EFFECT OF CREEP ON THE VIBRATIONS OF MECHANICAL SYSTEMS SUBJECTED TO RANDOM EXCITATION

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Abstract

Although mechanical systems subjected to random excitations have been extensively studied during the past few years, there appears to be no analysis which takes into account the possible creep deformation of the mechanical elements due to the loading on the system. As is well known, the creep in metals, being an energy absorbing process, introduces an equivalent damping into the system and hence attenuates the vibrations. A study of this creep damping in coiled springs has been made by Hoff[1]

In this paper, the study of mechanical systems idealised by a single degree of freedom system, which admits both elastic and creep deformations, has been undertaken. By assuming a stationary,* Gaussian, white noise type of Random excitation on the system, the statistical properties of the response have been obtained with the aid of Generalised Harmonic analysis[2]. Also, an estimate of the damping introduced into the system due to creep, has been made by evaluating the energy absorbed by the creep process.

Analysis

Consider a single degree of freedom system whose equation of motion can be written as

$$m\,(d^2x/dt^2) + F_s = mg + F_i \qquad [1]$$

where

- m is the mass
- x is the displacement
- F_s is the spring force
- g is the acceleration due to gravity
- F_i is the impressed force.

By Hooke's law

$$F_s = K_1\, x_{el} \qquad [2]$$

*The Random process assumed is not only stationary but also ergodic.

where,

x_{el} is the elastic part of displacement

K_1 is the spring constant.

Assuming the displacement or the spring deformation to consist of both elastic and creep deformations, and the creep velocity to follow a linear law of viscosity,

$$\frac{dx_{cr}}{dt} = \frac{F_s - F_i}{K_2} \quad [3]$$

where x_{cr} is the creep displacement and K_2 is the equivalent dashpot constant.

Now,

$$x = x_{el} + x_{cr}$$

Therefore,

$$\frac{dx}{dt} = \frac{d\,F_s/dt}{K_1} + \frac{F_s - F_i}{K_2} \quad [5]$$

The corresponding initial conditions at $t = 0$ are

$$x_{el} = mg/K_1, \; x_{cr} = 0, \; \frac{d\,x_{el}}{dt} = 0 \quad [6]$$

$$\frac{d\,x_{cr}}{dt} = \frac{F_s}{K_2} = \frac{mg}{K_2}$$

The particular solution for equation [1] and [5] under these conditions will be

$$x_p = \frac{mg}{K_1} + \frac{mg}{K_2}\,t \text{ (satisfying the I.C.)} \quad [7]$$

This represents the steady state motion without impressed force acting on the system.

Now making the transformation

$$\left.\begin{aligned} X &= x - x_p \\ \text{and} \qquad Q &= F_s - mg \end{aligned}\right\} \quad [8]$$

where the new variables X and Q represent the displacement and force in a system of zero gravitation.

$$\left.\begin{aligned} &\text{by [2] } X_{el} = Q/K_1 \\ &\text{by [3] } \frac{d\,X_{cr}}{dt} = \frac{Q - F_i}{K_2} \end{aligned}\right\} \quad [9]$$

$$\text{by [5]} \quad \frac{dX}{dt} = \frac{dQ/dt}{K_1} + \frac{Q - F_i}{K_2} \tag{10}$$

and
$$\frac{d^2Q/dt^2}{K_1} + \frac{dQ/dt - dF_i/dt}{K_2} = \frac{d^2X}{dt^2} = \frac{F_i - Q}{m}$$

i.e.
$$\frac{d^2Q}{dt^2} + 2\beta\frac{dQ}{dt} + W_0^2 Q = F(t) \tag{11}$$

where,

$$2\beta = \frac{K_1}{K_2}; \quad W_0^2 = \frac{K_1}{m} \tag{12}$$

$$F(t) = \left[\frac{F_i}{m} + \frac{dF_i/dt}{K_2}\right] \times K_1 = W_0^2 F_i + 2\beta\frac{dF_i}{dt} \tag{13}$$

By solving [10] and [11] we can get X and Q which in turn gives the solution for the response x and creep in spring, etc.

For this, at this stage it is necessary to specify the function $F(t)$ *i.e.* in turn $F_i(t)$.

$F_i(t)$ is taken as a Gaussion random variable stationary, mean zero, white noise type *i.e.*, constant spectral density type.

Hence $\frac{dF_i(t)}{dt}$ which can be thought of as a linear combination of two normal random variables with subsequent passage to the limit, can also be taken as a normal random variable with mean zero[3].

By similar argument $F(t)$ the linear combination $F_i(t)$ and $\frac{dF_i(t)}{dt}$ is also a normal random variable with mean zero.

Therefore the solution to equation [11] using the convolution integral will be

$$Q = \exp(-\beta t)\cdot(A \cos W_1 t + B \sin W_1 t) + \int_0^\infty h(\tau) F(t-\tau)\, d\tau \tag{14}$$

$$h(\tau) = [\exp(-\beta t)/W_1] \sin W_1\tau \tag{15}$$

$$W_1^2 = W_0^2 - \beta^2 \tag{16}$$

Therefore
$$Q(t) = \int_0^\infty h(\tau) F(t-\tau)\, d\tau \tag{17}$$

Mean or Expectation of $Q(t)$: $\langle Q(t) \rangle$:

As $\langle F_i(t) \rangle = 0$, $\langle F(t) \rangle = 0$,

Hence $\langle Q(t) \rangle = 0$ [18]

Mean Square Value of $Q(t)$: $\langle Q^2(t) \rangle$:

To determine $\langle Q^2(t) \rangle$, first, it is necessary to determine the auto correlation function $R_Q(t, \tau)$ which will be just $R_Q(\tau)$ if Q is stationary. As the system is linear and since $F(t)$ is stationary $Q(t)$ the output will be stationary and hence we can specify the auto correlation function in terms of τ only and it will be independent of t.

Now $R_Q(\tau) = \langle Q(t)\, Q(t+\tau) \rangle$ [19]

$$= \int_0^\infty h(\tau_1) \int_0^\infty h(\tau_2) \, \langle F(t-\tau_1)\, F(t+\tau-\tau_2) \rangle \, d\tau_1 \cdot d\tau_2$$

The terms in the angular brackets $= R_F(\tau + \tau_1 - \tau_2)$ [20]

$$R_F(\tau) = \langle F(t)\, F(t+\tau) \rangle =$$

by [13]] $= \{W_0^4 R_{Fi}(\tau) + 4\beta^2 R_{dFi/dt}(\tau) + 2\beta W_0^2 [R_{Fi\, dFi/dt}(\tau)$

$$+ R_{dFi/dt.Fi}(\tau)]\}$$ [21]

As $F_i(t)$ is Gaussion, random variable, mean zero of white noise type *i.e.*, constant spectral density type. $S_{Fi}(W) = S_0 =$ const. Where the letter S stands for spectral density.

$$R_{Fi}(\tau) = (S_0/2)\, \delta(\tau)$$ [22]

Therefore

$$R_{dFi/dt}(\tau) = \frac{d^2}{dt_1\, dt_2}[R_{Fi}(\tau)] = \frac{d\tau}{dt_1}\frac{d\tau}{dt_2}\frac{d^2}{d\tau^2}[R_{Fi}(\tau)$$

[as $\tau = (t_2 - t_1)$,] $= -\frac{1}{2} S_0\, \delta''(\tau)$ [23]

$$R_{Fi\, dFi/dt}(\tau) = 0$$ [24]

$$R_{dFi/dt\, Fi}(\tau) = 0$$ [25]

Using [22] through [25] in [21] and substituting

$$R_F(\tau) = (+S_0/2)\,[W_0^4\, \delta(\tau) + 4\beta^2\, \delta''(\tau)]$$ [26]

Substituting [26] and [20] in [19] and integrating

$$R_Q(\tau) = \frac{S_0}{8\beta W_1} \exp(-\beta\tau) \{(W_0^2 - 4\beta^2) W_1 \cos W_1\tau + (W_0^2 + 4\beta^2)\beta \sin W_1\tau\} \quad [27]$$

Therefore

$$\sigma_Q^2 = \langle Q^2(t) \rangle, \text{ as } \langle Q(t) \rangle = 0$$
$$= R_Q(0) = (S_0/8\beta)(W_0^2 - 4\beta^2) \quad [28]$$

Now

$$R_{dQ/dt}(\tau) = -(\partial^2/\partial\tau^2) R_Q(\tau)$$
$$= (S_0/8\beta W_1) \exp(-\beta\tau) \{(W_0^4 - 4\beta^2 W_0^2 + 16\beta^4) W_1 \cos W_1\tau - (W_0^4 - 12\beta^2 W_0^2 + 16\beta^4)\beta \sin W_1\tau\}$$

Similarly,

$$R_{dQ^2/dt^2}(\tau) = -(\partial^2/\partial\tau^2) R_{dQ/dt}(\tau)$$
$$= (S_0/8\beta W_1) \exp(-\beta\tau) \{(-W_0^6 + 8\beta^2 W_0^4 - 48\beta^4 W_0^2 + 64\beta^6) W_1 \cos W_1\tau + (-3W_0^6 + 24\beta^2 W_0^4 - 80\beta^4 W_0^2 + 64\beta^6)\beta \sin W_1\tau\} \quad [29]$$

Now

$$X(t) = (Q/K_1) + (1/K_2) \int (Q - F_i)\, dt \quad [30]$$

Mean:

$$\langle X(t) \rangle = \frac{\langle Q \rangle}{K_1} + \frac{\langle Q^I \rangle}{K_2} - \frac{\langle F_i^I \rangle}{K_2} = 0 \quad [31]$$

Here $Q^I = \int Q \cdot dt$, $F_i^I = \int F_i\, dt$ } integral transforms of Q and F_i

Mean Square: $\langle x^2(t) \rangle = R_X(0)$:

$$R_X(t, \tau) = \langle X(t) X(t+\tau) \rangle = \left\langle \left\{ \frac{Q(t)}{K_1} + \frac{Q^I(t)}{K_2} - \frac{F_i^I(t)}{K_2} \right\} \cdot \left\{ \frac{Q(t+\tau)}{K_1} + \frac{Q^I(t+\tau)}{K_2} - \frac{F_i^I(t+\tau)}{K_2} \right\} \right\rangle$$
$$= \frac{R_Q(\tau)}{K_1^2} + \frac{R_{Q^I}(\tau)}{K_2^2} + \frac{R_{F_i^I}(\tau)}{K_2^2} - \frac{R_{Q^I \cdot F_i^I}(\tau) + R_{F_i^I Q^I}(\tau)}{K_2^2} \quad [32]$$

where $R_Q(\tau)$ is given by [27]

$$R'_Q(\tau) = \int\int R_Q(\tau)\, dt\, dt, \quad R_{Q^I \cdot F_i^I}(\tau) = \int\int R_{QF_i}(\tau)\, dt\, dt$$

$$R_{F_i I}(\tau) = \int\int (S_0/2)\, \delta(\tau)\, dt\, dt, \quad R_{F_i{}^I Q^I}(\tau) = \int\int R_{F_i Q}'(\tau)\, dt\, dt \quad [33]$$

Here

$$R_{QF_i}(\tau) = \langle Q(t)\, F_i(t+\tau) \rangle$$

$$= \int_0^\infty h(\tau_1) \langle F(t-\tau_1)\, F_i(t+\tau) \rangle\, d\tau_1$$

$$= \frac{S_0}{2} \frac{\exp(-\beta\tau)}{W_1} \{ (\beta^2 - W_1^2) \sin W_1\tau - 2\beta W_1 \cos W_1\tau \} \quad [34]$$

and

$$R_{F_iQ}(\tau) = \langle F_i(t)\, Q(t+\tau) \rangle$$

$$= \int_0^\infty h(\tau_1) \langle F_i(t)\, F(t+\tau-\tau_1) \rangle\, d\tau_1$$

$$= (S_0/2) \exp(-\beta\tau) \{ (W_1^2 - \beta^2) \sin W_1\tau - 2\beta W_1 \cos W_1\tau \} \quad [35]$$

Hence all the quantities in [39] and in turn [32] are known. Hence $R_X(t, \tau)$ is known.

Now, $x = X + x_p$

$$\langle x \rangle = \langle X \rangle + \langle x_p \rangle$$

$\langle x_p \rangle = (mg/K_1) + (mg/K_2)\, t$ at time t

$$\langle x \rangle = (mg/K_1) + (mg/K_2)\, t \quad [36]$$

Mean square value: $\langle x^2(t) \rangle$:

$$R_x(t, \tau) = R_X(t, \tau) + \left\langle \frac{m^2 g^2}{K_1^2} + \frac{mg}{K_1} \frac{mg}{K_2} [t + (t+\tau)] + \frac{m^2 g^2}{K_2^2} t(t+\tau) \right\rangle$$

Therefore

$$\langle x^2(t) \rangle = R_X(0) + \left\langle 2 \frac{m^2 g^2}{K_1 K_2} t + \frac{m^2 g^2}{K_2^2} t^2 + \frac{m^2 g^2}{K_1^2} \right\rangle \quad [37]$$

To calculate the strain energy absorbed by the creep process:

W_{cr} = work done = strain energy absorbed by the creep process

$$= \frac{1}{K_2} \int_0^t (F_s - F_l)^2\, dt \quad [38]$$

But

$$F_s - F_l = m\,[g - d^2x/dt^2)\,]$$

Therefore

$$W_{cr} = \frac{m^2}{K_2} \int_0^t [g^2 - 2g(d^2x/dt) + (d^2x/dt^2)^2]\, dt \qquad [39]$$

$$\langle W_{cr}(t) \rangle = \frac{m^2}{K_2} \int_0^t [g^2 - 2g \langle (d^2x/dt) \rangle + \langle (d^2x/dt^2)^2 \rangle]\, dt$$

Here

$$(d^2x/dt^2) = (d^2X/dt^2)\ [\text{as } (d^2X/dt^2)_p = 0]$$

$$= \frac{(d^2Q/dt)^2}{K_1} + \frac{(dQ/dt) - (dF_i/dt)}{K_2}$$

As $\langle Q \rangle = \langle F_i \rangle = 0$,

$$\langle (d^2x/dt^2) \rangle = 0$$

Also,

$$\langle (d^2x/dt^2)^2 \rangle = \frac{\langle (d^2Q/dt^2)^2 \rangle}{K_1^2} + \frac{\langle (dQ/dt)^2 + (dF_i/dt)^2 \rangle}{K_2^2}$$

(other terms are being zero)

Using these expressions along with [29] gives

$$\langle W_{cr}(t) \rangle = Kt \qquad [40]$$

where

$$K = \frac{m^2}{K_2} \left[g^2 - \frac{1}{K_1^2} \cdot \frac{S_0}{8\beta} (\omega_0^6 - 8\beta^2 \omega_0^4 + 48\beta^4 \omega_0^2 - 64\beta^6) + \frac{1}{K_2^2} \left\{ \frac{S_0}{2} + \frac{S_0}{8\beta} (\omega_0^4 - 4\beta^2 \omega_0^2 + 16\beta^4) \right\} \right]$$

Now, the decrease in the potential energy of the system is

$$dE = mg \int_0^t \frac{F_s - F_i}{K_2}\, dt$$

Therefore $\langle dE \rangle = \dfrac{m^2 g^2 t}{K_2}$ [41]

$D(t)$, the dissipated energy is provided for by the reduction in kinetic energy and

$$D(t) = \langle W_{Cr}(t) \rangle - \langle dE \rangle = K't \quad [42]$$

Hence $D(t)$, the decrease in kinetic energy is a measure of the damping in the system as the difference between the squares of successive peak velocities is proportional to this decrease in kinetic energy.

CONCLUSIONS

From the analysis, it is seen that inclusion of linear creep deformation is equivalent to introduction of viscous damping into the system as the effect of both is the same, *viz.*, attenuation of the vibrations of the system But the similarity between them is limited to this attenuation property, as the nature of the response will be different for the two cases. As seen from the results (expressions [36] and [37]), the response of the system to a stationary random excitation (while remaining stationary for the case of viscous damping) becomes non-stationary for the case considered in this analysis. This difference in the statistical properties of the response can be attributed to the time dependent nature of the creep phenomenon.

The creep process being an irrecoverable process, absorbs energy, which is met in part by the reduction of the potential energy of the system. The decrease in kinetic energy supplies the rest of the energy absorbed by the creep process and this is a measure of damping as it is an index to the ratio of successive peaks of velocities. Since the excitation is random in narure, only a mean value of this energy dissipation can be determined, from which an estimate of the damping can be made.

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