PLANE COUETTE FLOW WITH SUCTION OR INJECTION AND HEAT TRANSFER FOR RIVLIN-ERICKSEN FLUID

By V. G. Tikekar

(*Department of Applied Mathematics, Indian Institute of Science, Bangalore-12, India*)

[Received; May, 1, 1968]

Abstract

Following the method due to Bhatnagar (P. L.) [Jour. Ind. Inst. Sci., 50, 1, 1, 1968], we have discussed the problem of suction and injection and that of heat transfer in a plane Couette flow for Rivlin-Ericksen fluid. By perturbation technique, regarding the elastic parameter as small, we have built the solutions on those obtained by Bhatnagar for Newtonian fluids, the latter forming the zeroth order solutions for the former. We have used certain properties of fundamental solutions, of differential equations and some transformations that enable us to solve the two-point boundary value and eigen-value problems without using the trial and error method. In fact, each integration provides us with a solution for a suction parameter and the corresponding Reynolds number without imposing the condition of smallness on them. Investigations on other non-Newtonian fluids and other bounding geometries will be published elsewhere.

1. Introduction

Bhatnagar[1] has given a method for solving the problem of suction and injection and of heat transfer for Newtonian fluid in a plane Couette flow without imposing the conditions of smallness on the suction parameter or such similar conditions on the Reynolds number to allow the series solution. In this paper we have extended these techniques to non-Newtonian fluid defined by the following constitutive equation given by Rivlin-Ericksen:

$$T_{ij} = -p\,\delta_{ij} + \phi_1 E_{ij} + \phi_2 D_{ij} + \phi_3 E_{im} E_{mj}, \qquad [1.1]$$

where T_{ij} is the stress tensor, δ_{ij} are the Kronecker deltas,

$$E_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \qquad [1.2]$$

is the rate of strain tensor,

$$D_{ij} = \frac{\partial a_i}{\partial x_j} + \frac{\partial a_j}{\partial x_i} + 2\,\frac{\partial u_m}{\partial x_i} \cdot \frac{\partial u_m}{\partial x_j} \qquad [1.3]$$

is the acceleration gradient tensor and ϕ_1, ϕ_2, ϕ_3 are respectively the co-efficients of viscosity, visco-elasticity and cross-viscosity.

The solutions obtained by Bhatnagar[1] form the zeroth order solutions for the present case. We have obtained the solutions for the Rivlin-Ericksen fluid using the perturbation technique regarding the elastic parameter as small. We have used certain properties of fundamental solutions, of differential equations and some transformations that enable us to solve the two-point boundary value and eigen-value problems without using the trial and error method. In fact, each integration provides us with a solution for a suction parameter and the corresponding Reynolds number without imposing the condition of smallness on them We have applied the suction or injection only on the fixed plate so that the usual boundary condition on the cross flow, namely the injection at one plate is equal to the suction at the other, has not been employed.

2. Basic Equations of the Problem

Let the infinite plate $y = 0$ be stationary, while the plate $y = a$ be moving with uniform velocity U_0 in the direction of the x-axis. We maintain these plates at constant temperatures T_0 and T_1 respectively. Moreover, uniform injection or suction with velocity $v = \pm v_0$ ($v_0 > 0$) is applied on the plane $y = 0$, while the upper plane is non-porous. Here the plus sign refers to injection and the minus sign to suction.

Since we have taken the suction or injection to be uniform, we assume that the cross-velocity v is a function of y alone. We shall use the dimensionless variables u, v, x, y, p, θ for

$$\frac{u}{U_0},\ \frac{v}{v_0},\ \frac{x}{aR},\ \frac{y}{a},\ \frac{p}{\rho U_0^2},\ \frac{T - T_0}{T_1 - T_0}$$

respectively and denote the suction parameter $v_0\, a\rho/\phi_1$, Reynolds number $a\, U_0\, \rho/\phi_1$, Prandtl number $\phi_1 C_p/k$, Eckert number $U_0^2/C_p(T_1 - T_0)$ and non-Newtonian parameters $\phi_2/\rho a^2$ and $\phi_3/\rho a^2$ by λ, R, P, E, K and S respectively.

In terms of the dimensionless parameters and variables, the equations of the problem and the boundary conditions reduce to the following:

$$\frac{\partial u}{\partial x} + \lambda v' = 0 \qquad [2.1]$$

$$u\frac{\partial u}{\partial x} + \lambda v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}$$

$$- K\lambda \left[v' \frac{\partial^2 u}{\partial y^2} + u v''' - v \frac{\partial^3 u}{\partial y^3} + 3 v'' \frac{\partial u}{\partial y} \right] - 2 S \lambda \frac{\partial u}{\partial y} v'' \qquad [2.2]$$

$$vv' = -\frac{R^2}{\lambda^2}\frac{\partial p}{\partial y} + \frac{1}{\lambda}v'' + K\left[13v'v'' + vv''' + 4\frac{R^2}{\lambda^2}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right]$$

$$+ S\left[8v'v'' + 2\frac{R^2}{\lambda^2}\frac{\partial u}{\partial y}\cdot\frac{\partial^2 u}{\partial y^2}\right] \qquad [2.3]$$

$$P\left[u\frac{\partial\theta}{\partial x} + \lambda v\frac{\partial\theta}{\partial y}\right] = \frac{1}{R^2}\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + EP\left[4\frac{\lambda^2}{R^2}(v')^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$+ KEP\left[\frac{2}{R^2}\frac{\partial u}{\partial x}\left\{2\left(\frac{\partial u}{\partial x}\right)^2 + \lambda v\frac{\partial^2 u}{\partial x\,\partial y}\right\} + 2\lambda v'\left\{\frac{\lambda^2}{R^2}(vv'' + 2(v')^2)\right.\right.$$

$$\left.\left. + \left(\frac{\partial u}{\partial y}\right)^2\right\} + \frac{\partial u}{\partial y}\left\{u\frac{\partial^2 u}{\partial x\,\partial y} + 3\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \lambda\left(v\frac{\partial^2 u}{\partial y^2} + v'\frac{\partial u}{\partial y}\right)\right\}\right] \qquad [2.4]$$

with

$$\left.\begin{array}{ll} y = 0: & u = 0,\ v = \pm 1,\ \theta = 0 \\ y = 1: & u = 1,\ v = 0,\ \ \theta = 1 \end{array}\right\} \qquad [2.5]$$

where a dash denotes differentiation with respect to y.

We note that the cross-viscosity does not contribute to the energy equation in the present case of two-dimensional plane motion.

3. Solution of the Flow Problem

From [2.1], we have

$$u(x, y) = -\lambda v' x + u_0(y), \qquad [3.1]$$

where $u_0(y)$ is an arbitrary function to be determined later. Equation [3.1] determines $u(x, y)$ in terms of $u_0(y)$ and $v(y)$.

Using [3.1] in [2.3] and integrating it, we get

$$p(x, y) = \frac{\lambda^2}{R^2}\left[-\frac{1}{2}v^2 + \frac{1}{\lambda}v'\right] + K\frac{\lambda^2}{R^2}\left[6(v')^2 + vv''\right] + 4S\frac{\lambda^2}{R^2}(v')^2$$

$$+ 2(2K + S)\left[\frac{\lambda^2}{2}(v'')^2 x^2 - \lambda x u_0' v'' + \frac{1}{2}(u_0')^2\right] + p_0(x), \qquad [3.2]$$

where $p_0(x)$ is another arbitrary function to be determined later. Equation [3.2] determines $p(x, y)$ in terms of $v(y)$, $u_0(y)$ and $p_0(x)$. We note that the cross-viscosity contributes to the pressure.

Using [3.1] and [3.2] in [2.2] and concentrating on the powers of x that occur in the resulting equation, we find that we should take the following expression for $dp_0(x)/dx$:

$$-dp_0(x)/dx = c_2 + 2\,c_1\,x, \tag{3.3}$$

where c_1 and c_2 are constants and then this equation breaks into the following two equations which are independent of x:

$$\lambda\, v\, u_0' - \lambda\, u_0\, v' + K\lambda\,[u_0''\, v' + u_0\, v''' - v\, u_0''' - u_0'\, v''] = u_0'' + c_2 \tag{3.4}$$

$$\lambda^2\,[vv'' - (v')^2] - \lambda\, v''' = -2c_1 + K\lambda^2\,[(v'')^2 - 2v'\, v''' + vv^{\text{iv}}] \tag{3.5}$$

Equation [3.5] determines v for prescribed values of λ, K and c_1, while equation [3.4] then determines the value u_0 for prescribed values of c_2.

Since $u = 0$ at $y = 0$ and $u = 1$ at $y = 1$ for all values of x, we have from [3.1] the following boundary conditions to be satisfied by u_0 and v':

$$\left.\begin{aligned} u_0(0) &= 0, \quad v'(0) = 0 \\ u_0(1) &= 1, \quad v'(1) = 0 \end{aligned}\right\} \tag{3.6}$$

Therefore the boundary conditions for [3.5] are

$$\left.\begin{aligned} y = 0 &: \ v = \pm 1, \quad v' = 0 \\ y = 1 &: \ v = 0, \quad\ \ v' = 0 \end{aligned}\right\} \tag{3.7}$$

while those for [3.4] are

$$u_0(0) = 0, \quad u_0(1) = 1. \tag{3.8}$$

We shall first concentrate on the equation [3.5] and take

$$v = v^{(0)} + K v^{(1)} \tag{3.9}$$

and regard K as small. Letting

$$c_1 = \bar{c}_0 - K\bar{c}_1, \tag{3.10}$$

where $\bar{c}_0$ and $\bar{c}_1$ are constants, from [3.5] we get the following equations determining $v^{(0)}$ and $v^{(1)}$:

$$\lambda^2\,[(v^{(0)})']^2 - \lambda^2\, v^{(0)}\,(v^{(0)})'' + \lambda\,(v^{(0)})''' = 2\,\bar{c}_0 \tag{3.11}$$

as tak

$$\lambda^2\,[v^{(0)}\,(v^{(1)})'' + v^{(1)}\,(v^{(0)})'' - 2\,(v^{(0)})'\,(v^{(1)})'] = 2\bar{c}_1 + \lambda^2\,[\{(v^{(0)})''\}^2 - 2\,(v^{(0)})'\,(v^{(0)})''' + v^{(0)}\,(v^{(0)})^{\text{iv}}] + \lambda\,(v^{(1)})''' \tag{3.12}$$

to be solved under the boundary conditions:

$$\left.\begin{array}{ll} y=0: v^{(0)}=\pm 1, & (v^{(0)})'=0 \\ y=1: v^{(0)}=0, & (v^{(0)})'=0 \end{array}\right\} \quad [3.13]$$

$$\left.\begin{array}{ll} y=0: v^{(1)}=0, & (v^{(1)})'=0 \\ y=1: v^{(1)}=0, & (v^{(1)})'=0 \end{array}\right\} \quad [3.14]$$

Equation [3.11] with boundary conditions [3.13] is exactly the problem that Bhatnagar[1] has solved with $A_1 = 2\bar{c}_0$ and as such we know $v^{(0)}$ for particular λ. This forms the zeroth order solution for our case.

Equation [3.12] with boundary conditions [3.14] is a two-point boundary value problem with eigen-value $\bar{c}_1$. This equation is of order three and we have four boundary conditions and hence the problem is fully determined. With the transformation

$$\left.\begin{array}{ll} v^{(0)} = (2\bar{c}_0)^{1/4}\, V^{(0)}/\lambda, & v^{(1)} = (2\bar{c}_0)^{1/4}\, V^{(1)}/\lambda, \\ 1-y = Y = \xi/(2\bar{c}_0)^{1/4}, & V^{(1)} = \bar{c}_1\, Z/\bar{c}_0 \end{array}\right\} \quad [3.15]$$

the equation [3.12] reduces to

$$\begin{aligned} Z''' + V^{(0)} Z'' - 2\, V^{(0)\prime} Z' + V^{(0)\prime\prime} Z & \\ = 1 + p^* \left[\{(V^{(0)})''\}^2 - 2\, V^{(0)\prime}\, V^{(0)\prime\prime\prime} + V^{(0)}\, V^{(0)iv}\right] & \\ = 1 + p^* f(\xi) & \end{aligned} \quad [3.16]$$

with the boundary conditions

$$\left.\begin{array}{l} \xi = 0: Z = 0,\; Z' = 0 \\ \xi = \xi_0 = (2\bar{c}_0)^{1/4}: Z = 0,\; Z' = 0 \end{array}\right\} \quad [3.17]$$

where

$$p^* = (2\bar{c}_0)^{3/2}/2\bar{c}_1 \quad [3.18]$$

and hence the right hand member of [3.16] is known from the solution of the zeroth order problem and where a dash now denotes differentiation with respect to ξ.

Solution of the equation [3.16] under the boundary conditions [3.17] is given by

$$Z = C_0 Z_3 + \sum_{i=1}^{3} f_i(\xi)\, Z_i + p^* \sum_{i=1}^{3} g_i(\xi)\, Z_i, \quad [3.19]$$

where $Z = Z_i$, $(i = 1, 2, 3)$ are the fundamental solutions of the homogeneous equation corresponding to [3.16] satisfying the usual boundary conditions at $\xi = 0$,

$$\left.\begin{aligned} f_1(\xi) &= \int_0^\xi \frac{Z_2 Z_3' - Z_3 Z_2'}{D(\xi)} d\xi, & g_1(\xi) &= \int_0^\xi \frac{f(\xi)(Z_2 Z_3' - Z_3 Z_2')}{D(\xi)} d\xi, \\ f_2(\xi) &= \int_0^\xi \frac{Z_3 Z_1' - Z_1 Z_3'}{D(\xi)} d\xi, & g_2(\xi) &= \int_0^\xi \frac{f(\xi)(Z_3 Z_1' - Z_1 Z_3')}{D(\xi)} d\xi, \\ f_3(\xi) &= \int_0^\xi \frac{Z_1 Z_2' - Z_2 Z_1'}{D(\xi)} d\xi, & g_3(\xi) &= \int_0^\xi \frac{f(\xi)(Z_1 Z_2' - Z_2 Z_1')}{D(\xi)} d\xi, \end{aligned}\right\} \quad [3.20]$$

$$D(\xi) = \exp\left[-\int_0^\xi V^{(0)}(\eta)\, d\eta\right]$$

and C_0 and p^* are the roots of the equations

$$\left.\begin{aligned} & C_0 Z_3(\xi_0) + p^* \sum_{i=1}^{3} g_i(\xi_0) Z_i(\xi_0) + \sum_{i=1}^{3} f_i(\xi_0) Z_i(\xi_0) = 0, \\ & C_0 Z_3'(\xi_0) + p^* \left[\sum_{i=1}^{3} g_i(\xi_0) Z_i'(\xi_0) + \sum_{i=1}^{3} g_i'(\xi_0) Z_i(\xi_0)\right] \\ & \qquad + \sum_{i=1}^{3} f_i(\xi_0) Z_i'(\xi_0) + \sum_{i=1}^{3} f_i'(\xi_0) Z_i(\xi_0) = 0 \end{aligned}\right\} \quad [3.21]$$

The solution of the equation [3.12] under boundary conditions [3.14] and the eigen value $\bar{c}_1$, for the suction parameter λ for which $v^{(0)}(\xi)$ are known, are given by

$$v^{(1)}(\xi) = 2\,\bar{c}_1\, Z(\xi)/\lambda \xi_0^3 \quad [3.22]$$

and

$$\bar{c}_1 = \xi_0^6 / 2p^*. \quad [3.23]$$

Equation [3.9] will give v in terms of $v^{(0)}$ and $v^{(1)}$ for small non-Newtonian parameter K.

We shall now discuss the equation [3.4]. In order to avoid the specific assumption about the numerical value of the constant c_2 occurring in it, we take

$$u_0 = U + \lambda^2 v'(y). \quad [3.24]$$

If further we use

$$c_2 = -2\lambda c_1 \qquad [3.25]$$

and write

$$U = U^{(0)} + KU^{(1)} \qquad [3.26]$$

for small K, we get the following equations determing $U^{(0)}$ and $U^{(1)}$:

$$(1/\lambda)\, U^{(0)''} - v^{(0)}\, U^{(0)'} + v^{(0)'}\, U^{(0)} = 0 \qquad [3.27]$$

with

$$U^{(0)}(0) = 0, \qquad U^{(0)}(1) = 1 \qquad [3.28]$$

and

$$(1/\lambda)\, U^{(1)''} - v^{(0)}\, U^{(1)'} - v^{(1)}\, U^{(0)'} + v^{(0)'}\, U^{(1)} + v^{(1)'}\, U^{(0)} - v^{(0)'}\, U^{(0)''} + v^{(0)''}\, U^{(0)'} + v^{(0)'''}\, U^{(0)} + v^{(0)}\, U^{(0)'''} = 0 \qquad [3.29]$$

with

$$U^{(1)}(0) = U^{(1)}(1) = 0, \qquad [3.30]$$

where a dash denotes differentiation with respect to y.

Solution of the equation [3.27] under boundary conditions [3.28] is known in terms of the solutions obtained in reference [ι]. Equation [3.29] under boundary conditions [3.30] is a two-point boundary value problem and its solution is given by

$$U^{(1)}(y) = \frac{U_\beta^{(1)}(1)}{U_\beta^{(1)}(1) - U_\alpha^{(1)}(1)}\, U_\alpha^{(1)}(y) - \frac{U_\alpha^{(1)}(1)}{U_\beta^{(1)}(1) - U_\alpha^{(1)}(1)}\, U_\beta^{(1)}(y) \qquad [3.31]$$

where $U_\alpha^{(1)}(y)$ and $U_\beta^{(1)}(y)$ are the solutions of equation [3 29] under the boundary conditions.

$$U^{(1)}(0) = 0, \quad U^{(1)'}(0) = \alpha \qquad [3.32]$$

and

$$U^{(1)}(0) = 0, \quad U^{(1)'}(0) = \beta \qquad [3.33]$$

respectively.

Then we obtain

$$u_0^{(1)} = U^{(1)} + \lambda^2 v^{(1)'} \qquad [3.34]$$

and u_0 is given by

$$u_0 = u_0^{(0)} + K u_0^{(1)} \qquad [3.35]$$

for small non-Newtonian parameter K.

4. Solution of Heat Transfer Problem

If we substitute the value of $u(x, y)$ from equation [3.1] in equation [2.4] and concentrate on the powers of x that occur in the resulting equation, we find that we must take

$$\theta(x, y) = \theta_0(y) + \theta_1(y)\,x + \theta_2(y)\,x^2 \quad [4.1]$$

If now, we equate the coefficients of various powers of x on the the two sides of the resulting equation, we get the following three equations in θ_0, θ_1, θ_2:

$$P[u_0\theta_1 + \lambda v\theta_0'] = (2/R^2)\,\theta_2 + \theta_0'' + EP[(4\lambda^2/R^2)(v')^2 + (u_0')^2] + KEP[(4\lambda^3/R^2)\,vv'v'' + \lambda v u_0' u_0'' - \lambda v'' u_0 (u_0')] \quad [4.2]$$

$$P[2u_0\theta_2 - \lambda v'\theta_1 + \lambda v\theta_1'] = \theta_1'' + EP[-2\lambda v'' u_0'] + \lambda^2 KEP[(v'')^2 u_0 + u_0' v' v'' - vv'' u_0'' - vv''' u_0'] \quad [4.3]$$

$$P[-2\lambda v'\theta_2 + \lambda v\theta_2'] = \theta_2'' + EP\lambda^2 (v'')^2 + \lambda^3 KEP[vv''v''' - v'(v'')^2] \quad [4.4]$$

$$\left.\begin{array}{ll} \text{with} & \theta_0(0) = \theta_1(0) = \theta_2(0) = 0 \\ \text{and} & \theta_0(1) = 1,\ \theta_1(1) = \theta_2(1) = 0 \end{array}\right\} \quad [4.5]$$

For small K, we set

$$\theta_i = \theta_i^{(0)} + K\theta_i^{(1)}, \quad (i = 0, 1, 2) \quad [4.6]$$

along with [3.9] and [3.35] in the above equations to get the following equations determining $\theta_i^{(0)}$ and $\theta_i^{(1)}$, $(i = 0, 1, 2)$:

$$P[u_0^{(0)}\theta_1^{(0)} + \lambda v^{(0)}\theta_0^{(0)\prime}] = (2/R^2)\,\theta_2^{(0)} + \theta_0^{(0)\prime\prime} + EP[(4\lambda^2/R^2)(v^{(0)\prime})^2 + (u_0^{(0)\prime})^2] \quad [4.7]$$

$$P[2u_0^{(0)}\theta_2^{(0)} - \lambda v^{(0)\prime}\theta_1^{(0)} + \lambda v^{(0)}\theta_1^{(0)\prime}] = \theta_1^{(0)\prime\prime} + EP[-2\lambda v^{(0)\prime\prime} u_0^{(0)\prime}] \quad [4.8]$$

$$P[-2\lambda v^{(0)\prime}\theta_2^{(0)} + \lambda v^{(0)}\theta_2^{(0)\prime}] = \theta_2^{(0)\prime\prime} + EP\lambda^2 (v^{(0)\prime\prime})^2 \quad [4.9]$$

$$\begin{array}{ll} \text{with} & \theta_0^{(0)}(0) = \theta_1^{(0)}(0) = \theta_2^{(0)}(0) = 0 \\ & \theta_0^{(0)}(1) = 1, \quad \theta_1^{(0)}(1) = \theta_2^{(0)}(1) = 0 \end{array} \quad [4.10]$$

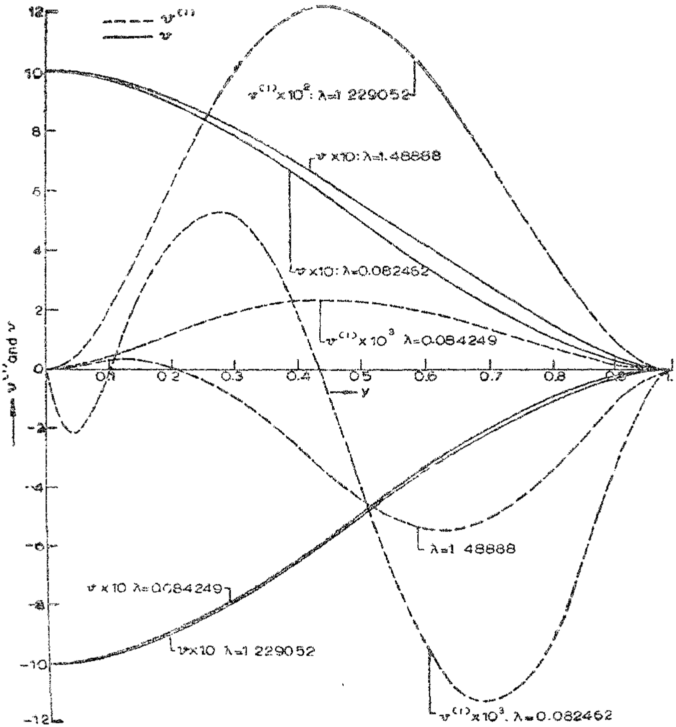

FIG. 1 (K=-0 01)

$$P\left[\theta_1^{(1)} u_0^{(0)} + \theta_1^{(0)} u_0^{(1)} + \lambda\left\{v^{(0)} \theta_0^{(1)\prime} + v^{(1)} \theta_0^{(0)\prime}\right\}\right]$$
$$= (2/R^2)\, \theta_2^{(1)} + \theta_0^{(1)\prime\prime} + EP\left[8\, (\lambda^2/R^2)\, v^{(0)\prime} v^{(1)\prime} + 2\, u_0^{(0)\prime} u^{(1)\prime}\right.$$
$$- \lambda\, v^{(0)\prime\prime} u_0^{(0)\prime} u_0^{(0)}$$
$$\left. + \lambda\, v^{(0)} u_0^{(0)\prime} u_0^{(0)\prime\prime} + (4\, \lambda^3/R^2)\, v^{(0)} v^{(0)\prime} v^{(0)\prime\prime}\right] \qquad [4.11]$$

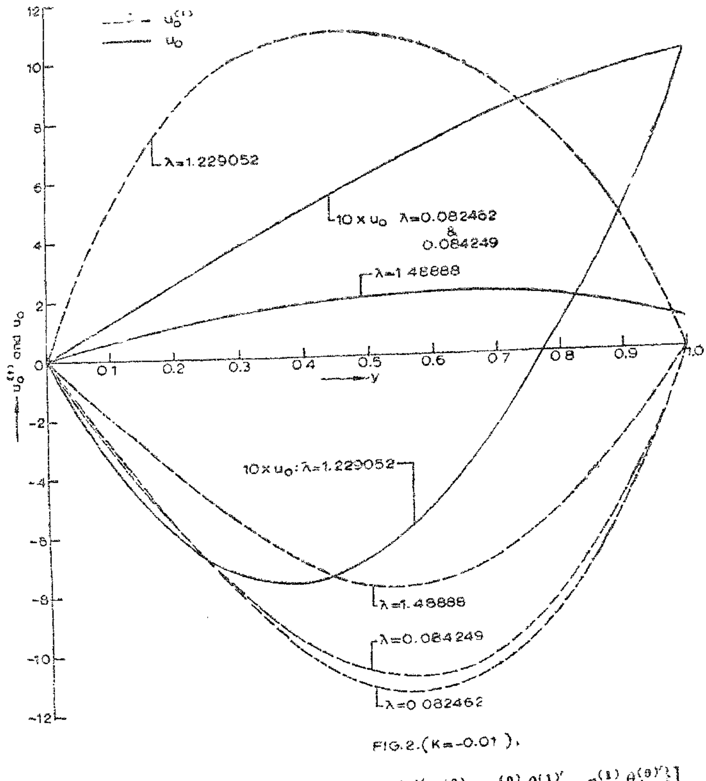

FIG. 2. (K=-0.01).

$$
\begin{aligned}
&P[2\, u_0^{(0)}\, \theta_2^{(1)} + 2\, u_0^{(1)}\, \theta_2^{(0)} - \lambda\{v^{(0)'}\, \theta_1^{(1)} + v^{(1)'}\, \theta_1^{(0)} - v^{(0)}\, \theta_1^{(1)'} - v^{(1)}\, \theta_1^{(0)'}\}] \\
&\quad = \theta_1^{(1)''} + EP\lambda\,[-2\{v^{(0)''}\, u_0^{(1)'} + v^{(1)''}\, u_0^{(0)'}\} + \lambda\{(v^{(0)''})^2\, u_0^{(0)} \\
&\qquad + u_0^{(0)'}\, v^{(0)'}\, v^{(0)''} - v^{(0)}\, v^{(0)''}\, u_0^{(0)''} - v^{(0)}\, v^{(0)'''}\, u_0^{(0)'}\}]
\end{aligned}
\qquad [4.12]
$$

$$
\begin{aligned}
&P\lambda\,[v^{(0)}\, \theta_2^{(1)'} + v^{(1)}\theta_2^{(0)'} - 2\{v^{(0)'}\, \theta_2^{(1)} + v^{(1)'}\, \theta_2^{(0)}\}] \\
&\quad = \theta_2^{(1)''} + EP\lambda^2\,[2\, v^{(0)''}\, v^{(1)''} + \lambda\{v^{(0)}\, v^{(0)''}\, v^{(0)'''} - v^{(0)'}(v^{(0)''})^2\}]
\end{aligned}
\qquad [4.13]
$$

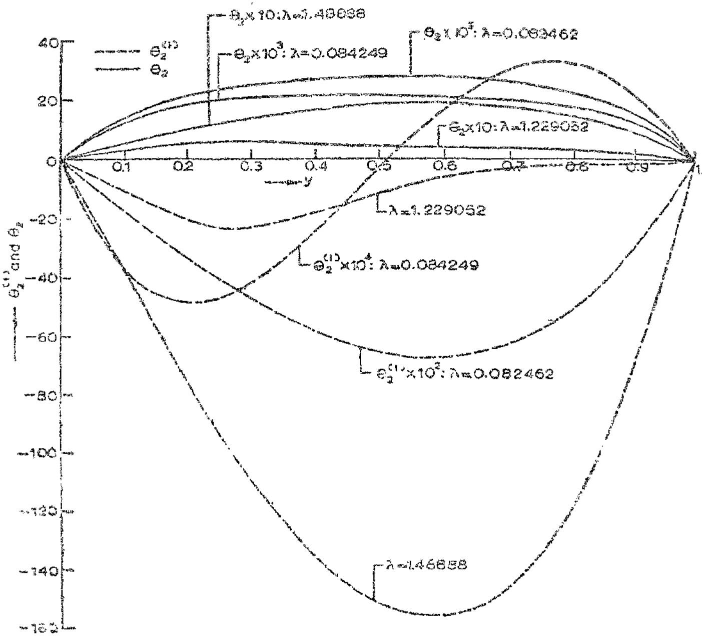

FIG. 3
(K=−0.01, E=5, P=0.8)

with $$\theta_i^{(1)}(0) = \theta_i^{(1)}(1) = 0, \; (i = 0, 1, 2) \qquad [4.14]$$

Each of the zeroth order equations, namely [4.7], [4.8], [4.9] under specified boundary conditions [4.10] has been solved in reference [1] and we use them to solve the first order equations, namely [4.11] [4.12] [4.13] under the boundary conditions given by [4.14]. Each one of these is also a two-point boundary value problem. We have the following solution for $\theta_i^{(1)}$, $(i = 0, 1, 2)$ from the equations [4.13], [4.12] and [4.11] respectively obtained by solving the equations in this order :

$$\theta_2^{(1)}(y) = \frac{\theta_{2\beta}^{(1)}(1)}{\theta_{2\beta}^{(1)}(1) - \theta_{2\alpha}^{(1)}(1)}\, \theta_{2\alpha}^{(1)}(y) - \frac{\theta_{2\alpha}^{(1)}(1)}{\theta_{2\beta}^{(1)}(1) - \theta_{2\alpha}^{(1)}(1)}\, \theta_{2\beta}^{(1)}(y) \qquad [4.15]$$

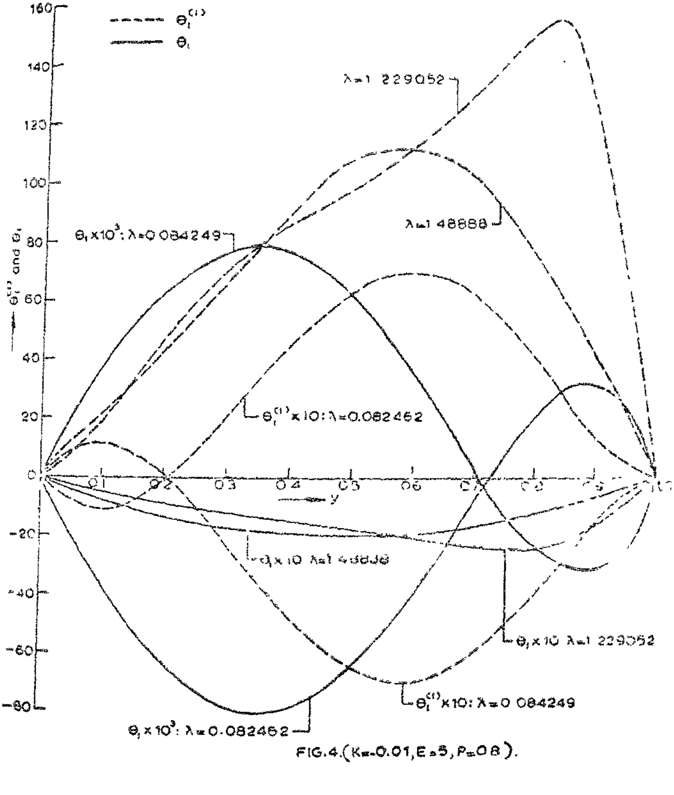

FIG. 4. (K=-0.01, E=5, P=0 8).

where $\theta_{2\alpha}^{(1)}$ and $\theta_{2\beta}^{(1)}$ are the solutions of the equation [4.13] under the boundary conditions,

$$\theta_2^{(1)}(0) = 0,\ \theta_2^{(1)\prime}(0) = \alpha \tag{4.16}$$

and

$$\theta_2^{(1)}(0) = 0,\ \theta_2^{(1)\prime}(0) = \beta \tag{4.17}$$

respectively.

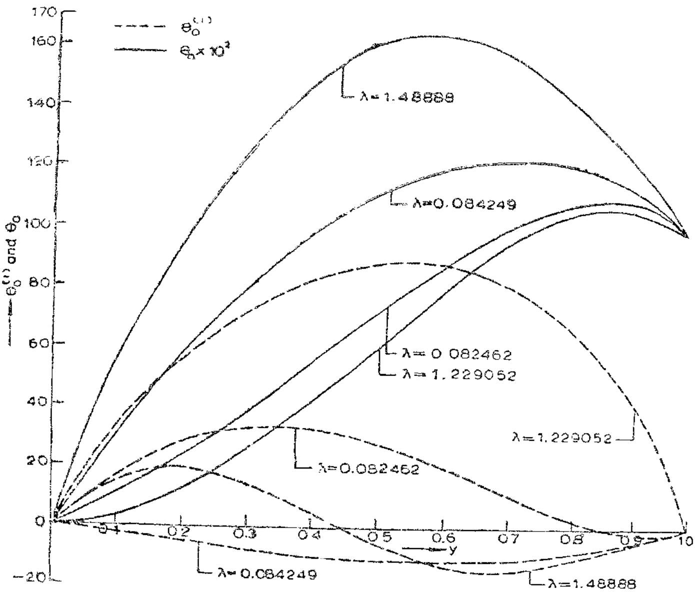

FIG. 5

(K = −0.01, E = 5, P = 0.8, R = 100)

Similarly

$$\theta_1^{(1)}(y) = \left\{\frac{\theta_{1\beta}^{(1)}(1)\,\theta_{1\alpha}^{(1)}(y) - \theta_{1\alpha}^{(1)}(1)\,\theta_{1\beta}^{(1)}(y)}{\theta_{1\beta}^{(1)}(1) - \theta_{1\alpha}^{(1)}(1)}\right\} \quad [4.18]$$

where $\theta_{1\alpha}^{(1)}$ and $\theta_{1\beta}^{(1)}$ are the solutions of [4.12] under the boundary conditions

$$\theta_1^{(1)}(0) = 0, \;\; \theta_1^{(1)\prime}(0) = \alpha \quad [4.19]$$

and

$$\theta_1^{(1)}(0) = 0, \;\; \theta_1^{(1)\prime}(0) = \beta \quad [4.20]$$

respectively and

$$\theta_0^{(0)}(y) = \left\{\frac{\theta_{0\beta}^{(1)}(1)\,\theta_{0\alpha}^{(1)}(y) - \theta_{0\alpha}^{(1)}(1)\,\theta_{0\beta}^{(1)}(y)}{\theta_{0\beta}^{(1)}(1) - \theta_{0\alpha}^{(1)}(1)}\right\} \quad [4.21]$$

where $\theta_{0\alpha}^{(1)}$ and $\theta_{0\beta}^{(1)}$ are the solutions of [4.11] under the boundary conditions.

$$\theta_0^{(1)}(0) = 0, \;\; \theta_0^{(1)\prime}(0) = \alpha \quad [4.22]$$

$$\theta_0^{(1)}(0) = 0, \; \theta_0^{(1)\prime}(0) = \beta \qquad [4.23]$$

respectively.

We note that the specific choice of the Prandtl number and Eckert number is required during the discussion of θ_2 and θ_1 equations, but that of Reynolds number does not come up till we discuss the θ_0 – equations.

5. Numerical Results

We have performed the numerical calculations of $v^{(1)}$, $u_0^{(1)}$, $\theta_2^{(1)}$, $\theta_1^{(1)}$, $\theta_0^{(1)}$ for those values of λ, R P,E, which have been used in [1] as our solutions are based on the solutions obtained therein. Figures 1 and 2 give the plots of $v^{(1)}$ and $u_0^{(1)}$ respectively. Figures 3 and 4 give the plots of $\theta_2^{(1)}$ and $\theta_1^{(1)}$ respectively. for $E = 5$, $P = 0.8$, while the Figure 5 gives the plot of $\theta_0^{(1)}$ for $E = 5$, $P = 0.8$ and $R = 100$. We have then calculated v, u_0, θ_2, θ_1, θ_0 by taking $K = -0.01$ and their plots are sketched on those graphs which give the plots of the corresponding first order solution Figure 1 shows $v^{(1)}$ with dotted plots, while v with continuous plots and same convention is followed in other figures also. The following table gives the values of λ and the corresponding eigen-values $\bar{c}_1$:

λ	1.4889	0.0825	1.2291	0.0842
$\bar{c}_1$	100.0	0.1612	– 53.35	–0.2545

For convenience of plotting, we have multiplied the values of the dependent variables by suitable quantities before plotting.

As a final remark we add that the cross-viscosity contributes only to the pressure but does not affect the other flow variables and the temperature distribution. This is in keeping with the remark which Bhatnagar[2] has made in connection with the boundary layer equation on a flat plate.

Acknowledgements

I gratefully acknowledge my indebtedness to Professor P. L. Bhatnagar for suggesting this problem and for his valuable guidance throughout the preparation of this paper. I am grateful to Miss S Swarnalatha Prabhu for programming this problem on Elliott 803 computer at H.A.L. Bangalore and for her help in drawing the graphs of this paper.

References

1. Bhatnagar, P. L. .. .. *J. Indian Inst. Sc.*, 1968, 50, 1.
2. ——— .. .. *Proc. Ind Acad. Sc.*, 1961, 53, 95.