1} = \frac{(-1)^n (2n+1)!}{(2\pi)^{2n}} \left\{ 1 - \frac{1}{2^{2n-1}} \right\} \zeta(2n), \quad [5.5]$$

where

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}. \quad [5.6]$$

Similarly taking the limit when $\alpha \to 0$, the relations [3.1], [3.3], [4.1] and [4.3] reduce to

$$\int_0^a \frac{e^{\beta t}\, t^{\gamma}}{\begin{smallmatrix}\cosh\\ \cos\end{smallmatrix}(t)}\, dt = \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!}\, I(a\,;\,\beta\,;\,2n+\gamma) \begin{bmatrix} 1 \\ (-1)^n \end{bmatrix}^* \quad [5.7]$$

and

$$\int_0^a \frac{e^{\beta t}\, t^{\gamma}}{\begin{smallmatrix}\sinh\\ \sin\end{smallmatrix}(t)}\, dt = -\sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!}\, I(a;\,\beta\,;2n-1+\gamma)\left(1 - \frac{1}{2^{2n-1}}\right) B_{2n} \begin{bmatrix} 1 \\ (-1)^n \end{bmatrix}^* \quad [5.$$

Some particular cases of the integrals like [3.1], [3.2], [4.1] and [4.2] have been evaluated in reference 1. Similarly, some integrals of the type [3.5] have been numerically evaluated in reference 2.

* 1 and $(-1)^n$ correspond respectively to $\cosh t$ and $\cos t$ in [5.7] and to $\sinh t$ and $\sin t$ in [5.8].

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