ON TRIANGULAR DECOMPOSITIONS TO EVALUATE THE DETERMINANT OF AN ARBITRARY SQUARE MATRIX A INCLUDING THE SOLUTION OF $Ax=b$ AND ON THE RELATED COMPUTATIONAL RECURRENCE RELATIONS

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1. ABSTRACT

The invariant nature of the determinant of a matrix under elementary transformations affords an easy determining way of obtaining the determinant of a matrix by converting it into a triangular form. The single LR decomposition does it still more easily, provided all the leading submatrices are non-singular. In fact, in case of any singular or near-singular leading submatrix or submatrices LR decomposition fails or produces inaccurate results. In this context, presented in this paper are four types of triangular decompositions (different from LR type) which usually succeed in such singular or near-singular cases. Also presented here is a discussion of all possible triangular decompositions, and their usefulness and uniqueness in presenting a square matrix. A solution of a system of linear algebraic equations $Ax=b$ through the easy inversion of the triangular ones is shown. Also presented are the simple explicit computational recurrence relations for easy automatic computations of the solution vector $\vec{x}$ of $A\vec{x}=\vec{b}$ by all possible triangular decompositions. A typical numerical example has been worked out as an illustration.

2. TRIANGULAR DECOMPOSITIONS

Four possible triangular matrices may be thought of for any square matrix. These triangular decompositions spring from the fact that any square matrix has two and only two diagonals, left and right. In any triangular decomposition, the diagonal elements of any of the two triangular matrices must be specified in order to determine them uniquely from a given square matrix. It is convenient here to define a few matrix terms to be used in the treatment.

(*a*) Triangular matrix is a square matrix having either a left or a right diagonal below or above which all elements are zeros, thus presenting a form of a bilateral triangle of elements, normally, non-zero in nature.

(*b*) Lower triangular matrix of left diagonal (L_l) is a square matrix, all the elements of which above the left diagonal are zeros.

(*c*) Upper triangular matrix of left diagonal (R_l) is a square matrix all the elements of which below the left diagonal, are zeros.

(*d*) Lower triangular matrix of right diagonal (L_r) is a square matrix, all the elements of which above the right diagonal are zeros.

(*e*) Upper triangular matrix of right diagonal (R_r) is a square matrix, all the elements of which below the right diagonal are zeros.

Since the conventional matrix multiplication is not commutative, the four triangular matrices L_l, R_l, L_r and R_r have twelve different product combinations taking any two at a time. Out of these, only five product combinations can represent a given square matrix uniquely, They are $L_l R_l$, $R_l L_l$, $L_l R_r$, $L_r L_l$ and $R_r R_l$. The rest of the possible combinations $R_r L_l$, $L_l L_r$, $R_l R_r$, $R_l L_r$, $L_r R_l$, $R_r L_r$ and $L_r R_r$ only give rise to triangular matrices and are therefore of no use in presentation of a given square matrix uniquely.

It can be seen that of the five decompositions cited above the first two can also represent a square matrix uniquely even if they are multiplied in the reverse way. The last three of these, since they fail in this respect, cannot be used for finding eigen values of a square matrix. But these three decompositions as also the $R_l L_l$ decomposition have the advantage over the conventional $L_l R_l$ decomposition in that when the leading submatrix or submatrices of a square matrix is singular or near-singular they succeed in complete decompositions It can also be seen that triangular martices of the right diagonal only *e.g.*, $L_r R_r$ or $R_r L_r$ cannot represent a square matrix.

3. COMPUTATIONAL RECURRENCE RELATIONS

Simple recurrence relations for finding L_l, R_l, L_r and R_r for the five useful decompositions have been derived explicitly from a given square matrix using conventional matrix multiplication rules (*i.e.* defining unit matrix as a left diagonal matrix with unity in its left diagonal terms, instead of right diagonal terms).

Consider the square matrix

$$A=\{a_{ij}\} \quad i=1, 2, 3, \cdots, n-1, n$$
$$j=1, 2, 3, \cdots, n-1, n$$

and the triangular matrices

$$L_l=\{l_{ij}\} \quad i=1, 2, 3, \cdots, n-1, n$$
$$j=1, 2, 3, \cdots, i-1, i$$

$$R_l = \{r_{ij}\} \quad j = 1, 2, 3, \cdots, n-1, n$$
$$i = 1, 2, 3, \cdots, j-1, j$$

$$L_r = \{l'_{ij}\} \quad i = 1, 2, 3, \cdots, n-1, n$$
$$j = \{n-(i-1)\}, \ \{n-(i-1)\}+1,$$
$$\{n-(i-1)\}+2, \cdots, n-1, n$$

and

$$R_r = \{r'_{ij}\} \quad i = 1, 2, 3, \cdots, n-1, n$$
$$j = 1, 2, 3, \cdots, \{n-(i-1)\}$$

It can be seen that the sequence of values of the subscripts i and j (to represent matrix elements) carries the following meaning all through out, though in some cases brief note will be added for easy and quick access to the recurrence relations. The sequence of subscripts i and j in

$$\text{“}\, i = 1, 2, 3, \cdots, n-1, n$$
$$j = 1, 2, 3, \cdots, n-1, n\,\text{”}$$

indicate that i is to be taken as 1 (fixed) first and j is to be varied from 1 to n at the interval of 1. Then $i=2$ (fixed) and $j=1, 2, 3, \cdots, n-1, n$. Next $i=3$ (fixed), $j=1, 2, 3, \cdots, n-1, n$, and so on.

The square matrix A when expressed as the product of L_l and R_l matrices, the left diagonal elements of L_l being specified as unity, we get the following recurrence relations.

$$\left.\begin{array}{l} r_{ij} = a_{ij} - \sum_{p=1}^{i-1} l_{ip}\, r_{pj}, \quad l_{ii} = 1 \text{ for all } i \\ i \leqq j \\ l_{ij} = (a_{ij} - \sum_{p=1}^{j-1} l_{ip}\, r_{pj})/r_{jj} \\ i > j \end{array}\right\} \qquad [1.1]$$

$$j = 1, 2, 3, \cdots, n-1, n; \ i = 1, 2, 3, \cdots, n-1, n$$

We first take $j=1$ (fixed) and go on varying i from 1 to n at an interval of 1. As a result we get r_{11} from the first relation and then $l_{21}, l_{31}, l_{41}, l_{51}, \cdots, l_{n1}$ from the second relation. Next we take $j=2$ (fixed), and vary i from 1 to n as usual at an interval of 1 and find r_{12}, r_{22} from the first relation and $l_{32}, l_{42}, l_{52}, \cdots, l_{n2}$ from the second relation and so on. Lastly we take $j=n$ (fixed) and vary i from 1 to n at an interval of 1 and consequently we get $r_{1n}, r_{2n}, r_{3n}, \cdots, r_{nn}$.

$$\text{Det } A = \prod_{i=1}^{n} r_{ii} \qquad [1.2]$$

We replace the system of equations $Ax = b$ by

$$L_l R_l x = b, \text{ hence } R_l x = L_l^{-1} b = c \text{ (say)}$$

The elements of L_l^{-1} matrix are

$$l_{ij} = -\sum_{p=j}^{i-1} l_{ip} l_{pj}. \quad j=1, 2, 3, \cdots, n-1; \; i=j+1, j+2, j+3, \cdots, n \quad [1.3]$$

This recurrence relation has the unique feature of demanding no separate storage for its elements and it moreover, converts L_l matrix to L_l^{-1} matrix in the same locations as those of L_l, with maximum possible automation. All the above elements l_{ij} are but the elements of L_l^{-1} matrix; the elements of L_l matrix get destroyed. In all the subsequent presentation of the inverse triangular matrix, the same property and nature can be observed except at those places where different letters have been used.

The elements of the column vector c are

$$c_j = \sum_{p=1}^{j} l_{jp} b_p \quad j = 1, 2, 3, \cdots, n-1, n \qquad [1.4]$$

The roots are

$$x_i = c_i / r_{ii} \quad i = n \qquad [1.5]$$

$$x_i = (c_i - \sum_{p=i+1}^{n} r_{ip} x_p)/r_{ii}, \quad i = n-1, n-2, \cdots, 1$$

This evaluation of the roots are evidently carried out by back substitution method.

$$\text{If } A = R_l L_l, \text{ then} \qquad [2]$$

$$\left.\begin{array}{l} r_{ij} = a_{ij} - \sum_{p=j+1}^{n} r_{ip} l_{pj}, \quad l_{ii} = 1 \text{ for all } i \\ i \leq j \\ l_{ij} = (a_{ij} - \sum_{p=i+1}^{n} r_{ip} l_{pj})/r_{ii} \\ i > j \end{array}\right\} \qquad [2.1]$$

$$i = n, n-1, n-2, \cdots, 2, 1$$

$$j = n, n-1, n-2, \cdots, 2, 1$$

$$\text{Det } A = \prod_{i=1}^{n} r_{ii} \qquad [2.2]$$

The equations $Ax=b$ are written as

$$R_l L_l x=b, \text{ i.e., } L_l x=R^{-1} b=c \text{ (say)}$$

The elements of the R_l^{-1} matrix are

$$\left.\begin{array}{ll} q_{ij}=-[\sum_{p=i+1}^{j} r_{ip} q_{pj}]/r_{ii} & i=n, n-1, \cdots, 2,1 \\ i<j & j=i, i+1, \cdots, n \\ q_{ij}=\dfrac{1}{r_{ij}} & \\ i=j & \end{array}\right\} \quad [2.3]$$

It can be seen here that extra $n(n+1)/2$ locations have been used for the R_l^{-1} matrix. Since the diagonal elements of R_l matrix are not unity, exactly similar recurrence relations as in [1.3] are not feasible. It can be noted that q_{ij}'s are to be taken zeros for $i>j$.

The elements of c are, therefore,

$$c_j=\sum_{p=j}^{n} q_{jp} b_p \quad j=1, 2, 3, \cdots, n \quad [2.4]$$

and the roots are

$$x_i=c_i-\sum_{p=1}^{i-1} l_{ip} x_p \quad i=1, 2, 3, \cdots, n \quad [2.5]$$

If $A=L_l R_r$, then [3]

$$i=1, 2, 3, \cdots, n-1, n \quad [3.1]$$

$$l_{ij}=(a_{i,\,n-j+1}-\sum_{p=1}^{j-1} l_{ip} r'_{p,\,n-j+1})/r'_{j,\,n-j+1}, \quad l_{ii}=1 \text{ for all } i$$

$$j=1, 2, 3, \cdots, i-1$$

$$r'_{ij}=a_{ij}-\sum_{p=1}^{i-1} l_{ip} r'_{pj}, \quad j=1, 2, 3, \cdots, (n-i), (n-i+1).$$

We first find for $i=1$, $r'_{11}, r'_{12}, r'_{13}, \cdots, r'_{1n}$; then for $i=2$, we find l_{21}; $r'_{21}, r'_{22}, r'_{23}, \cdots, r'_{2,\,n-1}$. Next for $i=3$, we find l_{31}, l_{32}; $r'_{31}, r'_{32}, r'_{33}, \cdots, r'_{3,n-2}$ and so on. Lastly we find for $i=n$, $l_{n1}, l_{n2}, l_{n3}, \cdots, l_{n,\,n-1}$; r'_{n1}.

$$\text{Det } A=(-1)^{\text{Int } n/2} \prod_{i=1}^{n} r'_{i,\,n-i+1} \quad [3.2]$$

where Int $n/2$ is the integral part of $n/2$.

Since $L_l R_r x = b$ i.e. $R_r x = L_l^{-1} b = c$ (say) we have the elements of L_l^{-1} matrix as

$$l_{ij} = -\sum_{p=j}^{i-1} l_{ip}\, l_{pj} \qquad \begin{array}{l} j=1,\ 2,\ 3,\ \cdots,\ n-1 \\ i=j+1,\ j+2,\ \cdots,\ n \end{array} \qquad [3.3]$$

The elements of the column vector c, then, are

$$c_j = \sum_{p=1}^{j} l_{jp}\, b_p \qquad j=1,\ 2,\ 3,\ \cdots\ n \qquad [3.4]$$

and the roots are

$$x_i = [c_{n-i+1} - \sum_{p=1}^{i-1} r'_{n-i+1,\ p}\, x_p]/r'_{n-i+1,\ i} \qquad [3.5]$$

$$i=1,\ 2,\ 3,\ \cdots,\ n$$

When $A = L_r L_l$, we have [4]

$$q=1,\ 2,\ 3,\ \cdots,\ n \qquad [4.1]$$

$$l'_{i,j} = a_{ij} - \sum_{p=j+1}^{n} l'_{ip}\, l_{pj},\ l_{ii}=1 \text{ for all } i$$

$$j=n-q+1\ ;\ i=n-j+1,\ n-j+2,\ n-j+3,\ \cdots,\ n$$

$$l_{ji} = (a_{n-j+1,\ i} - \sum_{p=j+1}^{n} l'_{n-j+1,\ p}\, l_{pi})/l'_{n-j+1,\ j}$$

$$j=n-q+1\ ;\ i=1,\ 2,\ 3,\ \cdots,\ j-1.$$

We first find for $q=1$, l'_{1n}, l'_{2n}, l'_{3n}, $\cdots$, l'_{nn}; then l_{n1}, l_{n2}, l_{n3}, $\cdots$ $l_{n,\ n-1}$. Next for $q=2$, we find $l'_{2,\ n-1}$; $l'_{3,\ n-1}$; $l'_{4,\ n-1}$; $\cdots$ $l'_{n,\ n-1}$; then $l_{n-1,\ 1}$; $l_{n-1,2}$; $l_{n-1,3}$; $\cdots$; $l_{n-1,\ n-2}$ and so on. Lastly for $q=n$, we find l'_{n1}.

$$\text{Det } A = (-1)^{\text{Int } n/2} \prod_{i=1}^{n} l'_{i,\ n-i+1} \qquad [4.2]$$

The equations $Ax=b$ can be written as $L_r L_l x = b$ *i.e.* $L_l x = L_r^{-1} b = c$ (say) The elements of L_r^{-1} matrix are

$$q_{ij} = \frac{1}{l'_{n-i+1,\ i}} \qquad \text{when } i+j > n$$

otherwise

$$q_{ij} = -(\sum_{p=i+1}^{n-j+1} l'_{n-i+1,\ p}\, q_{p,j})/l'_{n-i+1,\ i}$$

$$i=n,\ n-1,\ n-2,\ \cdots,\ 1\ ;\ j=1,\ 2,\ 3,\ \cdots,\ n-i+1$$

We first take $i=n$, $j=1$ and find $q_{n,1}$ from the first relation. Next we take $i=n-1$, $j=1$ and find $q_{n-1,1}$ from the second relation. Then we take $i=n-1$, $j=2$ and calculate $q_{n-1,2}$ from the first and so on,

The elements of the column vector c are

$$c_j=\sum_{p=1}^{n-j+1} q_{jp}\, b_p\,,\quad j=1,\ 2,\ 3,\ \cdots,\ n \qquad [4.4]$$

Roots are then given by

$$x_i=c_i-\sum_{p=1}^{i-1} l_{ip}\, x_p,\quad i=1,\ 2,\ 3,\ \cdots,\ n \qquad [4.5]$$

If $A=R_r R_l$, tren [5]

$j=1,\ 2,\ 3,\ \cdots,\ n$ [5.1]

$$r_{ij}=(a_{n-i+1,\,j}-\sum_{p=1}^{i-1} r'_{n-i+1,\,p}\, r_{pj})/r'_{n-i+1,\,i}$$

$i=1,\ 2,\ 3,\ \cdots\ j-1$

$$r'_{ij}=a_{ij}-\sum_{p=1}^{j-1} r'_{ip}\, r_{pj}$$

$i=n-j+1,\ n-j,\ n-j-1,\ \cdots,\ 1$

From these recurrence formulae we find first (for $j=1$) r'_{n1}; $r'_{n-1,1}$; $r'_{n-2,1}\cdots$, r'_{11} in a sequence; next we fin' (for $j=2$) r_{12}; followed by $r'_{n-1,2}$; $r_{n-2,2}$; $r'_{n-3,2}$; $\cdots$; r'_{12}. Next (for $j=3$), we find r_{13}; r_{23}; followed by $r'_{n-2,3}$; $r'_{n-3,3}$; $r'_{n-4,3}$; $\cdots$, r'_{13} and so on. Proceeding in this way we find at the end (for $j=n$) r_{1n}; r_{2n}; r_{3n}; $\cdots$; $r_{n-1,n}$; followed by r'_{1n}.

$$\text{Det } A=(-1)^{\text{Int } n/2}\ \prod_{i=1}^{n} r'_{i,\,n-i+1} \qquad [5.2]$$

The equations $Ax=b$ can be written as $R_r R_l\, x=b$ *i.e.* $R_l\, x=R_r^{-1}\, b=c$ (say) The elements of R_r^{-1} matrix are

$$q_{ij}=1/r'_{ji},\quad i=n,\ n-1,\ n-2,\ \cdots,\ 1;\ j=n-i+1$$

$$q_{ij}=-\,[\sum_{p=n-j+1}^{i-1} r'_{n-i+1,p}\, q_{pj}]/r'_{n-i+1,\,i} \qquad \begin{matrix} j=2,\ 3,\ 4,\ \cdots,\ n \\ i=n,\ n-1,\ n-2,\ \cdots,\ j \end{matrix} \qquad [5\ 3]$$

From the first recurrence relation we find q_{n1}; $q_{n-1,\,2}$; $q_{n-2,\,3}$; $\cdots$; q_{1n} in a sequence, then we move on to second relation and find q_{n2}; $q_{n-1,\,3}$; $q_{n-2,\,4}$; $\cdots$; q_{2n} sequentially; followed by q_{n3}; $q_{n-1,\,4}$; $q_{n-2,\,5}$; $\cdots$, q_{3n} and so on. Proceeding in this way we find at the end q_{nn}.

The elements of the column vector c are

$$c_j = \sum_{p=n-j+1}^{n} q_{jp}\, b_p, \quad j=1,\ 2,\ 3,\ \cdots,\ n \qquad [5.4]$$

and the roots are

$$x_i = \Big[c_i - \sum_{p=i+1}^{n} r_{ip}\, x_p\Big]/r_{ii} \qquad [5.5]$$

$$i=n,\ n-1,\ n-2,\ \cdots,\ 1$$

4. *Example*

$$A = \begin{pmatrix} 2 & 4 & 3 & 2 \\ 3 & 6 & 5 & 2 \\ 2 & 5 & 2 & -3 \\ 4 & 5 & 14 & 14 \end{pmatrix}$$

Well-conditioned with respect to inverse and having a determinant value of -5, the above matrix, however, fails to oblige $L_l R_l$ decomposition, since in the course of decomposition r_{22} turns out to be zero, and as a result, l_{32} and l_{42} cannot be determined.

A can be written as, when $A = R_l L_l$,

$$A = \begin{pmatrix} -\frac{1}{23} & \frac{29}{14} & 1 & 2 \\ & \frac{23}{14} & 3 & 2 \\ & & 5 & -3 \\ & & & 14 \end{pmatrix} \begin{pmatrix} 1 & & & \\ \frac{10}{23} & 1 & & \\ \frac{4}{7} & \frac{17}{14} & 1 & \\ \frac{4}{14} & \frac{5}{14} & 1 & 1 \end{pmatrix} \qquad [2.1\text{ ex}]$$

When $A = L_l R_r$, the elements of L_l and R_r are calculated rowwise as follows :—

$$r'_{11}=2,\quad r'_{12}=4,\quad r'_{13}=3,\quad r'_{14}=2 \qquad [3.1\text{ ex}]$$

$$l_{21}=a_{24}/r'_{14}=2/2=1\ ;\ r'_{21}=a_{21}-l_{21}\ r'_{11}=1\ ;\ r'_{22}=a_{22}-l_{21}\ r'_{12}=2$$

$$r'_{23}=a_{23}-l_{21}\ r'_{13}=5-1.3=2$$

$$l_{31}=a_{34}/r'_{24}=-3/2\ ;\ l_{32}=(a_{33}-l_{31}\ r'_{13})/r'_{23}=13/4$$

$$r'_{31}=a_{31}-l_{31}\ r'_{11}-l_{32}\ r'_{21}=2+\tfrac{3}{2}\cdot 2-\tfrac{13}{4}\cdot 1=\tfrac{7}{4}$$

$$r'_{32}=a_{32}-l_{31}\, r'_{12}-l_{32}\, r'_{22}=5+\tfrac{3}{2}\cdot 4-\tfrac{13}{4}\cdot 2=\tfrac{9}{2}$$
$$l_{41}=a_{44}/r'_{14}=14/2=7\ ;\ l_{42}=(a_{43}-l_{41}\, r'_{13})/r'_{23}=-\tfrac{7}{2}$$
$$l_{43}=(a_{42}-l_{41}\, r'_{12}-l_{42}\, r'_{22})/r'_{32}=(5-7\cdot 4+\tfrac{7}{2}\cdot 2)/\tfrac{9}{2}=-\tfrac{32}{9}$$
$$r'_{41}=a_{41}-l_{41}\, r'_{11}-l_{42}\, r'_{21}-l_{43}\, r'_{31}=4-7.2+\tfrac{7}{2}\cdot 1+\tfrac{32}{9}\cdot\tfrac{7}{4}=-\tfrac{5}{18}$$

Therefore

$$L_l\, R_r=\begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ -\tfrac{3}{2} & 13/4 & 1 & \\ 7 & -7/2 & -32/9 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & 3 & 2 \\ 1 & 2 & 2 & \\ 7/4 & 9/2 & & \\ -5/18 & & & \end{pmatrix}$$

$$\text{Det } A=r'_{14}\, r'_{23}\, r'_{32}\, r'_{41}=2.2\cdot\tfrac{9}{2}\,(-)\,\tfrac{5}{18}=-5 \qquad [3.2\text{ ex}]$$

The elements of L_l^{-1} matrix are

$$l_{21}=-l_{21}\, l_{11}=-1\ ;\ l_{31}=-l_{31}\, l_{11}-l_{32}\, l_{21}=+\tfrac{3}{2}+\tfrac{13}{4}\cdot 1=\tfrac{19}{4}$$
$$l_{41}=-l_{41}\, l_{11}-l_{42}\, l_{21}-l_{43}\, l_{31}=-7-\tfrac{7}{2}\cdot 1+\tfrac{32}{9}\times\tfrac{19}{4}=\tfrac{115}{18}$$
$$l_{32}=-l_{32}\, l_{22}=-\tfrac{13}{4}\ ;\ l_{42}=-l_{42}\, l_{22}-l_{43}\, l_{32}=-\tfrac{145}{18}$$
$$l_{43}=-l_{43}\, l_{33}=+\tfrac{32}{9}$$

Therefore L_l^{-1} matrix is

$$\begin{pmatrix} 1 & & & \\ -1 & & & \\ \tfrac{19}{4} & -\tfrac{13}{4} & 1 & \\ \tfrac{115}{18} & -\tfrac{145}{18} & \tfrac{32}{9} & 1 \end{pmatrix}$$

When $A=L_r\, L_l$ and $l_{ii}=1$ for all i, we can rewrite A as [4.2 ex]

$$\begin{pmatrix} & & & 2 \\ & & 2 & 2 \\ & \tfrac{9}{2} & \tfrac{13}{2} & -3 \\ -\tfrac{5}{18} & -16 & -7 & 14 \end{pmatrix}\begin{pmatrix} 1 & & & \\ \tfrac{7}{18} & 1 & & \\ \tfrac{1}{2} & 1 & 1 & \\ 1 & 2 & \tfrac{3}{2} & 1 \end{pmatrix}$$

and $$L_r^{-1}=\begin{pmatrix} -23 & 29 & -\tfrac{64}{5} & -\tfrac{18}{5} \\ \tfrac{19}{18} & -\tfrac{13}{18} & \tfrac{2}{9} & \\ -\tfrac{1}{2} & \tfrac{1}{2} & & \\ \tfrac{1}{2} & & & \end{pmatrix}$$

5. CONCLUSION

The motivation of such triangular decomposition lies in the fact that, unlike a square matrix, the inverse of a triangular matrix can always be determined very easily. Furthermore computational recurrence relations can easily be formulated for triangular matrices from the matrix operation rules, using simple expressions of double subscripts of the matrix elements and of the order of the matrix ; these computational relations actually mechanize the whole problem for computational purposes, irrespective of programming languages used. It can be seen that the last three decompositions *i.e.* $L_l R_r$, $L_r R_l$ and $R_r L_l$, in their reverse multiplication, give rise to a triangular matrix of the right diagonal, and not of the left diagonal. As a result, the possiblity of direct computation of eigen values which should be the left diagonal elements of a diagonal or a triangular matrix is remote. Furthermore it may be noted that L_r^{-1} matrix is an upper triangular matrix of right diagonal and R_r^{-1} matrix is a lower triangular matrix of the right diagonal. This is unlike the L_l^{-1} and R_l^{-1} matrices, in which case, L_l^{-1} matrix is a lower triangular matrix of the left diagonal and R_l^{-1} matrix is an upper triangular matrix of left diagonal.

It can be seen that all these triangular Algorithms along with those of Gaussian type produce the identical results[2, 7] within the limitations of computing precission and rounding errors of arithmetic calculations, whenever all of them succeed well for an arbitrary matrix. The last four Algorithms ($R_l L_l$, $L_l R_r$, $L_r L_l$ and $R_r R_l$), however, usually succeed in those cases also where $L_l R_l$ as well as Gaussian Algorithms fail. For a matrix having one or more leading submatrices singular, Gaussian Algorithms along with that of $L_l R_l$ fail unless we use row intercharging technique[5]. Thus the importance of using triangular matrices of right diagonal along with that of left one need not be stressed.

It can be, furthermore, seen that the last four Algorithms involve divisions in their computational recurrence relations. It is, therefore, a a problem to find the class of matrices where these denominators become zero. In this class of matrices, the $L_l R_l$ as also Gaussian Algorithms probably will succeed. For example, the $R_l L_l$ algorithm though suceeds when some leading principal minors vanish, fails when some trailing minors vanish. The $L_l R_l$ algorithm is, on the other hand, immune to such some vanishing trailing minors.

Lastly, the $R_l L_l$ Algorithm is the one which can be used not only for finding the solution vector $\vec{x}$ of $A\vec{x} = \vec{b}$, but also for eigenvalues of an arbitrary matrix having one or more leading submatrices singular or even the original matrix singular[7].

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