SECONDARY FLOW OF A ELASTICO-VISCOUS FLUID BETWEEN TWO CO-AXIAL CONES OSCILLATING ABOUT A FIXED AXIS

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Abstract

The present paper investigates the secondary flow induced in a Rivlin-Ericksen fluid contained between two coaxial cones, due to rotational oscillations of the cones about their common axis. The secondary motion consists of a steady distribution as well as a periodic motion with frequency twice that of the primary motion. The steady part of this flow is studied in detail for the following cases: (1) The outer cone is oscillating with fou times the angular velocity of the inner cone in the same phase, and (2) the other cone is oscillating with four times the angular velocity of the inner cone in the opposite phase for cones with semi-vertical angles (i) $\theta_1 = \pi/4$ *and* $\theta_2 = \pi/2$ *and (ii)* $\theta_1 = \pi/4$ *and* $\theta_2 = \pi/3$. *We notice that the secondary flow is strongly dependent on the frequency of the oscillation and also on the state of relative motion of the cones.*

1. Introduction

The present paper seeks to discuss the problem of secondary flows induced in an elastico-viscous fluid confined between two co-axial cones oscillating about their common axis. Oscillatory motion of the boundaries is of particular interest in the above class of fluids, because it is here that we can observe the effects of elasticity conicuously as indicated by the general theorems established by Bhatnagar (P.L.) for cylindrical and spherical gecmetries[1]. So far discussion has been confined to steady rotational problems in cone-cone geometry due to certain difficulties in solving the first order motion. We have overcome this difficulty by assuming that the frequency of oscillation is small so that the Reynolds number can be used as a perturbation parameter. In earlier papers[2,3] the secondary flow generated in an elastico-viscous fluid contained between two concentric oscillating spheres has been studied. In these investigations the authors have restricted their discussion to the steady component of the secondary flow field. We have also done like-wise because the calculation af the fluctuating component is extremely cumbersome.

2. Formulation of the Problem

Let us consider a mass of Rivlin-Ericksen (R.E.) fluid contained between two coaxial cones. The cones are represented by $\theta=\theta_1$, and $\theta=\theta_2$ $(>\theta_1)$ in spherical polar coordinates (r, θ, ϕ) with the origin at the common vertex of the two cones. The two cones perform oscillations about the axis $\theta=0$ with the same frequency $n/2\pi$ but different angular velocities Ω_1 and Ω_2. If u, v, w are physical components of the velocity vector, the boundary conditions of the problem are :

$$\left.\begin{array}{l} u=0,\ v=0,\ w=r\,\Omega_1\ \sin\theta_1\ e^{int}\ \text{ on } \theta=\theta_1\ \forall r, \\ u=0,\ v=0,\ w=r\,\Omega_2\ \sin\theta_2\, e^{int}\ \text{ on } \theta=\theta_2\ \forall\ r \end{array}\right\} \qquad [2.1]$$

The constitutive equations, momentum equations and continuity equation are as follows :

$$T_{ij}=-p\delta_{ij}+P_{ij}\,, \qquad [2.2]$$

$$P_{ij}=\phi_1\,E_{ij}+\phi_2\,D_{ij}+\phi_3\,E^m_i\,E_{mj}\,, \qquad [2.3]$$

$$E_{ij}=\tfrac{1}{2}\,(u_{i,\,j}+u_{j,\,i})\,, \qquad [2.4]$$

$$D_{ij}=A_{i,\,j}+A_{j,\,i}+2\,u_{m,\,i}\,u^m_{,j}\,, \qquad [2.5]$$

$$A_i=u^m\,u_{i,\,m}+(\partial\,u_i/\partial\,t), \qquad [2.6]$$

$$\rho\,[(\partial u_i/\partial\,t)+u_m\,u_{i,\,m}]=T_{ij,\,j}\,, \qquad [2.7]$$

$$u_{i,\,i}=0\,, \qquad [2.8]$$

where the symbols have the usual meaning and a suffiix following a comma denotes covariant differentiation.

Introducing the nondimensional quantities through the relations :

$$\left.\begin{array}{llll} r=L^*r'\,, & t=n^{-1}\,t'\,, & u=L^*n\,u'\,, & v=L^*n\,v'\,, \\ w=L^*n\,w', & \Omega_1=n\,\Omega_1'\,, & \Omega_2=n\,\Omega_2'\,, & \\ p_{ij}=n\,\phi_1\,p_{ij}'\,, & p=\rho\ L^*n^2\,p' & & \end{array}\right\} \qquad [2.9]$$

where L^* is a characteristic length.

The boundary conditions of this problem reduce :

$$\left.\begin{array}{l} u=0,\quad v=0,\quad w=r\,\Omega_1\sin\theta_1\ e^{it}\ \text{ on } \theta=\theta_1\,,\ \forall\ r \\ u=0,\quad v=0,\quad w=r\,\Omega_2\sin\theta_2\ e^{it}\ \text{ on } \theta=\theta_2\,,\ \forall\ r \end{array}\right\} \qquad [2.10]$$

Here and after, the prime denoting non-dimensional quantitie is dropped for simplicity and the real parts are to be understood wherever complex expressions are quoted for physical quantities.

3. Due to nonlinearity of the equation of state, we restrict ourselves to solve the problem approximately by assuming the parameter $\Omega\,(=|\Omega_1|+|\Omega_2|)$ to be small so that we can express the velocity components, stress components and isotropic pressure as a power series in Ω in the form :

$$\begin{aligned}
u &= \Omega^2 g(r, \theta, t) + \cdots, \\
v &= \Omega^2 h(r, \theta, t) + \cdots, \\
w &= \Omega f(r, \theta)\, e^{it} + \Omega^3 f_1(r, \theta, t) + \cdots, \\
p_{rr} &= \Omega^2 G(r, \theta, t) + \cdots, \\
p_{\theta\theta} &= \Omega^2 H(r, \theta, t) + \cdots, \\
p_{\phi\phi} &= \Omega^2 M(r, \theta, t) + \cdots, \\
p_{r\theta} &= \Omega^2 L(r, \theta, t) + \cdots, \\
p_{\theta\phi} &= \Omega\, Q(r, \theta)\, e^{it} + \Omega^3 Q_1(r, \theta, t) + \cdots, \\
p_{r\phi} &= \Omega\, T(r, \theta)\, e^{it} + \Omega^3 T_1(r, \theta, t) + \cdots, \\
p &= \Omega^2 N(r, \theta, t) + \cdots.
\end{aligned} \tag{3.1}$$

Substituting these expressions in the equations of state equations of motion and continuity and separating the terms in Ω and Ω^2 and neglecting higher order terms in Ω, the following system of linear partial differential equations is obtained :

$$Q = (1/r)\,[\partial f/\partial \theta - f \cot\theta]\,(1+ik), \tag{3.2}$$

$$T = [\partial f/\partial r - f/r]\,(1+ik), \tag{3.3}$$

$$fi = \frac{1}{R}\left[\frac{1}{r}\,\frac{\partial Q}{\partial \theta} + \frac{2Q\cot\theta}{r} + \frac{\partial T}{\partial r} + \frac{3T}{r}\right], \tag{3.4}$$

$$G = 2\left(1 + k\,\frac{\partial}{\partial t}\right)\frac{\partial g}{\partial r} + (2k + S)\left\{e^{it}\left(\frac{\partial f}{\partial r} - \frac{f}{r}\right)\right\}^2 \tag{3.5}$$

$$H = 2\left(1 + k\,\frac{\partial}{\partial t}\right)\left(\frac{1}{r}\,\frac{\partial h}{\partial \theta} + \frac{g}{r}\right) + \frac{2k+S}{r^2}\left\{e^{it}\left(\frac{\partial f}{\partial \theta} - f\cot\theta\right)\right\}^2 \tag{3.6}$$

$$M = 2\left(1 + k\,\frac{\partial}{\partial t}\right)\left(\frac{g}{r} + \frac{h}{r}\cot\theta\right) + S\left[\frac{1}{r^2}\left\{e^{it}\left(\frac{\partial f}{\partial \theta} - f\cot\theta\right)\right\}^2 + \left\{e^{it}\left(\frac{\partial f}{\partial r} - \frac{f}{r}\right)\right\}^2\right] \tag{3.7}$$

$$L=\left(1+k\frac{\partial}{\partial t}\right)\left(\frac{\partial h}{\partial r}+\frac{1}{r}\frac{\partial g}{\partial r}-\frac{h}{r}\right)+(2k+S)\left[e^{it}\left[\frac{\partial f}{\partial r}-\frac{f}{r}\right)\times\right.$$

$$\left.e^{it}\left(\frac{\partial f}{\partial\theta}-f\cot\theta\right)\right], \quad [3.8]$$

$$\frac{\partial g}{\partial t}-\frac{(e^{it}f)^2}{r}=-\frac{\partial N}{\partial r}+\frac{1}{R}\left[\frac{\partial G}{\partial r}+\frac{2G}{r}+\frac{1}{r}\frac{\partial L}{\partial\theta}+\frac{L}{r}\cot\theta-\frac{H+M}{r}\right] \quad [3.9]$$

$$\frac{\partial h}{\partial t}-\frac{(e^{it}f)^2}{r}\cot\theta=-\frac{1}{r}\frac{\partial N}{\partial\theta}+\frac{1}{R}\left[\frac{\partial L}{\partial r}+\frac{3L}{r}+\frac{1}{r}\frac{\partial H}{\partial\theta}+\frac{H-M}{r}\cot\theta\right] \quad [3.10]$$

$$\frac{\partial}{\partial r}(r^2 g\sin\theta)+\frac{\partial}{\partial\theta}(rh\sin\theta)=0 \quad [311]$$

where

$R=(L^{*2}n\,\rho/\phi_1)$ is the Reynolds number for the flow

$k=(n\phi_2/\phi_1)$, $S=n\,\phi_3/\phi_1$ are the dimensionless parameters, specifying the effects of viscoelasticity and cross-viscosity on the flow field respectively.

The boundary conditions in terms of dimensionless quantities are

$$\left.\begin{aligned} g&=0,\ h=0,\ f=m_1 r\sin\theta_1 \text{ on } \theta=\theta_1\ \forall r\\ g&=0,\ h=0,\ f=m_2 r\sin\theta_2 \text{ on } \theta=\theta_2\ \forall r\end{aligned}\right] \quad [3.12]$$

$$m_1=\frac{\Omega_1}{|\Omega_1|+|\Omega_2|},\quad m_2=\frac{\Omega_2}{|\Omega_1|+|\Omega_2|}. \quad [3.13]$$

4. Solution of the Equations

(a) Primary motion

To obtain the primary motion, we eliminate Q and T from [3.2]–[3.4]. Thus, we get the following second order differential equation[1] determining f:

$$\left.\begin{aligned} Rf\delta^2&=\frac{1}{r^2}\left(\frac{\partial^2 f}{\partial\theta^2}+\cot\theta\frac{\partial f}{\partial\theta}-f\cot^2\theta+f\right)+\frac{\partial^2 f}{\partial r^2}\\ &\quad+\frac{2}{r}\frac{\partial f}{\partial r}-\frac{2f}{r^2}\end{aligned}\right\} \quad [4.1]$$

where

$$\delta^2=\frac{i}{1+k\,i}$$

The boundary conditions to be satisfied by f are :

$$\left.\begin{aligned} f&=m_1\, r \sin\theta_1 \text{ on } \theta=\theta_1 \ \forall r \\ f&=m_2\, r \sin\theta_2 \text{ on } \theta=\theta_2 \ \forall r \end{aligned}\right] \quad [4.2]$$

We notice that we cannot determine a function $q(\theta)$ such the $f=r\,q(\theta)$ satisfies the equation [4.1] and the boundary conditions [4.3]. Hence we adopt a perturbation method, assuming the Reynolds number of the motion to be small, so that we assume f of the form $f=W_0+R\,W_1$, neglecting the higher order terms in R.

$$0=\frac{1}{r^2}\left(\frac{\partial^2 W_0}{\partial\theta^2}+\cot\theta\frac{\partial W_0}{\partial\theta}-W_0\cot^2\theta+W_0\right)+\frac{\partial^2 W_0}{\partial r^2}$$

$$+\frac{2}{r}\frac{\partial W_0}{\partial r}-\frac{2}{r^2}W_0, \quad [4.4]$$

$$W_0\,\delta^2=\frac{1}{r^2}\left(\frac{\partial^2 W_1}{\partial\theta^2}+\cot\theta\frac{\partial W_1}{\partial\theta}-W_1\cot^2\theta+W_1\right)+\frac{\partial W_1}{\partial r^2}$$

$$+\frac{2}{r}\frac{\partial W_1}{\partial r}-\frac{2W_1}{r^2} \quad [4.5]$$

With boundary conditions :

$$\left.\begin{aligned} W_0&=m_1\, r\sin\theta_1, \quad \text{on } \theta=\theta_1 \quad \forall r \\ W_0&=m_2\, r\sin\theta_2, \quad \text{on } \theta=\theta_2 \quad \forall r \end{aligned}\right\} \quad [4.6]$$

$$\left.\begin{aligned} W_1&=0 \quad \text{on} \quad \theta=\theta_1, \quad \forall r, \\ W_1&=0 \quad \text{on} \quad \theta=\theta_2, \quad \forall r, \end{aligned}\right\} \quad [4.7]$$

The solution of the equation [4.4] is

$$W_0=r\sin\theta\,[A\,(-\cot\theta\,\mathrm{cosec}\,\theta+ln\tan\theta/2)+B] \quad [4.8]$$

where A and B are arbitrary constants and are given by using boundary condition [4.6]

$$\left.\begin{aligned} A&=\frac{m_1-m_2}{[ln\tan(\theta_1/2)-ln\tan(\theta_2/2)-\cot\theta_1\,\mathrm{cosec}\,\theta_1+\cot\theta_2\,\mathrm{cosec}\,\theta_2]} \\ B&=\frac{m_1[\cot\theta_2\,\mathrm{cosec}\,\theta_2-ln\tan(\theta_2/2)]+m_2[ln\tan(\theta_1/2)-\cot\theta_1\,\mathrm{cosec}\,\theta_1]}{[ln\tan(\theta_1/2)-ln\tan(\theta_2/2)-\cot\theta_1\,\mathrm{cosec}\,\theta_1+\cot\theta_2\,\mathrm{cosec}\,\theta_2]} \end{aligned}\right\} [4.9]$$

The solution of the equation [4.5] is given by

$$W_1 = r^3 s(\theta) \sin\theta \qquad [4.10]$$

where

$$s(\theta) = (\delta^2/10)\ \omega(\theta) + (1-5\cos^2\theta)\{C - (D/32)\,[\operatorname{cosec}\theta \cot\theta - 6\ ln \tan(\theta/2) + (25\cos\theta)/(1-5\cos^2\theta)]\}\ , \qquad [4.11]$$

$$\omega(\theta) = A\,[-\cot\theta \operatorname{cosec}\theta + ln \tan(\theta/2)] + B. \qquad [4.12]$$

C and D are arbitrary constants and are given by (using the boundary conditions [4.7])

$$\left.\begin{aligned}
D &= \frac{k}{5(1+k^2)E}\left[\frac{m_1}{(1-5\cos^2\theta_1)} - \frac{m_2}{(1-5\cos^2\theta_2)}\right], \\
E &= \frac{1}{8}\left[\frac{1}{2}\left(\frac{\cos\theta_1}{\sin^2\theta_1} - \frac{\cos\theta_2}{\sin^2\theta_2}\right) - 3\ ln\left\{\frac{\tan\theta_1/2}{\tan\theta_2/2}\right\}\right. \\
&\qquad \left. + \frac{25}{2}\left(\frac{\cos\theta_1}{1-5\cos^2\theta_1} - \frac{\cos\theta_2}{1-5\cos^2\theta_2}\right)\right] \\
&\text{and} \\
C &= \frac{D}{16}\left[\frac{1}{2}\ \frac{\cos\theta_1}{\sin^2\theta_1} - 3\ ln \tan\frac{\theta_1}{2} + \frac{25}{2}\ \frac{\cos\theta_1}{1-5\cos^2\theta_1}\right] \\
&\qquad - \frac{k\,m_1}{10(1+k^2)(1-5\cos^2\theta_1)}
\end{aligned}\right\} \qquad [4.13]$$

Hence we get

$$f = r\sin\theta\ \omega(\theta) + Rr^3 \sin\theta\ F(\theta) + [(R\,r^3 \sin\theta/10)\,k\,]\,[\omega(\theta)/(1+k^2)] + i\,[R\,r^3 \sin\theta\ \omega(\theta)/10\,(1+K^2)\ , \qquad [4.14]$$

where

$$F(\theta) = (1-5\cos^2\theta)\ [C - (D/32)\operatorname{cosec}\theta\cot\theta - 6\ ln \tan\theta/2 + (25\cos\theta)/(1-5\cos^2\theta)\ ]. \qquad [4.15]$$

(*b*) *Secondary motion* :

Having determined f, we use the set of equation [3.5] — [3.11] to discuss the second order motion, which we shall call the secondary motion. An examination of the equations [3.5] to [3.10] reveals that purley periodic primary motion is associated with an additional steady velocity distribution as well as a periodic motion with twice the frequency of the primary motion. Therefore to separate the steady and unsteady parts, we set

$$\left.\begin{aligned} g &= g_1(r,\theta)+g_2(r,\theta)\exp(2it),\\ G &= G_1(r,\theta)+G_2(r,\theta)\exp(2it),\\ M &= M_1(r,\theta)+M_2(r,\theta)\exp(2it),\\ N &= N_1(r,\theta)+N_2(r,\theta)\exp(2it),\\ h &= h_1(r,\theta)+h_2(r,\theta)\exp(2it),\\ H &= H_1(r,\theta)+H_2(r,\theta)\exp(2it),\\ L &= L_1(r,\theta)+L_2(r,\theta)\exp(2it), \end{aligned}\right\} \tag{4.16}$$

We shall concentrate our attention here only on the steady part of the secondary flow generated.

Substituting [4.16] in equations [3.5]–[3.11] and equating the time independent parts we get :

$$G_1 = 2\frac{\partial g_1}{\partial r} + \frac{2k+S}{2}\left|\frac{\partial f}{\partial r}-\frac{f}{r}\right|^2, \tag{4.17}$$

$$H_1 = 2\left(\frac{1}{r}\frac{\partial h_1}{\partial \theta}+\frac{g_1}{r}\right) + \frac{2k+S}{r^2}\left|\frac{\partial f}{\partial \theta}-f\cot\theta\right|^2, \tag{4.18}$$

$$M_1 = 2\left(\frac{g_1}{r}+\frac{h_1\cot\theta}{r}\right) + \frac{5}{2}\left[\frac{1}{r^2}\left|\frac{\partial f}{\partial\theta}-f\cot\theta\right|^2 + \left|\frac{\partial f}{\partial r}-\frac{f}{r^2}\right|^2\right], \tag{4.19}$$

$$L_1 = \left(\frac{\partial h_1}{\partial r}+\frac{1}{r}\frac{\partial g_1}{\partial\theta}-\frac{h_1}{r}\right) + \frac{2k+S}{r^2}\,Rl\left[\left(\frac{\partial f}{\partial r}-\frac{f}{r}\right)\overline{\left(\frac{\partial f}{\partial\theta}-f\cot\theta\right)}\right], \tag{4.20}$$

$$-\frac{f\bar{f}}{2r} = -\frac{\partial N_1}{\partial r} + \frac{1}{R}\left[\frac{\partial G_1}{\partial r}+\frac{2G_1}{r}+\frac{1}{r}\frac{\partial L_1}{\partial\theta}+\frac{L_1}{r}\cot\theta-\frac{H_1+M_1}{r}\right], \tag{4.21}$$

$$-\frac{f\bar{f}}{2r}\cot\theta = -\frac{\partial N_1}{\partial\theta} + \frac{1}{R}\left[r\frac{\partial L_1}{\partial r}+3L_1+\frac{\partial H_1}{\partial\theta}+(H_1-M_1)\cot\theta\right] \tag{4.22}$$

$$\partial(r^2 g_1\sin\theta)/\partial r + \partial(r\,h_1\sin\theta)/\partial r = 0. \tag{4.23}$$

The bar over an expression denotes its complex conjugate.

Eliminating N_1 from [4.21] and [4.22], then using the relations [4.14], [4.17]–[4.20] and introducing a stream function $\psi\,(r,\theta)$ defined by

$$\left.\begin{aligned} r\,g_1 \sin\theta &= \partial(\psi\, r\sin\theta)/\partial\theta, \\ \text{and} \quad r\,h_1 &= -\partial\,(r\,\psi)/\partial r \end{aligned}\right\} \qquad [4.24]$$

we get the equation satisfied by $\psi\,(r,\theta)$ to be

$$\bar{L}^2\,\psi\,(r,\ \theta) = [2\,(k+S)/r^2]\ F_1\,(\theta) + R\,F_2\,(\theta) + (2\,k+S)\ F_3\,(\theta) + SR\,F_4\,(\theta) + R^2 r^2 F_5\,(\theta) \qquad [4.25]$$

where

$$\bar{L} \equiv \left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\ \frac{\partial}{\partial r} + \frac{1}{r^2}\ \frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{r^2}\ \frac{\partial}{\partial\theta} - \frac{1}{r^2\sin^2\theta}\right] \qquad [4.26]$$

and the boundary conditions :

$$\left.\begin{aligned} \partial\,(\psi\sin\theta)/\partial\theta &= 0 \quad \text{on} \quad \theta=\theta_1 \ \text{and}\ \theta=\theta_2 \\ \partial\,(\psi r)/\partial r &= 0 \quad \text{on} \quad \theta=\theta_1 \ \text{and}\ \theta=\theta_2 \end{aligned}\right\} \qquad [4.27]$$

$$F_1\,(\theta) = -8\,A^2 \operatorname{cosec}^4\theta\ \cot\theta, \qquad [4.28]$$

$$F_2\,(\theta) = -2\,A\ \operatorname{cosec}\theta\ \{\,A\,[\ln\tan(\theta/2) - \cot\theta\operatorname{cosec}\theta] + B\}, \qquad [4.29]$$

$$F_3\,(\theta) = -2\,0\,AC\operatorname{cosec}\theta\ (1+\cos^2\theta) + \frac{AD}{4}\left[\frac{-15\,(1+\cos^2\theta)}{\sin\theta}\ln\tan\ \theta/2 + \frac{17\cos\theta + 10\cos^3\theta - 15\cos^5\theta}{\sin^5\theta}\right] + 20\,A\left[\,A(-\cot\theta\operatorname{cosec}\theta + \ln\tan\theta/2) + B\right]\operatorname{cosec}\theta - 48\,A^2\cot\theta\operatorname{cosec}^4\theta, \qquad [4.30]$$

$$F_4\,(\theta) = -20\,AC\operatorname{cosec}\theta\ (1+\cos^2\theta) + \frac{AD}{4}\times \left[\frac{-15\,(1+\cos^2\theta)}{\sin\theta}\ \ln\tan(\theta/2) + \frac{17\cos\theta + 10\cos^3\theta - 15\cos^5\theta}{\sin^5\theta}\right] - 48\,A^2\cot\theta\operatorname{cosec}^4\theta, \qquad [4.31]$$

$$F_5(\theta)=2\,AC\,[-\text{cosec}\,\theta+9\cos^2\theta\,\text{cosec}\,\theta-2\sin 2\theta\,\ln\tan\theta/2]$$
$$+(AD/8)\,\{-6\sin 2\theta\,(\ln\tan\theta/2)^2$$
$$+[(39\cos^2\theta-11)/\sin\theta]\,\ln\tan\theta/2$$
$$+(21\cos\theta-27\cos^3\theta)/\sin^3\theta\}$$
$$+(BD/2)[-3\sin\theta\cos\theta\,\ln\tan(\theta/2)+(3\cos^2\theta-2)/\sin\theta]. \qquad [4.32]$$

Let us put

$$\psi(r,\theta)=2(k+S)\,\psi_1(r,\theta)+R\,\psi_2(r,\theta)+(2k+S)\,R\,\psi_3(r,\theta)$$
$$+SR\,\psi_4(r,\theta)+R^2\,\psi_5(r,\theta) \qquad [4.33]$$

then

$$\left.\begin{array}{l}\bar{L}^2\psi_1(r,\theta)=F_1(\theta)/r_2,\quad \bar{L}^2\psi_2=F_2(\theta),\quad \bar{L}^2\psi_3=F_3(\theta)\\ \bar{L}^2\psi_4(r,\theta)=F_4(\theta)\ \text{and}\ \bar{L}^2\psi_5=r^2F_5(\theta)\end{array}\right\} \qquad [4.34]$$

To solve the equation [4.25] we shall put

$$\left.\begin{array}{l}\psi_1=r^2\,\text{cosec}\,\theta\,\chi_1(\theta),\quad \psi_2=r^4\,\text{cosec}\,\theta\,\chi_2(\theta),\\ \psi_3=r^4\,\text{cosec}\,\theta\,\chi_3(\theta),\quad \psi_4=r^4\,\text{cosec}\,\theta\,\chi_4(\theta),\\ \text{and}\\ \psi_5=r^6\,\text{cosec}\,\chi_5(\theta)\end{array}\right\} \qquad [4.35]$$

and using the method of variation of parameters, we shall get the solution of the equation as follows :

$$\chi_1(\theta)=a_1+b_1\cos\theta+c_1\,\xi(\theta)+d_1\,\xi_1(\theta)+(A^2/2)\,\xi_2(\theta), \qquad [4.36]$$

$$\chi_2(\theta)=a_2\,\xi(\theta)+b_2\,\xi_1(\theta)+c_2\,\xi_3(\theta)+d_2\,\xi_4(\theta)$$
$$-(A^2/2)\,\xi_5(\theta)-AB/60, \qquad [4.37]$$

$$\chi_3(\theta)=a_3\,\xi(\theta)+b_3\,\xi_1(\theta)+c_3\,\xi_3(\theta)+d_3\,\xi_4(\theta)+5\,A^2\,\xi_5(\theta)$$
$$(AD/32)\,\xi_6(\theta)+(5\,AC/12)\,\xi_7(\theta)-(A^2/16)\,\xi_{13}(\theta)+AB/6, \qquad [4.38]$$

$$\chi_4(\theta)=a_4\,\xi(\theta)+b_4\,\xi_1(\theta)+c_4\,\xi_3(\theta)+d_4\,\xi_4(\theta),\ +5A^2\,\xi_5(\theta)+(AD/32)\,\xi_6(\theta)$$
$$+(5/12)\,AC\,\xi_7(\theta)-(A^2/16)\,\xi_{13}(\theta)+AB/6 \qquad [4.39]$$

$$\chi_5(\theta)=c_5\,\xi_8(\theta)+b_5\,\xi_9(\theta)+c_5\,\xi_3(\theta)+d_5\,\xi_4(\theta)+(AC/3079)\,\xi_{10}(\theta)$$
$$(AD/128)\,\xi_{11}(\theta)+(BD/8192)\,\xi_{12}(\theta)+(BC/15)\,\xi_{14}(\theta) \qquad [4.40]$$

where

$$\xi\ (\theta) = \cos^3\theta - \cos\theta,$$

$$\xi_1(\theta) = \xi\ (\theta)\ ln \tan\theta/2 + \cos^2\theta - \tfrac{2}{3},$$

$$\xi_2(\theta) = \xi\ (\theta)\ (ln \tan\theta/2)^2 + (2\cos^2\theta - 1)\ (ln\tan\theta/2 + \cos\theta),$$

$$\xi_3(\theta) = 7\cos^5\theta - 10\cos^3\theta + 3\cos\theta,$$

$$\xi_4(\theta) = \xi_3(\theta)\ ln\tan\theta/2 + 7\cos^4\theta - \tfrac{23}{3}\cos^2\theta + \tfrac{16}{15},$$

$$\xi_5(\theta) = (-\tfrac{9}{8}\cos^5\theta + \tfrac{7}{4}\cos^3\theta - \tfrac{5}{8}\cos\theta)\ (ln\tan\theta/2)^2 + (-\tfrac{9}{4}\cos^4\theta + \tfrac{11}{4}\cos^2\theta - \tfrac{1}{2})\ ln\tan\theta/2 - \tfrac{63}{80}\cos^5\theta + \tfrac{2}{7}\cos^3\theta + \tfrac{211}{560}\cos\theta,$$

$$\xi_6(\theta) = (-25\cos^5\theta + 34\cos^3\theta - 9\cos\theta)\ (ln\tan\theta/2)^2 + (-50\cos^4\theta + \tfrac{149}{3}\cos^2\theta - \tfrac{17}{3})\ ln\tan\theta/2 - \tfrac{40}{9}\cos^5\theta + \tfrac{34}{9}\cos^3\theta - \tfrac{26}{3}\cos\theta,$$

$$\xi_7(\theta) = (-6\cos^5\theta + 7\cos^3\theta - \cos\theta)\ ln\tan\theta/2 - 6\cos^4\theta + \tfrac{13}{3}\cos^2\theta,$$

$$\xi_8(\theta) = 33\cos^7\theta - 63\cos^5\theta + 35\cos^3\theta - 5\cos\theta,$$

$$\xi_9(\theta) = \xi_8(\theta)\ ln\tan\theta/2 + 33\cos^6\theta - 52\cos^4\theta + \tfrac{103}{5}\cos^2\theta - \tfrac{128}{105},$$

$$\xi_{10}(\theta) = (305\cos^7\theta - 335\cos^5\theta + 19\cos^3\theta + 11\cos\theta)\ ln\tan\theta/2 + 305\cos^6\theta - \tfrac{700}{3}\cos^4\theta + 11\cos^2\theta,$$

$$\xi_{11}(\theta) = (-145\cos^7\theta + 295\cos^5\theta - 179\cos\theta + 29\cos\theta)\times (ln\tan\theta/2)^2 + (-290\cos^6\theta + \tfrac{1480}{3}\cos^4\theta - \tfrac{656}{3}\cos^2\theta + \tfrac{46}{3})\ ln\tan\theta/2 + (-\tfrac{5369}{15}\cos^7\theta + \tfrac{91913}{12}\cos^5\theta - \tfrac{21138}{9}\cos^3\theta + \tfrac{46}{3}\cos\theta),$$

$$\xi_{12}(\theta) = (35\cos^9\theta - \tfrac{1383}{3}\cos^7\theta + \tfrac{1869}{3}\cos^5\theta - \tfrac{592}{3}\cos^3\theta)\ ln\tan\theta/2 - \tfrac{2135}{27}\cos^8\theta + \tfrac{8405}{27}\cos^6\theta - \tfrac{2312}{9}\cos^4\theta - 22\cos^2\theta,$$

$$\xi_{13}(\theta) = (-270\cos^5\theta + 396\cos^3\theta - 126\cos\theta)\ (ln\tan\theta/2)^2 + (-862\cos^4\theta + 612\cos^2\theta + 96)\ ln\tan\theta/2 - 643\cos^5\theta + 718\cos^3\theta - 105\cos\theta.$$

$$\xi_{14}(\theta) = \cos^7\theta - \cos^5\theta \qquad [4.41]$$

a_i, b_i, c_i and d_i are the arbitrary constants and are determined by the boundary conditions :

$$\left.\begin{array}{ll} \partial/\partial\theta\ (\psi_i \sin\theta) = 0 & \text{on } \theta = \theta_1 \text{ and } \theta = \theta_2 \\ \partial/\partial r\ (\psi_i r) = 0 & \text{on } \theta = \theta_1 \text{ and } \theta = \theta_2 \end{array}\right\} \qquad [4.42]$$

$i = 1,\ 2,\ 3,\ 4$ and 5.

5. DISCUSSION

We have studied in particular the flow field for a fluid specified by $k/R = -0.2$ and $S/R = 0.6$, for a particular Reynolds number $R = 0.5$. In case I (a) – I (b) we have taken $\theta_1 = \pi/4$ and $\theta_2 = \pi/2$. While in II (a), II (b) we have taken $\theta_1 = \pi/4$ and $\theta_2 = \pi/3$. In I (a), II (a) we have taken $m_1 = 0.2$, $m_2 = 0.8$ and I (b), II (b) we have taken $m_1 = 0.2$, $m = -0.8$.

Case 1

No separation is observed for Newtonian fluids in cases I (a) and I (b). In case I (a) the fluid is drawn in near the cone and thrown out near the plate. In the case of the above R. E. fluid a separating stream line is present extending from the cone to the plate in I (a). In the inner regions closed loops are formed, while in the outer the flow resembles the Newtonian pattern. As the Reynolds number increases the separating stream line will move away

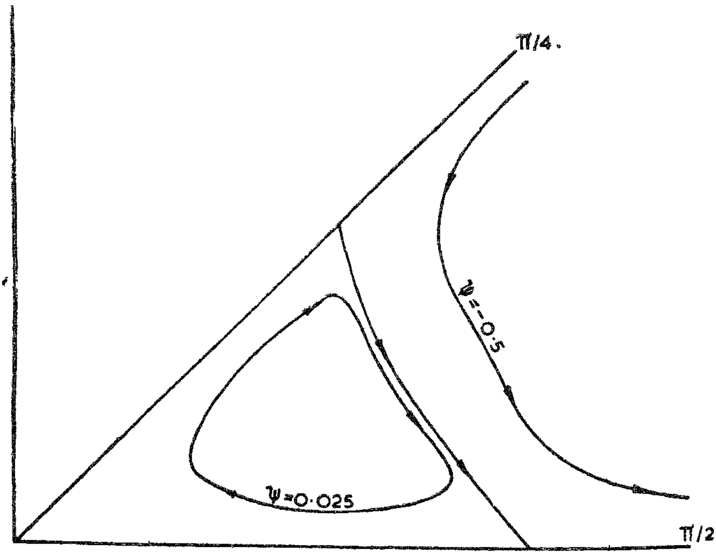

FIG. 1

Stream Lines of Secondary flow when $m_1 = 0.2$, $m_2 = 0.8$, $R = 0.5$, $k = 0.1$, $S = 0.3$.

from the vertex. In I (b) the stream line pattern is opposite to that for Newtonian fluid for the same Reynolds number. This reversal of the secondary flow pattern is characteristic of these fluids.

Case 2

When the cones oscillate in the same phase, the outer oscillating with four times the angular amplitude of the inner, we notice that the separation of the secondary flow has just set in for the above fluid when $R=0.5$. In meridian plane the fluid is drawn in near the inner cone and thrown out at the outer resembling the Newtonian pattern. In I (b) the secondary flow field resembles that of the Newtonian fluid with fluid drawn in near the outer cone and thrown out at the inner cone, when compared with I (b) we notice that the separation sets in earlier for the same fluid at the same Reynolds number when the oscillating boundaries have semi-vertical angles $\pi/4$ and $\pi/2$ when compared with semi-vertical angles $\pi/4$ and $\pi/3$.

The secondary flow in the above cases resembles that when the boundaries rotate because W_0 in the primary motion for oscillating boundaries is identical with the primary motion for rotating boundaries.

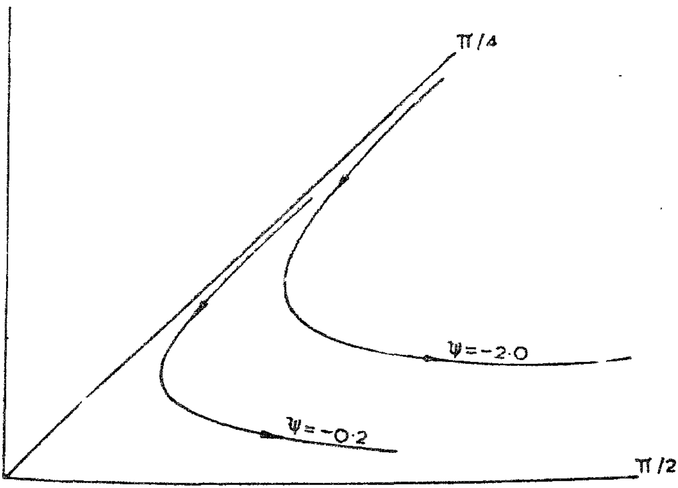

FIG. 2

Stream Lines of Secondary flow when $m_1=0.2$, $m_2=-0.8$, $R=0.5$, $k=-0.1$, $S=0.3$.

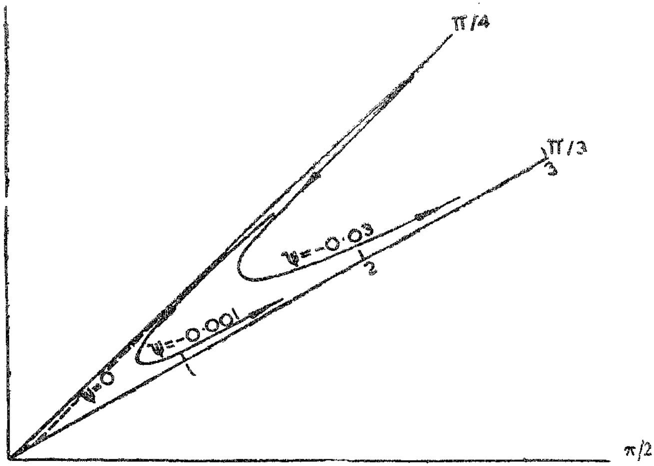

FIG. 3
Stream Lines of Secondary flow when $m_1=0.2$, $m_2=0.8$, $R=0.5$, $k=-0.1$, $S=0.3$.

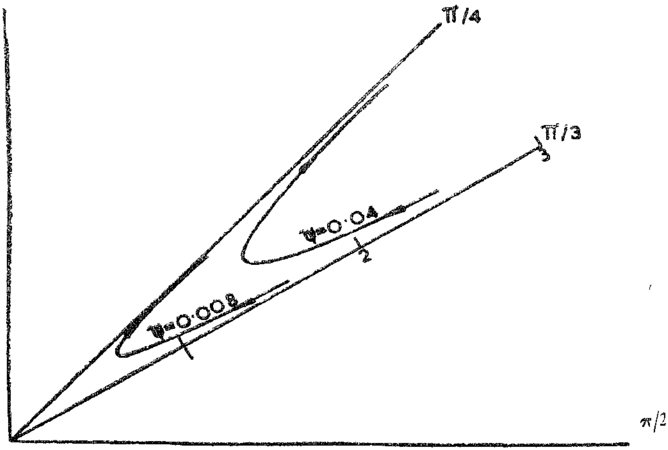

FIG. 4
Stream Lines of Secondary flow when $m_1=0.2, m_2=-0.8$, $R=0.5$, $k=-0.1$, $S=0.3$.

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