ANALYTICAL SOLUTION FOR A SHRINK-FIT PROBLEM

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Abstract

The problem of stress distribution resulting from the Shrink–fit applied on an infinitely long cylinder has been analysed. Using Fourier integral method, the problem is reduced to the solution of a pair of dual integral equations, whieh are reduced to series solution of a Fredholm equation finally. Bousinesq-Neuber three function theory has been employed for stress-analysis.

1. Introduction

Many boundary value problems connected with a solid cylinder have heen solved using classical techniques. Solutions for the infinite solid elastic cylinder with the given tractions of various types have been given by Filon[1], Foeppl[2] and Freudenthal[3]. Tranter and Craggs[4] have given the solution for an infinite solid cylinder subjected to a normal stress over half of its length which is a fundamental problem in elasticity. In all the above problems surface tractions are prescribed. Sparenberg[5] has recently solved a different kind of problem – a shrink-fit or indentation problem in which an infinite cylinder is subjected to an unit displacement over half of its length and traction-free over the remainder. It is the purpose of the present analysis to present a solution for an infinite solid shaft, the surface of which is traction free except for a band which is subjected to a prescribed radial displacement.

2. The Problem

The coordinate system employed is shown in Fig. 1. According to the Boussines-Neuber three function theorem [6.7], the displacements will satisfy the equalibrium conditions of elasticity theory if they are represented by the potential fields ϕ_1, ϕ_2 and ϕ_3 as follows:

$$2Gu = -\partial F/\partial z$$

$$2Gv = -(1/r)(\partial F/\partial \theta) - 4(1-\mu)(\phi_2 \sin\theta - \phi_3 \cos\theta)$$

$$2Gw = -\partial F/\partial r + 4(1-\mu)(\phi_2 \cos\theta + \phi_3 \sin\theta) \qquad [1]$$

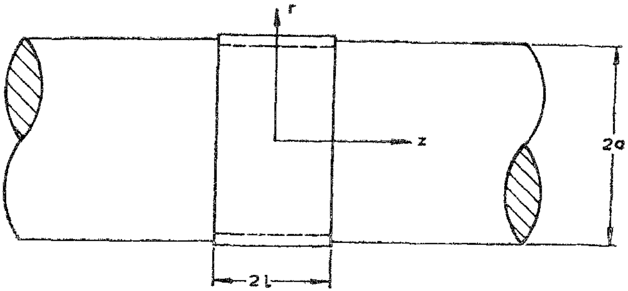

FIG. 1
Geometry and co-ordinate axes

$$F = \phi_1 + r\,(\phi_2 \cos\theta + \phi_3 \sin\theta)$$

where $\nabla^2 \phi_1 = \nabla^2 \phi_2 = \nabla^2 \phi_3 = 0$

(u, v, w) ; Displacements in r, θ, z directions

μ, G : Poisson's Ratio & Shear modulus

The stress field associated with Eqs. 1 & 2 is of the form

$$\sigma_r = \mu\,\nabla^2 F - (\partial^2 F/\partial r^2) + 4\,(1-\mu)\,[(\partial\,\phi_2/\partial r)\cos\theta + (\partial\,\phi_3/\partial r)\sin\theta]$$
$$\sigma_\theta = \mu\,\nabla^2 F - (1/r)\,(\partial F/\partial r) + [4\,(1-\mu)/r]\,(\phi_2 \cos\theta + \phi_3 \sin\theta)$$
$$\sigma_z = \mu\,\nabla^2 F - (\partial^2 F/\partial z^2)$$
$$\tau_{rz} = -(\partial^2 F/\partial r\,\partial z) + 2\,(1-\mu)\,[(\partial\phi_2/\partial z)\cos\theta + (\partial\,\phi_3/\partial z)\sin\theta) \qquad [3]$$

On the the curved boundary, $r = a$, throughout the length, it is assumed that

$$\tau_{rz} = 0 \qquad [4]$$

$$w = \delta,\ a \text{ constants for } |z| \leq l \text{ and } \sigma_r = 0 \text{ for } |z| > l \qquad [5]$$

i.e., the curved boundary has a shrink-fit type axisymmetric radial displacement for a length $2l$.

3. THE SOLUTION

The three-functions ϕ_1, ϕ_2, ϕ_3 are considered in the following form:

$$\phi_1 = \int_0^\infty (1/\alpha^2)\,A(\alpha)\,I_0(\alpha r)\cos\alpha z\,d\alpha$$

$$\phi_2 = \int_0^\infty (1/\alpha)\, B(\alpha)\, I_1(\alpha r) \cos \alpha z \cos \theta \, d\alpha$$

$$\phi_3 = \int_0^\infty (1/\alpha)\, B(\alpha)\, I_1(\alpha r) \cos \alpha z \sin \theta \, d\alpha \qquad [6]$$

in which $A(\alpha)$, $B(\alpha)$ are functions of α, and $I_0(\alpha r)$, $I_1(\alpha r)$ are Modified Bessel functions of first kind and order zero and one respectively. The expressions for stresses and displacements, using Eqs. [6], [1] and [3] are :

$$\sigma_r = \int_0^\infty [A(\alpha)\{-I_0(\alpha r) + I_1(\alpha r)/\alpha r\} - B(\alpha)\{(4\overline{1-\mu}/\alpha r + \alpha r)\, I_1(\alpha r) - (1+2\overline{1-\mu})\, I_0(\alpha r)\}] \cos \alpha z \, d\alpha$$

$$\tau_{rz} = \int_0^\infty [A(\alpha)\, I_1(\alpha r) - B(\alpha)\, \{2(1-\mu)\, I_1(\alpha r) - \alpha r\, I_0(\alpha r)\}] \sin \alpha x \, d\alpha$$

$$\sigma_z = \int_0^\infty [A(\alpha)\, I_0(\alpha r) + B(\alpha)\, \{\alpha r\, I_1(\alpha r) + 2\mu\, I_0(\alpha r)\}] \cos \alpha z \, d\alpha$$

$$\sigma_\theta = \int_0^\infty [A(\alpha) \cdot \{-I_1(\alpha r)/\alpha r\} + B(\alpha)\, \{4(1-\mu)/\alpha r\, I_1(\alpha r) + (2\mu - 1)\, I_0(\alpha r)\}] \cos \alpha z \, d\alpha \qquad [7]$$

$$2Gu = \int_0^\infty (1/\alpha)\, [A(\alpha)\, I_0(\alpha r) + B(\alpha)\, \alpha r\, I_1(\alpha r)] \sin \alpha z \, d\alpha \; ;$$

$$2Gw = \int_0^\infty (1/\alpha)\, [-A(\alpha)\, I_1(\alpha r) + B(\alpha)\, \{4(1-\mu)\, I_1(\alpha r) - \alpha r\, I_0(\alpha r)\}] \cos \alpha z \, d\alpha \qquad [8]$$

Substituting the boundary condition Eq. [4],

$$A(\alpha) = B(\alpha)\, [2(1-\mu) - \alpha a \cdot I_0(\alpha a)/I_1(\alpha a)] \qquad [9]$$

Putting in tne boundary condition that

$$\sigma_r\big|_{r=a} = 0 \quad \text{for} \quad |z| > l \text{ one gets}$$

$$\int_0^8 \xi(\alpha) \cos \alpha z \, d\alpha = 0 \quad \text{for} \quad |z| > l \qquad [10]$$

where

$$\xi(\alpha) = B(\alpha)\, [\{2(1-\mu) - \alpha a\, I_0(\alpha a)/I_1(\alpha a)\}\, \{-I_0(\alpha a) + I_1(\alpha a)/\alpha a\} - \{(\overline{4-4}\mu/\alpha a + \alpha a)\, I_1(\alpha a) - (3-2\mu)\, I_0(\alpha a)\}]$$

The remaining condition to be satisfied is

$$w]_{r=a} = \delta \quad \text{for} \quad |z| \leq l$$

Then upon substitution, one gets,

$$\int_0^\infty \lambda(\alpha)\,\xi(\alpha)\cos\alpha z\,d\alpha=\delta \quad \text{for} \quad |z|\leq l \tag{11}$$

where

$$\lambda(\alpha)=(1/2\,G\alpha)\,[-I_1(\alpha a)\,\{2(1-\mu)-\alpha a\,J_0(\alpha a)/I_1(\alpha a)\}$$
$$+\{4(1-\mu)\,I_1(\alpha a)-\alpha a\,I_0(\alpha a)\}]-[\{2(1-\mu)$$
$$-\alpha a\,I_0(\alpha a)/I_1(\alpha a)\}\{-I_0(\alpha a)+I_1(\alpha a)/\alpha a\}-\{\langle(4-4\mu)/\alpha a$$
$$+\alpha a\rangle\,I_1(\alpha a)-(3-2\mu)\,I_0(\alpha a)\}]$$

Hence the problem reduces to the solution of the pair of dual integral Eqs. [10] & [11]

3.1 Solution of the Dual Integral Equations

For simplicity a and l are taken to be unities. In Eqs. 10 and 11 the unknown is $\xi(\alpha)$ or $A(\alpha)$. Since the surface traction $\sigma_r|_{r=1}$ and not $\xi(\alpha)$ is of direct interest, the former is expanded in series of Legendre Polynomials[8] as follows[9] :

$$\sigma_r|_{r=1}=\sum_{n=0}^{\infty} a_n\,P_n(1-2z^2) \quad \text{for} \quad |z|\leq 1$$
$$=0 \qquad\qquad \text{for} \quad |z|>1 \tag{12}$$

The problem is considered solved when the constants a_n are determined. Eq. 12 identically satisfies the first integral equation 10. Then it follows[10] that

$$\xi(\alpha)=\sum_{n=0}^{\infty} a_n\,(-1)^n\,J_{(n+\frac{1}{2})}(\alpha/2)\,J_{-(n+\frac{1}{2})}(\alpha/2) \tag{13}$$

The second integral equation then becomes

$$\delta=\int_0^\infty \lambda(\alpha)\sum_{n=0}^{\infty} a_n(-1)^n\,J_{(n+\frac{1}{2})}(\alpha/2)\,J_{-(n+\frac{1}{2})}(\alpha/2)\cos\alpha z\,d\alpha \quad \text{for } |z|\leq 1 \tag{14}$$

Putting

$$\lambda_n(z)=\int_0^\infty \lambda(\alpha)\,(-1)^n\,J_{(n+\frac{1}{2})}(\alpha/2)\,J_{-(n+\frac{1}{2})}(\alpha/2)\cos\alpha z\,d\alpha$$

Eq. 14 becomes

$$\sum_{n=0}^{\infty} a_n\,\lambda_n(z)=\delta \tag{15}$$

The dual integral eqs, 10 and 11 have now been reduced to Eq. 15 in which the a's are the unknowns. Eq. 15 is identical in form with the series solution of a Fredholm integral equation of the first kind and Schmidt method can be adopted for its solution.

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