SOME STUDIES ON DYNAMIC VIBRATION ABSORBERS

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ABSTRACT

One of the main goals of study of engineering vibrations is to suppress or eliminate unwanted vibrations. Vibrations in mechanical systems can be appreciably quenched by the provision of Dynamic Vibration Absorbers (DVA). A DVA differs from a damper in that the motion from the main system is transferred to an auxiliary system, the total energy being conserved. On the other hand in the case of a damper a part of the vibrational energy is abstracted and dissipated as heat or other forms of energy thereby lowering the total energy of the system.

DVA are necessary to limit excessive vibration of a machine element subjected to steady vibrating loads. A perfectly tuned DVA can be designed for a single degree of freedom system subjected to mono-frequency excitation of a harmonic nature. But when the frequency varies the design of DVA becomes complicated owing to the introduction of damping, (introduced to extend the frequency range of of the DVA). In some cases a nonlinear spring coupling aids in extending the range of usefulness of the DVA.

When a forcing function of an impulsive type acts on a system, vibration absorbers can be designed so that if failure occurs it will be that of the absorber (because it is made to bear the brunt of shock) and not that of the main system. (The main system is protected from the ravages of the shock). The DVA functions as a protection, against shock.

In this paper a study is undertaken on the design of DVA for various types of excitations, steady, transcient in nature, and it will be the purpose of this investigation to study how effiecient these DVA are, in quenching unwanted oscillations.

INTRODUCTION

The possibility of providing dynamic vibration absorbers for mechanical systems that can be idealised by undamped single degree of freedom systems is investigated, when the system is subjected to different types of excitations.

In all the cases considered the motion from the main system is transferred to some auxiliary system so that the main system is protected from the harmful effects of vibration.

In section (1) it is shown that the main system could be completely isolated from the vibrations, by providing a dynamic vibration absorber, only if the excitation is sinusoidal. In section (2) the above problem is extended to the case when the system is resting on a flexible mounting and subjected to ground excitation. Section (3) may be considered as an extension of the section (1) wherein two absorber units instead of one are used to eliminate the vibrations of the main mass. This scheme will be particularly useful if the size of single absorber unit becomes impractical. The case of dynamic vibration absorber for a single degree of freedom system subjected to harmonic excitations of two different frequencies is dealt with in section (4). Section (5) considers the case of the same system subjected to triple frequency excitation. Extension of this concept to multiple frequency excitation is considered in section (6) where the system is assumed as being subjected to a train of pulses.

In all the above cases when more than one absorber has been used, the absorber units have been assumed to be attached to the main system in series. The same system of absorbers could also be attached in parallel as indicated in section (7) and (8).

Section (9) deals with the effect of providing dynamic absorber to a single degree of freedom system subjected to an impulsive input. It is shown that a reduction in the amplitude of the main system by about 20% is possible with an absorber, whose mass is of the same order as the main mass. Section (10) indicates a method to suppress the main mass vibration when subjected to transient ground motion. Section (11) deals with the provision of dynamic vibration absorbers to continuous systems. Two distinct cases are examined under this, *viz.*, (*a*) Self-excited oscillations of a cutting tool, idealised by a cantilever (*b*) Aerodynamic oscillations of a transmission line span idealised by a beam with hinged ends.

Section (12) deals with the provision of a nonlinear dynamic vibration absorber for a single degree of freedom system subjected to step function transient.

1. *Possibility of providing a dynamic vibration absorber for suppressing the vibrations of an undamped single degree of freedom system subjected to a forcing function $F(t)$:*

Consider the system shown schematically in Fig. (1.1). The equations of motion are

$$m_1 (d^2 x_1/dt^2) + k_1 x_1 + k_2 (x_1 - x_2) = F(t) \quad [1.1]$$

$$m_2 (d^2 x_2/dt^2) + k_2 x_2 - k_2 x_1 = 0 \quad [1.2]$$

If the vibrations of the mass m_1 are completely eliminated

then $x_1 = 0, \quad d x_1/dt = 0, \quad d^2 x_1/dt^2 = 0$

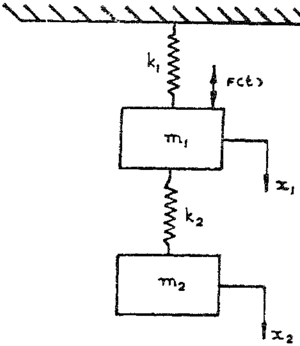

FIG. 1.1

Let $$\frac{k_1}{m_1} = p_{11}^2,\ \frac{k_2}{m_1} = p_{21}^2,\ \frac{k_2}{m_2} = p_{22}^2;\ \frac{F}{m} = F_1$$

∴ From (1.1) and (1.2)

$$F_1''(t) + p_{22}^2 F_1(t) = 0 \qquad [1.3]$$

When this condition is satisfied by the excitation, a dynamic absorber can be provided. It can be immediately seen from the solution of the differential equation [1.3] that the forcing function will have to be sinusoidal. The tuning condition will be given by

$$p_{22}^2 = k_2/m_2$$

where p_{22} is the frequency of the exciting force,

2. *Vibration absorber for a system on a flexible mounting and subjected to base excitation*: $X_0 = B \sin \omega t$:

In the fig. shown (Fig. 2.1) (k_2, m_2) constitutes, the main system resting on a flexible mounting. The absorber has a mass m_3 and stiffness k_3.

The equations of motion are

$$m_1 (d^2 x_1/dt^2) + k_1 (x_1 - x_0) + k_2 (x_1 - x_0) = 0 \qquad [2.1]$$

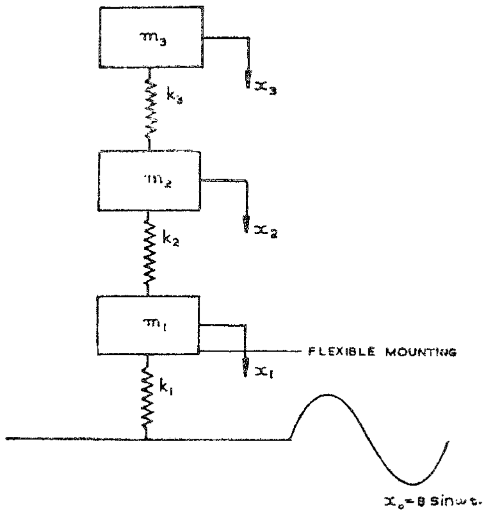

FIG. 2.1

$$m_2 (d^2 x_2/dt^2) + k_2 (x_2 - x_1) + k_3 (x_2 - x_3) = 0 \qquad [2.2]$$

$$m_3 (d^2 x_3/dt^2) + k_3 (x_3 - x_2) = 0 \qquad [2.3]$$

Putting $x_2 = 0$, $(d^2 x_2/dt^2) = 0$,

$$p_{12}^2 = \frac{k_1}{m_1},\ p_{21}^2 = \frac{k_2}{m_1},\ p_{22}^2 = \frac{k_2}{m_2},\ p_{32}^2 = \frac{k_3}{m_2},\ p_{33}^2 = \frac{k_3}{m_3}$$

from [2.1]

$$x_1 = \frac{p_{11}^2 B \sin \omega t}{-\omega^2 + p_{11}^2 + p_{21}^2} \qquad [2.4]$$

From [2.2 and 2.4]

$$x_3 = \frac{-p_{22}^2}{p_{32}^2} \left(\frac{p_{11}^2 B \sin \omega t}{p_{11}^2 + p_{21}^2 - \omega^2} \right) \qquad [2.5]$$

Therefore, $\omega^2 = p_{33}^2 = k_3/m_3$ is the tuning condition.

3. *Use of two dynamic vibration absorbers in series with a single degree of freedom system subjected to harmonic excitation* :

Fig. (3.1) shows the arrangement for this scheme. (The subscript 1 refers to the main system while the subscripts 2 and 3 refer to the two absorber systems). The equation of motion are

$$m_1 (d^2 x_1/dt^2) + (k_1 + k_2) x_1 - k_2 x_2 = B \sin \omega t \qquad [3.1]$$

$$m_2 (d^2 x_2/dt^2) + (k_2 + k_3) x_2 - k_2 x_1 - k_3 x_3 = 0 \qquad [3.2]$$

$$m_3 (d^2 x_3/dt^2) + k_3 x_3 - k_3 x_2 = 0 \qquad [3.3]$$

Putting $x_1 = 0$, $(d^2 x_1/dt^2) = 0$,

$$\frac{k_1}{m_1} = p_{11}^2,\ \frac{k_2}{m_1} = p_{21}^2,\ \frac{k_2}{m_2} = p_{22}^2,\ \frac{k_3}{m_2} = p_{32}^2,\ \frac{k_3}{m_3} = p_{33}^2$$

from [3.1]

$$x_2 = -(B/m_2 p_{21}^2) \sin \omega t \qquad [3.4]$$

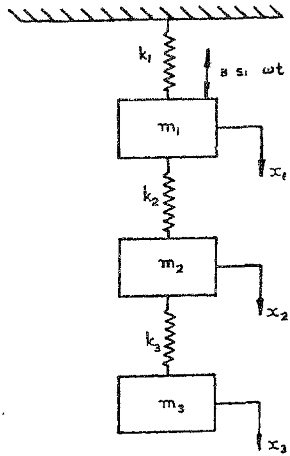

FIG. 3.1

from [3.2] and [3.4]

$$x_3 = (B/m_1 p_{21}^2 p_{32}^2)\,(\omega^2 - p_{22}^2 - p_{32}^2)\sin \omega t \qquad [3.5]$$

Therefore from [3.4] and [3.5],

$$\omega^4 - \omega^2\,(p_{22}^2 + p_{32}^2 + p_{33}^2) + p_{22}^2\, p_{33}^2 \qquad [3.6]$$

Further letting

$$\frac{k_2}{k_3} = K,\ \frac{m_2}{m_3} = M,\ \ \frac{p_{22}}{\omega} = \bar{p},\ \text{from [3.6]},$$

$$M = (K/\bar{p}^2) + 1/\bar{p}^2 - 1 \qquad [3.7]$$

This straight line relationship is shown plotted in Fig. 3.2.

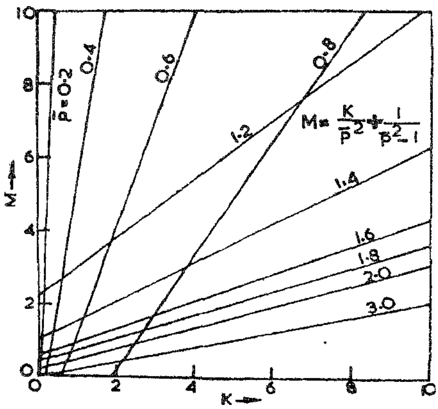

FIG. 3.2

Absorber parameter curves for single degree of freedom system with single frequency excitation.

4. *Dynamic vibration absorber for a single degree of freedom system subjected to bi-frequency excitation,*

Fig. 4.1 shows the idealised system subjected to an excitation $F(t) = b_1 \sin \omega t + b_3 \sin \omega t$. Two absorber units are attached in series with the system. The equations of motion are

$$m_1 (d^2 x_1/dt^2) + k_1 x_1 + k_2 (x_1 - x_2) = b_1 \sin \omega t + b_3 \sin 3\,\omega t \qquad [4.1]$$

$$m_2 (d^2 x_2/dt^2) + (k_2 + k_3)\, x_2 - k_2 x_1 - k_3 x_3 = 0 \qquad [4.2]$$

$$m_3 (d^2 x_3/dt^2) + k_3 x_3 - k_2 x_2 = 0 \qquad [4.3]$$

For complete isolation of m_1, $x_1 = 0$, $d^2 x_1/dt^2 = 0$

Let, $\dfrac{k_1}{m_1} = p^2_{11}$, $\dfrac{k_2}{m_1} = p^2_{21}$, $\dfrac{k_2}{m_2} = p^2_{22}$, $\dfrac{k_3}{m_2} = p^2_{32}$, $\dfrac{k_3}{m_3} = p^2_{33}$,

$$\frac{k_2}{k_3} = K,\ \frac{m_2}{m_3} = M,\ \bar{p} = \frac{p_{22}}{\omega}$$

From [4.1]

$$x_2 = -\frac{b_1}{m_1 p^2_{21}} \sin \omega t - \frac{b_3}{m_1 p^2_{21}} \sin 3\,\omega t \qquad [4.4]$$

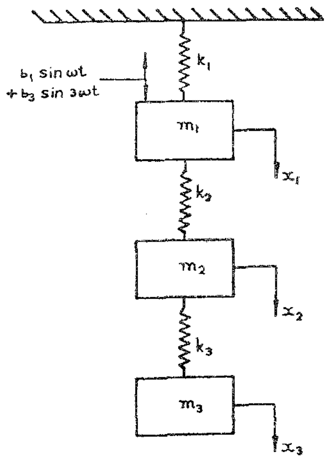

FIG. 4.1

From [4.2] and [4 4]

$$x_3 = \frac{1}{p_{32}^2} \cdot \frac{b_1}{m_1 \bar{p}_{12}^2} (\omega^2 - p_{22}^2) \sin \omega t + \frac{b_3}{m_1 p_{21}^2} (9\omega^2 - p_{22}^2) \sin 3\omega t \qquad [4.5]$$

Substituting [4.5] in [4.3] and equating coefficients of $\sin \omega t$ and $\sin 3\omega t$ separately to zero and rearranging,

$$K = \alpha [1 + M] \qquad [4.6]$$

where

$$\alpha = \bar{p}^2/[10 - \bar{p}^2] \qquad [4.7]$$

For real systems $K > 0$, $M > 0$, therefore only positive values of α are admissible. This implies

$$0 \leq \bar{p} \leq \sqrt{10}$$

Eqn. [4·6] is plotted in Fig. 4 2

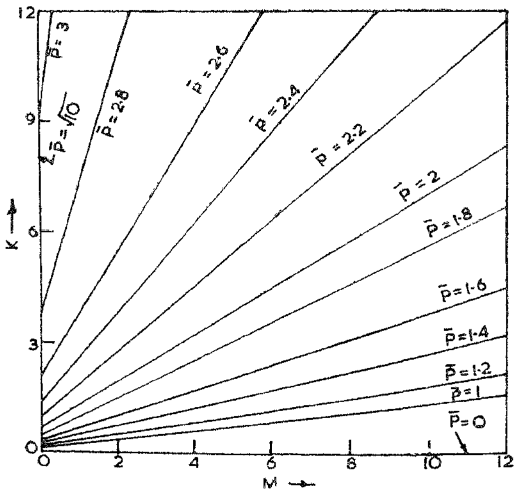

FIG. 4.2

Absorber parameter curves for single degree of freedom system subjected to bifrequency excitation.

5. *Dynamic Vibration absorber for single degree of freedom system with triple frequency excitation.*

This is an extension of case (4) : Here

$$F(t) = b_1 \sin \omega t + b_3 \sin 3\,\omega t + b_5 \sin 5\,\omega t \qquad [5.1]$$

The equations of motions will be

$$m_1\,(d^2x_1/dt^2) + k_1x_1 + k_2\,(x_1 - x_2) = F(t) \qquad [5.1]$$

$$m_2\,(d^2x_2/dt^2) + k_2x_2 + k_3x_2 - k_2x_1 - k_3x_3 = 0 \qquad [5.2]$$

$$m_3\,(d^2x_3/dt^2) + (k_3 + k_4)x_3 - k_3x_2 - k_4x_4 = 0 \qquad [5.3]$$

$$m_4\,(d^2x_4/dt^2) + k_4\,(x_4 - x_3) \qquad = 0 \qquad [5.4]$$

For complete isolation of m_1, $x_1 = 0$, $(d^2x_1/dt^2) = 0$,

Let $\dfrac{k_1}{m_1} = p_{11}^2$, $\dfrac{k_2}{m_1} = p_{21}^2$, $\dfrac{k_2}{m_2} = p_{22}^2$, $\dfrac{k_3}{m_2} = p_{32}^2$, $\dfrac{k_3}{m_3} = p_{33}^2$, $\dfrac{k_4}{m_3} = p_{43}^2$,

$\dfrac{k_4}{m_4} = p_{44}^2$

$$\frac{1}{\bar{p}} = \frac{\omega}{p_{22}}, \quad \alpha_1 = \frac{k_2}{k_3}, \quad \alpha_2 = \frac{k_2}{k_4}, \quad \beta_1 = \frac{m_2}{m_3}, \quad \beta_2 = \frac{m_2}{m_4}$$

Eqn. [5.1] gives x_2, Eqn. [5.2] gives x_3 and Eqn. [5.3] gives x_4. Substituting these into Eqn. [5.4] and equating coefficients of $\sin \omega t$, $\sin 3\omega t$, $\sin 5\omega t$, separately to zero, we find after considerable algebraic simplification

$$\bar{p}^2 \left[\frac{\alpha_1 + 1 + \beta_1}{\alpha_1} + \frac{\beta_1 + \beta_2}{\alpha_2} \right] = 35 \qquad [5.5]$$

In section (4) it was shown that for a bifrequency excitation

$$\frac{10}{\bar{p}^2} = \left[\frac{\alpha_1 + 1 + \beta_1}{\alpha_1} \right], \quad \alpha_1 = k, \; \beta_1 = M$$

If α_1, and β_1 are chosen from fig. (4.2) then for a suitable α_2, β_2 can be calculated from [5.5].

The same argument can be extended to multiple frequency excitations. See Fig. 5.1.

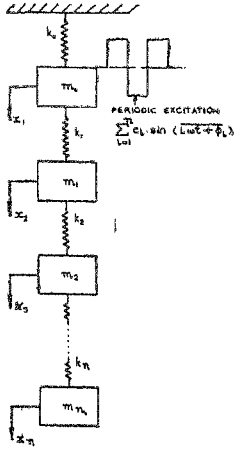

FIG. 5.1

6. *Single degree of freedom system subjected to pulse train of unit height, duration 2, and period T.*

Referring to Fig. 6.1,

$$f(t)=\begin{cases} 0, & -T/2 < t < -T \\ 1, & -T < t < +T \\ 0, & +T < t < +T/2 \end{cases}$$

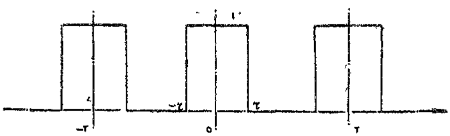

FIG. 6.1

Expressing as a Fourier Series

$$f(t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{2\pi nt}{T}\right), \text{ as } f(t) \text{ is even} \qquad [6.1]$$

where

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\, dt,\ a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t) \cos\left(\frac{2\pi nt}{T}\right) dt$$

For narrow pulses $a_n \simeq 4/T$, and for impulses $a_0 = (2/T)$, $a_n \simeq (2/T)$ (area under pulse is unity)

A good approximation may be obtained by taking a finite number of terms in the fourier-expansion and providing an equal number of appropriately tuned dynamic vibration absorbers.

7. *Parallel vibration absorbers for bifrequency excitation*

Fig. 7.1 shows the schematic arrangement, where $F(t) = b_1 \sin \omega t + b_3 \sin 3\omega t$.

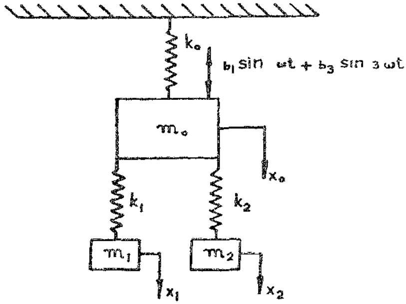

FIG. 7.1

The equations of motion are

$$m_0\,(d^2x_0/dt^2) + k_0\,x_0 + (k_1 + k_2)\ x_0 - k_1\,x_1 - k_2\,x_2 = F(t) \qquad [7.1]$$

$$m_1\,(d^2x_1/dt^2) + k_1\,x_1 - k_1\,x_0 = 0 \qquad [7.2]$$

$$m_2\,(d^2x_2/dt^2) + k_2\,x_2 - k_2\,x_0 = 0 \qquad [7\ 3]$$

As before we put, $x_0 = 0$, $(d^2x_0/dt^2) = 0$

$$\frac{k_0}{m_0} = p_{00}^2, \quad \frac{k_1}{m_0} = p_{10}^2, \quad \frac{k_2}{m_0} = p_{20}^2, \quad \frac{k_1}{m_1} = p_{11}^2, \quad \frac{k_2}{m_2} = p_{22}^2$$

Assuming $x_1 = A_1 \sin \omega t$, $x_3 = A_3 \sin 3\omega t$

We get $A_1 = -\dfrac{b_1}{k_1}$, $A_3 = -\dfrac{b_3}{k_2}$

and the tuning conditions will be $\omega^2 = p_{11}^2$, $9\omega^2 = p_{22}^2$ [7.4]

Generalisation

If the excitation can be represented by

$$F(t) = \sum_{n=1}^{\infty} C_n \sin (n\,\omega t + \phi_n)$$

then n parallel absorbers can be provided with the n^{th} absorber tuned to the n^{th} harmonic, *viz.*, $p_{nn}^2 = n^2\,\omega^2$

The amplitude of the n^{th} absorber mass would be $|A_n| = |C_n/k_n|$

8. *Parallel dynamic vibration absorbers for torsional oscillations*

Consider a flywheel rotating at a constant speed of ω, Fig. 8.1. Let a steady disturbing torque of $\sum_{i=1}^{n} M_i \sin i\,\omega t$ act on the flywheel.

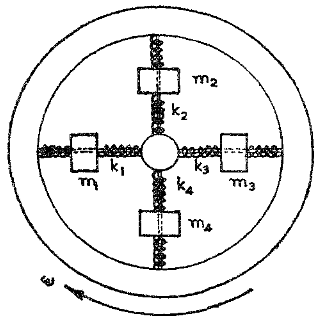

FIG. 8.1

We shall assume that the flywheel is provided with n spokes. Along each spoke is mounted an absorber unit comprising of a spring and mass.

Let m_i = mass mounted on the i^{th} spoke

k_i = spring connecting m_i to the hub.

The kinetic energy of the system is given by

$$T=\frac{1}{2}\ I\,(\omega+d\phi/dt)^2+\sum_{i=1}^{n}\frac{1}{2}\ m_i\ [(S_i+x_i)^2\ (\omega+d\phi/dt)^2+(d^2x_i/dt)^2] \qquad [8.1]$$

Where I = mass moment of inertia of the flywheel

S_i = equilibrium position of m_i

x_i = radial displacement coordinate of m_i

The potential energy is

$$V=\sum_{i=1}^{n}\ [(S_i-a_i+x_i)^2-(S_i-a_i)^2] \qquad [8.2]$$

a_i, being the free length of the spring k_i

The differential equations of motion are given in the Lagrangian form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\,(dx_i/dt)}\right)-\frac{\partial T}{\partial x_i}+\frac{\partial V}{\partial x_i}=0, \text{ for coordinate } x_i,\ i=1,2,3,\cdots n \qquad [8.3]$$

and

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\phi}}\right)-\frac{\partial T}{\partial\phi}+\frac{\partial V}{\partial\phi}=0, \text{ for coordinate } \phi \qquad [8.4]$$

$$(I+\sum_{1}^{n}2m_i\,S_i)^2\ \ddot{\phi}+\sum_{1}^{n}n\ \omega\ m_i\ S_i\ (dx_i/dt)\ =\ \sum_{1}^{n}M_i\ \sin\ i\ \omega\ t \qquad [8.5]$$

and $m_i\ (d^2x_i/dt^2)=m_1\ \omega^2\ (S_i+x_i)-2m_i\ \omega\ S_i\ \phi+k_i\ (S_i-a_i+x_i)=0,\ i=1,2,\cdots n$ [8.6]

The equilibrium configuration is defined by

$$m_i\ \omega^2\ S_i=k_i\,(S_i-a_i) \qquad [8.7]$$

$$\therefore\ m_i\ (d^2x_i/dt^2)+(k_i-m_i\ \omega_i^2)\ x_i-2\ m_i\ \omega\ S\ \dot{\phi}=0 \qquad [8.8]$$

If the torsional vibrations of the flywheel are to be completely suppressed

$$\phi=C,\ \ \dot{\phi}=0,\ \ \ddot{\phi}=0$$

This yields from [8 5] and [8 8]

$$\sum_{i=1}^{n} n\,\omega\, m_i\, S_i\,(dx_i/dt) = \sum_{i=1}^{n} M_i \operatorname{Sin} i\,\omega\, t \qquad [8.9]$$

and $$(d^2x_i/dt^2) + [K_i/m_i - \omega^2]\, x_i = 0, \quad i = 1, 2, \cdots n \qquad [8\ 10]$$

[8.10] yields the tuning condition

$$(K_i/m_i) = (i^2+1)\ \omega^2 \qquad [8\ 11]$$

for the i^{th} absorber unit from [8.10]

$$x_i = a_i \cos i\,\omega\, t \qquad [8.12]$$

∴ from [8 9] and [8.12]

$$a_i = \left| -M_i/4m_i\, S_i\, i\,\omega^2 \right| \text{ and phase } 180^\circ \qquad [8.13]$$

gives the amplitude of the i^{th} absorber mass.

9. *Vibration absorber for a single degree of freedom system subjected to an impulsive excitation.*

The single degree of freedom system shown in fig. 9 1 is subjected to an impulsive excitation $Hu'(t)$, where $u'(t)$ is the dirac-delta function. The possibility of reducing the vibrations of the main mass, by attaching a single absorber unit is investigated.

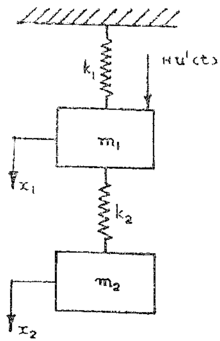

FIG. 9.1

The equations of motion are

$$(d^2x_1/dt^2)+p_{11}^2\ x_1+p_{21}^2\ x_1-p_{21}^2\ x_2=H_1\ u'\ (t) \quad [9.1]$$

$$(d^2x_2/dt^2)+p_{22}^2\ (x_2-x_1)=0 \quad [9.2]$$

where $p_{11}^2=\frac{K_1}{m_1}$, $p_{21}^2=\frac{K_2}{m_1}$, $p_{22}^2=\frac{K_2}{m_2}$, $H_1=\frac{H}{m_1}$

Applying laplace transform to equation [9.1] and [9.2] and assuming initial Conditions,

$$x_1\ (0)=0,\ (dx_1/dt)\ (0),\ x_2\ (0)=0,\ (dx_2/dt)\ (0)=0$$

$$(s^2+p_{11}^2+p_{21}^2)\ \bar{X}_1\ (s)-p_{21}^2\ \bar{X}_2\ (s)=H_1 \quad [9.3]$$

$$-p_{22}^2\ \bar{X}_1\ (s)+(s^2+p_{22}^2)\ \bar{X}_2\ (s)=0 \quad [9.4]$$

Solving for $\bar{X}\ (s)$

$$\bar{X}_1\ (S)=\frac{(s^2+p_{22}^2)\ H_1}{s^4+s^2\ (p_{11}^2+p_{21}^2+p_{22}^2)+p_{11}^2\ p_{22}^2} \quad [9.5]$$

The denominator of (9.5) is

$$s^4+s^2\ (p_{11}^2+p_{21}^2+p_{22}^2)+p_{11}^2\ p_{22}^2 \quad [9.6]$$

Since $p_{11}>0$, $p_{21}>0$, $p_{22}>0$, and $\bar{x}_1\ (s)$ is even *i.e.* $\bar{x}_1\ (-s)=\bar{x}_1\ (s)$, the denominator when equated to zero will have only two pair of complex conjugate roots, but no real roots.

Let the roots be

$$\left.\begin{aligned} s_1&=\alpha_1+i\mu_1 & s_2&=\alpha_1-i\mu_1 \\ s_3&=\alpha_2+i\mu_2 & s_4&=\alpha_2-i\mu_2 \end{aligned}\right\} \quad [9.7]$$

The roots will be purely imaginary because

$$s_1+s_2+s_3+s_4=0, \quad \therefore\ \alpha_1+\alpha_2=0$$

$$s_1\ s_2\ s_3+s_1\ s_2\ s_4+s_2\ s_3\ s_4+s_1\ s_3\ s_4=0$$

$$\therefore\ (\alpha_1^2+\mu_1^2)\ \alpha_2+(\alpha_2^2+\mu_2^2)\ \alpha_1=0$$

or $$\alpha_1\ (\alpha_1^2+\mu_1^2)-(\alpha_2^2+\mu_2^2)=0$$

As the expression [9.6] is not a perfect square it will not have repeated roots

$$\therefore\ (\alpha_1^2+\mu_1^2)-(\alpha_2^2+\mu_2^2)\neq 0$$

So $\alpha_1 = 0$ and $\therefore \alpha_2 = 0$.

$$\therefore s_1 = i\,\mu_1,\ \ s_2 = -i\,\mu_1,\ \ s_3 = i\,\mu_2,\ \ s_4 = -i\,\mu_2 \tag{9.8}$$

Inverse transform of [9.5] yields

$$\frac{x_1(t)}{H_1} = \frac{(p_{22}^2 - \mu_1^2)}{\mu_1(\mu_2^2 - \mu_1^2)} \sin \mu_1 t - \frac{(p_{22}^2 - \mu_2^2)}{\mu_2(\mu_2^2 - \mu_1^2)} \sin \mu_2 t \tag{9.9}$$

This gives the motion of m_1. Next we determine μ_1 and μ_2 in terms of, the system parameters p_{11}, p_{21}, p_{22}. We have from [9.8],

$$s_1\, s_2\, s_3\, s_4 = p_{11}^2\, p_{22}^2 \tag{9.10}$$

$$s_1(s_2 + s_3 + s_4) + s_2(s_3 + s_4) + s_3(s_4) = p_{11}^2 + p_{21}^2 + p_{22}^2 = \mu_1^2 + \mu_2^2 \tag{9.11}$$

Let
$$\frac{K_1}{K_2} = K,\ \ M = \frac{m_1}{m_2}$$

$$\therefore \mu_{1,2}^2 = \frac{p_{11}^2}{2}\left\{\frac{K+M+1}{K} \mp \sqrt{\left[\left(\frac{K+M+1}{K}\right)^2 \mp 4\left(\frac{M}{K}\right)\right\}}\ p_{22}^2 = \frac{M}{K} p_{11}^2 \tag{9.12}$$

[9.9] becomes,

$$(x_1(t) p_{11}/H_1) = A \sin \mu_1 (t) - B \sin \mu_2 t \tag{9.13}$$

Where

$$A = \frac{(M/K) - \frac{1}{2}\left\langle (K+M+1)/K - \sqrt{\{[(K+M+1)/K]^2 - 4\,(M/K)\}}\right\rangle}{\frac{1}{\sqrt{2}}\left\{\sqrt{\left\langle \frac{K+M+1}{K} - \sqrt{\left[\left(\frac{K+M+1}{K}\right)^2 - 4\,\frac{M}{K}\right]}\right\rangle}\right\}\left\{\sqrt{\left[\left(\frac{K+M+1}{K}\right)^2 - 4\,\frac{M}{4}\right]}\right\}} \tag{9.14}$$

$$B = \frac{\langle M/K) - \frac{1}{2}\{(K+M+1)/K + \sqrt{\{[(K+M+1)/K]^2 - 4\,(M/K)\}}\rangle}{\frac{1}{\sqrt{2}}\left\{\sqrt{\left\langle \frac{K+M+1}{K} + \sqrt{\left[\left(\frac{K+M+1}{K}\right)^2 - 4\,\frac{M}{K}\right]}\right\rangle}\right\}\left\{\sqrt{\left[\left(\frac{K+M+1}{K}\right)^2 - 4\,\frac{M}{K}\right]}\right\}} \tag{9.15}$$

For physically real systems $\mu_1 > 0,\ \mu_2 > 0$.

It can also be noted that for the case $\mu_1 = \mu_2$. Eqn. [9.13] becomes zero identically. However from [9.12] for $\mu_1 = \mu_2$ we have

$$[(K + M + 1)/K]^2 = 4\,(M/K)$$

or
$$K^2 + M^2 + 2KM + 2K + 2M + 1 = 4KM$$

But this is not permissible in view of the fact that x_1 in [9 9] becomes infinite.

The coefficients A and B were calculated on a "FERRANTI-SIRIUS" digital computer, for the following range of K and M values

K 1 to 10

M 1 to 10

For the range of parameters investigated B was always < 0

$\therefore [x_1(t)p_{11}/H_1]$ can attain a max. value of $|A|+|B|$ when $\sin i\,\mu_1 t=1$ and $\sin \mu_2 t=1$, simultaneously.

If $\sin \mu_1 t=1$, then $\mu_1 t=(4p+1)(\pi/2)$, $p=0,1,2,$

$\sin \mu_2 t=1$ then $\mu_2 t=(4q+1)(\pi/2)$, $q=0,1,2,$

$(\mu_2/\mu_1)=(4p+1)/(4q+1)=\lambda$, $1<\lambda<7$, for the range of parameters investigated.

If we can find two positive integers p and q such that the above condition is satisfied then $x_1(t)(p_{11}/H_1)$ will attain a maximum value of $A+B$.

Next consider the table (9.1) where λ is calculated for various values of p and q.

TABLE 9.1 Values of λ

p \ q	0	1	2	3	4	5	Common Diff. (5)−(4)
1	1	5	9	13	17	21	4.0000
2	...	1	1.8	2.6	3.4	4.2	0.8000
3	...	...	1	1.444	1.889	2.334	0.4450
4	...	...	...	1	1.3077	1.6154	0.3077
5	...	...	...	...	1	1.2353	0.2363

The above table indicates that two integers p and q can always be found to cover the entire range of values, $1<\lambda<7$.

It is clear from table 9.2 that the minimum value of X occurs at $K=0.1$ and $M=1$, in the range investigated. This value is 0.798. For a single degree of freedom system without absorber the maximum value of displacement $|x|_{max}=1.0$.

Hence a reduction of nearly 21% in the amplitude of the main mass could be attained by attaching the auxiliary system whose parameter values are $K=0.1$ and $M=1$.

Further reduction is possible theoretically, but the size of the absorber mass relative to the main mass makes further improvement prohibitive.

TABLE 9.2

Shows the value of $\chi = |A| + |B|$ for the following range of K and M values,
$1 < M < 5$ $0.1 < K < 10$, $\chi = |A| + |B|$

M	Sl. No.	1	2	3	4	5	6	7	8	9	10
1	K	0.10	0.11	0.125	0.143	0.167	0.2000	0.250	0.33	0.5	9.0
	χ	0.7980	0.8020	0.8050	0.8110	0.8170	0.8250	0.8350	0.8450	0.8650	0.8950
2	K	0.20	0.22	0.250	0.286	0.335	0.400	0.5000	0.667	1 000	2.0
	χ	0.883	0.885	0.8880	0.8890	0.8950	0.9000	0.9050	0.9150	0.9250	0.9450
3	K	0.30	0.33	0.3750	0.429	0.501	0 600	0.7500	1.000	1.500	3 0
	χ	0 917	0.918	0.922	0.9250	0.9250	0.9300	0.9350	0.9400	0.9470	0.9600
4	K	0.40	0.44	0.5000	0.572	0.667	0.800	1.0000	1.333	2.000	4 0
	χ	0.937	0.938	0.939	0.9395	0.945	0.946	0.950	0.9956	0.9600	0 9710
5	K	0.50	0.55	0.6250	0.715	0.833	1.000	1.2500	1.667	2.5000	5.0
	χ	0.947	0.950	0.9510	0.9520	0.9560	0 9560	0.9590	0.9640	0.9690	0.9780
6	K	0.60	0.66	0.7500	0 858	1.000	1.200	1.5000	2 000	3.000	6.0
	χ	0.958	0.957	0.9590	0 9600	0.9600	0.9650	0.9670	0.9680	0 9740	0.9800

10. *Vibration Absorber for Transient or Random Base Excitation of a Single Degree of Freedom System. See Fig. 10.1.*

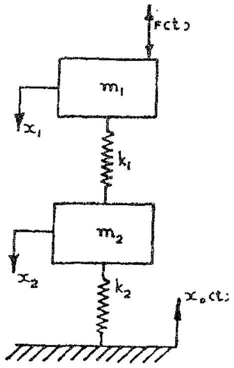

FIG. 10 1

(k_2, m_2) represents a spring mass system subjected to an arbitrary base excitation $X_0(t)$. (k_1, m_1) constitutes an absorber system attached to the main system. The equation of motion of the mass m_2 is

$$m_2\,(d^2x_2/dt^2)+K_1\,(X_2-X_1)+K_2X_2=K_2X_0\,(t)$$

If the vibrations of m_2 are completely suppressed then $X_2=0$, $(d^2x_2/dt^2)=0$

$$\therefore\ X_1=-(K_2/K_1)\ X_0\,(t) \qquad [12.1]$$

The equation of motion of the mass m_1 will be

$$m_1\,(d^2x_1/dt^2)+K_1\,(x_1-x_2)=F(t) \qquad [12.2]$$

where $F(t)$ is an external force to be applied to m_1. From (12.1) and (12.2)

$$-m_1\,(K_2/K_1)\,(d^2x_0/dt^2)\,(t)-K_2\,X_0\,(t)=F\,(t) \qquad [12.3]$$

Thus [12.3] represents the force to be applied to the main mass so that the mass m_2 remains stationary.

This force can be applied in the following way. The base excitation and its acceleration picked up by an accelerometer and an integrating circuit are properly amplified as required by equation [12.3] and applied to the mass m_1 through an electro-mechanical transducer.

11. *Vibration Absorber for Self Excited Vibrations of Continuous Systems.*

(*a*) *Lateral Vibrations of a Cantilever*: This type of vibration is very common in metal cutting tools with considerable overhang as in the case of boring bars. The self excited vibration (more commonly known as "Tool Chatter") is caused by the negative damping effect produced by the variation of frictional force with relative velocity, between the rubbihg aurfaces. For small relative velocities the frictional force aids any lateral motion of the tool while for large relative velocities the frictional force oppose the tool vibration. This results in the tool vibrating at the "limit cycle" amplitude and marring the surface finish of the work. The energy needed to excite the bar iuto vibration is drawn from the steadily rotating wotkpiece.

Consider the cutting tool idealised by a cantilever as in Fig. 11.1. The natural frequency of the cantilever with an attached mass and spring is first determined by the method followed in (1)*. Next an auxiliary mass is attached to the lower end of the spring and the tuning condition is derived.

Let L = length of the cantilever beam

M = Mass, attached rigidly to the beam, at a distance 'h' from fixed end.

K = spring constant of the absorber spring.

The frequencies of free vibration of the cantilever, without the mass and springs are determined by the method of normal modes as follows. The equation of motion is,

$$EI\,(\partial^4 y/\partial x^4) + \rho\,A\,(\partial^2 y/\partial t^2) = 0 \qquad [11\ 1]$$

where EI = flexural rigidity of the cantilever

ρ = mass density of the beam

A = area of c.s. of the bar

The end conditions are

$$\left.\begin{array}{ll}\text{(i)}\ (y)_{x=0}=0, & \text{(ii)}\ (\partial y/\partial x)_{x=0}=0,\\ \text{(iii)}\ (\partial^2 y/\partial x^2)_{x=L}=0 & \text{(iv)}\ (\partial^3 y/\partial x^3)_{x=L}=0\end{array}\right\} \qquad [11.2]$$

The general solution is assumed in the form

$$y = C_1(\cos\beta x + \cos h\,\beta x) + C_2(\cos\beta x - \cosh\beta x) + C_3(\sin\beta x + \sinh\beta x) + C_4(\sin\beta x - \sinh\beta x) \qquad [11.3]$$

where $\beta^4 = (m\omega^2/EIL),\ m = \rho\,A$

---

(1)* Vibrations of a beam with concentrated mass, spring and dashpot by D. Young, pp. 65, Jan. 1968, V. 15.

Equation [11.2]–(i) gives $C_1=0$, –(ii) gives $C_3=0$, –(iii) gives,

$$C_2 (\cos\beta L+\cosh\beta L)+C_4 (\sin\beta L+\sinh B L)=0\,;$$

$$C_2 (\sin\beta L+\sinh\beta L)+C_4 (\cos\beta L+\cosh\beta L)=0 \qquad [11.4]$$

Eliminating C_2 and C_4 the frequency equation is obteined as

$$1+\cos\beta L\cosh\beta L=0 \qquad [11.5]$$

∴ the normal function for the n-th mode becomes,

$$y_n=-C_2 [\cosh\beta_n x-\cos\beta_n x-\alpha_n (\sinh\beta_n x-\sin\beta_n x) \qquad [11.6]$$

where

$$\alpha_n=-\frac{C_4}{C_2}=\frac{\cosh\beta_n L+\cos\beta_n L}{\sinh\beta_n L+\sin\beta_n L} \qquad [11.7]$$

The constant C_2 is arbitrary and for the sake of convenience can be set equal to –1. ∴ [11.6] can be rewritten in the normalised form as

$$\phi_n=\cosh\beta_n x-\cos\beta_n x-\alpha_n (\sinh\beta_n x-\sin\beta_n x) \qquad [11.8]$$

Next we consider the mass M as being attached to the cantilever beam by a rigid link. When the system is vibrating freely, there will be a force F in the link joining the mass M and the beam which may be expressed as $F=F_0 \sin\omega t$, where ω=undetermined natural frequency of the composite system shown in fig. (11.1).

The system is next imagined to be cut through the link so that we have two parts (*a*) a beam acted upon by a hormonically varying force $F=F_0 \sin\omega t$ (*b*) a spring supported mass subjected to an equal but opposite force as in fig. (11.2).

The deflection of the beam at the location of the force *i.e.* at $x=h$ is given by

$$(y)_{x=h}=(w)_{x=h}\sin\omega t$$

$$=\frac{F_0 \sin\omega t}{m\,\omega_1^2}\sum_{n=1}^{\infty}\frac{[\phi_n (h)]^2}{\omega_n^2/\omega_1^2-\omega^2/\omega_1^2} \qquad [11.9]$$

The equation of motion of the mass is

$$M (d^2 Z_1/dt^2)+KZ_1=-F_0 \sin\omega t,$$

the steady state solution of which

$$Z_1=-F_0 \sin\omega t/(K-M\,\omega^2) \qquad [11.10]$$

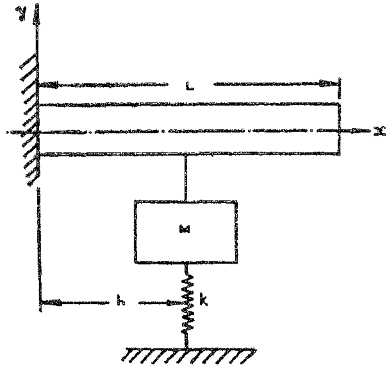

FIG. 11.1

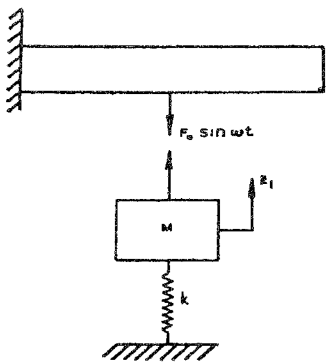

FIG. 11.2

for any position $x=h$, (M/m) Vs. (k/K_0), $K_0=(3EI/L^3)$, can be plotted for values of the parameter (ω/ω_1). This has been done for the cases (i) $h/L=0.78$, (ii) $h/L=0.50$. (Fig. 11.8) (11.10),

Next consider the auxiliary mass M_1, attached to the initially fixed end of the spring. (Fig. 11.3)

The equations of motion are

$$\left.\begin{aligned} M\,(d^2Z_1/dt^2)+K\,(Z_1-Z_2) &= -F_0 \sin \omega t \\ M_1\,(d^2Z_2/dt^2)+K\,(Z_2-Z_1) &= 0 \end{aligned}\right\} \qquad [11.11]$$

which for $Z_1=0=(d^2Z_1/dt^2)$ gives the tuning condition

$$\omega^2=(K/M_1) \qquad [11.12]$$

Thus for a given K the absorber mass M, required to suppress the lateral vibrations of the cantilever can be found.

Calculations for plotting

Roots of frequeucy equation $1+\cos\beta L\cosh\beta L=0$, are

$\beta_1 L=1.87510$, $\beta_2 L=4.69409$, $\beta_3 L=7.85476$, $\beta_4 L=10.99554$, $\beta_5 L=14.13717$

$$m=(k_b/\epsilon_1\,\omega_1^2),\ \epsilon=3/(\beta_1 L)^4=0.24267$$

$$(\omega_1^2/\omega_1^2)=1,\ (\omega_2^2/\omega_1^2)=39.2739,\ (\omega_3^2//\omega_1^2)=307.914,\ (\omega_4^2/\omega_1^2)=1182.40,$$
$$(\omega_5^2/\omega_1^2)=3231.08,\ (\omega_6^2/\omega_1^2)=7210,\ (\omega_7^2/\omega_1^2)=14{,}070.$$

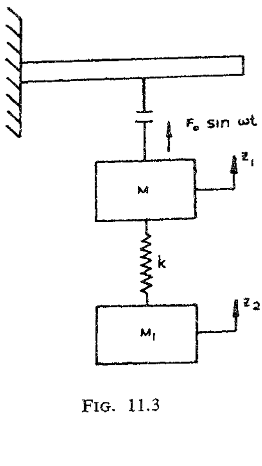

FIG. 11.3

TABLE 11.1

$(M/m) = A\ k/K_b + B$

ω/ω_1	$h/L = 0.78$		$h/L = 0\ 50$	
	A	B	A	B
2	0.06067	−0.38933	0.06067	−2.6000
3	0.02700	= 0.467504	0.02700	13.1800
4	0.01520	−0.504110	0.01520	1.1430
5	0.00970	−0.52747	0.00970	0.3260
6	0.00674	−0.56121	0.00674	0.0470
7	0.00495	−0.58343	0.00495	−0 1010
8	......	......	0.003791	−0.1955
9	......	......	0.002996	−0 2320
10	......	......	0.0024267	−0.2980

11 (*b*) *Analysis*: The span of a transmission line is idealised by a beam with hinged ends. The unstable cross section is provided by means of a semicircular cylindrical wooden piece running throughout the length of the beam (Fig. 11 4). The ends of the wooden piece however are free so that it will only act as an additional UDL on the beam, without in any way affecting the flexural rigidity of the main beam. The analysis is first done by assuming a concentrated mass to be rigidly attached to the beam at mid-span, the other end of the mass being tied to the ground through a spring. The composite frequency of the system is calculated by the method adopted in (1)*. The fixed end of the spring is next released, and an additional mass is attached and the tuning condition derived. Since the system oscillates almost at its natuual frequency during self excited oscillation, we see from the tuning condition, that, when the auxiliary mass-spring system is tuned to this frequency, the vibrations of the main beam will be suppressed. (Referring to Fig. 11.5).

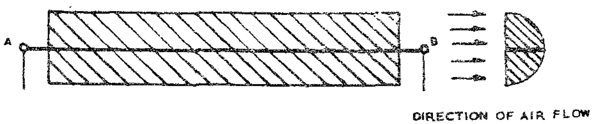

FIG. 11.4

Let AB be the beam with hinged ends, carrying a UDL of w/unit length due to the unstable section, in addition to its self weight of $A\gamma$ unit length, where A = Area of c. s. of beam alone and γ = specific weight of beam materil. The lateral vibrations of the beam are referred to the coordinates (x, y).

The equation of small lateral vibrations of the beam are given by (neglecting the effects of rotatory inertia and shear deflection).

$$\left(\frac{\partial^2 y}{\partial t^2}\right) + a^2\left(\frac{\partial^4 y}{\partial x^4}\right) = 0, \quad \text{where} \quad a^2 = \left[\frac{EIg}{A\gamma + w}\right] \qquad [11.13]$$

The solution can be assmed in the form

$$y = X\,(A \cos pt + B \sin pt), \qquad [11.14]$$

when the beam is performing a normal mode of vibration Substituting in [11.13]

$$(d^4X/dx^4) = (p^2/a^2)\ X \qquad [11.15]$$

the general solution of which is

$$X = C_1\ (\cos kx + \cosh kx) + C_2\ (\cos kx - \cosh kx) + C_3\ (\sin kx + \sinh kx)$$
$$+ C_4\ (\sin kx - \sinh kx) = 0 \qquad [11.16]$$

where $K^4 = (p^2/a^2) = (p^2/EIg)\ [A\gamma + w]$ where $C_1, C_2, C_3, C_4,$ are arbitrary constants to be determined from the end conditions.

The end conditions are

$$\left.\begin{array}{ll} \text{(i)}\ (y)_{x=0} = 0 & \text{(iii)}\ (y)_{x=L} = 0 \\ \text{(ii)}\ (d^2y/dx^2)_{x=0} = 0 & \text{(iv)}\ (d^2y/dx^2)_{x=L} = 0 \end{array}\right\} \qquad [11.17]$$

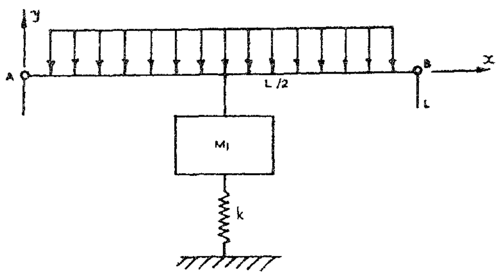

FIG. 11.5

[11.17] – (i) yields $C_1=0$

– (ii) yields $C_2=0$

– (iii) yields C_3 (sin kL + sinh kL) + C_4 (sin kL – sinh kL) = 0

– (iv) yields C_3 (– sin kL + sinh kL) + C_4 (– sin kL – sinh kL) = 0

Eliminating, C_3 and C_4, sin KL sinh KL = 0
since KL ≠ 0, ∴ sinh $kL \neq 0$. So sin $kL = 0$ is the frequency equation [11 18]

The roots of the frequency equations are

$$K_i L = i\pi, \quad i = 1, 2, 3, \ldots$$

The frequencies of the natural modes are given by

$$P_i^2 = a^2 K_i^4 = (i^4 \pi^4 / L^4) \; [EIg/(A\gamma + w)] \qquad [11.19]$$

The normal function for the n-th mode is

$$X_n = C_3 \left\{ (\sin K_n x + \sinh K_n x) - \left[\frac{\sin k_n L + \sinh k_n L}{\sin k_n L - \sinh k_n L} \right] (\sin k_n x - \sinh K_n x) \right\} \qquad [11.20]$$

Substituting [11,18] in [11.20] and putting $C_3 = 1$ for convenience (alternately this constant can be absorbed in the part "$A \cos pt + B \sin pt$" of the solution) we get

$$X_n = 2 \sin K_n x \qquad [11.21]$$

Beam with mass (M_1) *and spring* (k) *attaehed at mid-span* (Ref. Fig. 11.6).

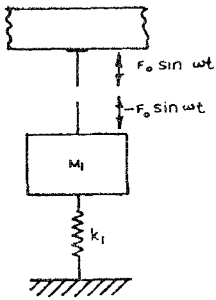

FIG. 11.6

We assume that the rigid link connecting the beam and the mass M_1 is cut through so that there is an alternating force $F_0 \sin \omega t$ acting on the beam, as well as the mass-spring system (M_1, K). We determine the displacement of each of the two systems at this point and determine the frequency ω of the composite system. (This ensures compatibility of displacements),

The response of the beam at $x = L/2$ is

$$y\,(L/2) = \frac{F_0 \sin \omega t}{m p_1^2} \sum_{n=1}^{\infty} \left\{ \frac{(X_n)^2{}_{x=L/2}}{p_n{}^2/p_1^2 - \omega^2/p_1^2} \right\}, \text{ where } m = \frac{A\gamma + w}{g} \qquad [11.22]$$

The response of the spring mass system is

$$y = \frac{-F_0 \sin \omega t}{K_1 - M_1 \omega^2} \qquad [11.23]$$

Equating [11.22] and [11.23] and rearranging, we get, the equation for composite frequency ω, as,

$$\left\{ \frac{1}{M_1/m - K_1/m\omega^2} \right\} = \frac{\omega^2}{p_1^2} \sum_{n=1}^{\infty} \frac{\{(X_n)^2{}_{L/2}\}}{\{p_n{}^2/p_1^2 - \omega^2/p_1^2\}} \qquad [11.24]$$

Again $K/m\omega^2 = (EI\,K_1\,p_1^2/\beta_1^4\,\omega^2$, β = Eigen value in the frequency equation (formerly k). But $\beta_1^4 = \pi^4$ and putting K_b = stiffness of beam due to a concentrated load at midspan = $(384\ EI/5\ L^3)$

$$\frac{k_1}{m\omega^2} = \epsilon_1 \frac{k_1}{k_b} \frac{p_1^2}{\omega^2} \text{ where } \epsilon_1 = \frac{384}{5\,\beta_1^4\,L^4} = 0.7880$$

Also $p_n{}^2/p_1^2 = n^4/l^4 = n^4$

$(X_n)_{L/2} = 2\,X \sin(n\pi/2) = 0$ if n is even

$= \pm 2$ if n is odd, $+2$ if $n = 4m+1$ $m = 0,1,2$

-2 if $n = 4m-1$ $m = 1,2.$

In any case $(X_n{}^2)_{L/2} = 4$, for n odd $\therefore$ equation [11.24] becomes

$$\frac{1}{\{(M_1/m) - \epsilon_1\,(K/k_b)\,(p_1^2/\omega^2)\}} = 4\,(\omega/p_1)^2 \sum_{n=1,3,5}^{\infty} \frac{1}{\{n^4 - \omega^2/p_1^2\}}$$

The right hand sides is convergent for all ω/p_1 except when $\omega^2/p_1^2 = n^4$ in which case it is infinite.

A plot of (M_1/m) Vs. (K/K^b) for various parameter values (ω/p_1) is shown (Fig. 11.8)

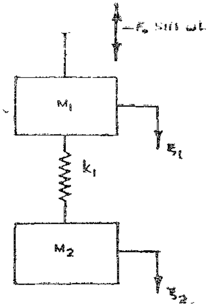

FIG. 11.7

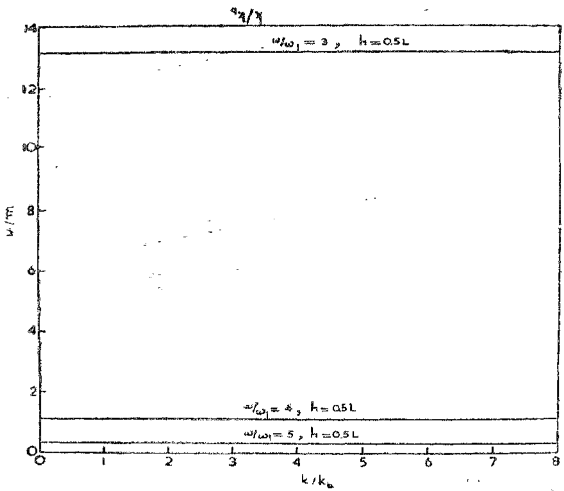

FIG. 11.8

Absorber Parameters for a vibrating cantilever.

These plots are made according to the following equations :

$M_1/m = A\ (K/K_b) + B$

ω/p_1	A	B
0.50	3.15200	+0.74300
2.00	0.19700	−0 19650
3.00	0.08756	−0 25550
4.00	0.04925	−0.31990
5.00	0.03125	−0.46920

Tuning condition :

Let M_2 = absorber mass. The equation of motion of M_1 and M_2 are

$$M_1\ \ddot{\xi}_1 + K_1\ \xi_1 - \xi_2 = -F_0 \sin \omega t$$

$$M_2\ \ddot{\xi}_2 + K_1\ (\xi_2 - \xi_1) = 0$$

To suppress the motion of M_1 and hence of the beam $\xi_1 = 0$,

$$\ddot{\xi}_1 = 0, \quad \therefore \xi_2 = (F_0 \sin \omega t/K_1)$$

$\therefore$ $\omega = \sqrt{(K_1/M_2)}$, *is the tuning condition.*

The procedure for the selection of the absorber may be briefly summarised as follows .

1. Choose the convenient values of M_1 and k_1 (the possibility of $M_1 = 0$ is not absolutely ruled out, it is possible as long as the lines in Fig. 11.8 cut the abscissa). Obtain (ω/p_1) from Fig. 11.8.

2. Then $M_2 = k_1\ \omega^2$.

12. *Nonlinear Dynamic Vibration Absorber for an Undamped Single Degree of Freedom Subjected to Step Function Excitation.*

Consider the single degree of freedom shown in Fig. 12.1. Let m_1 = mass, k_1 = stiffness of spring. Let the system be subjected to a step function excitation $F_1\ (t)$. Then the equation of motion becomes

$$m_1\ (d^2x_1/dt^2) + K_1x_1 = F_1 \qquad [12.1]$$

Dividing throughout by m_1 and putting $(K_1/m_1) = p_{11}^2$, $(F_1/m_1) = F_{11}$

$$(d^2x_1/dt^2) + p_{11}^2\ x_1 = F_{11} \qquad [12.2]$$

Assuming initial conditions $x_1\ (0) = 0$, $(dx_1/dt)\ (o) = 0$, the solution of (2) will be

$$x_1 = (F_{11}/p_{11}^2)\ (1 - \cos p_{11}\ t) \qquad [12.3]$$

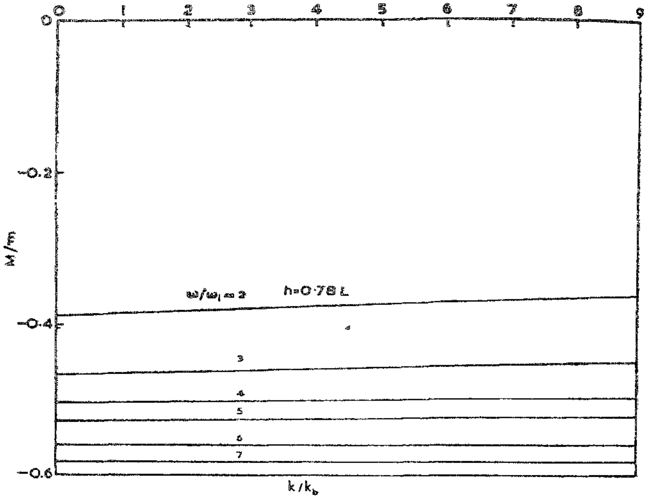

FIG. 11.9
Absorber parameters for a vibrating cantilever

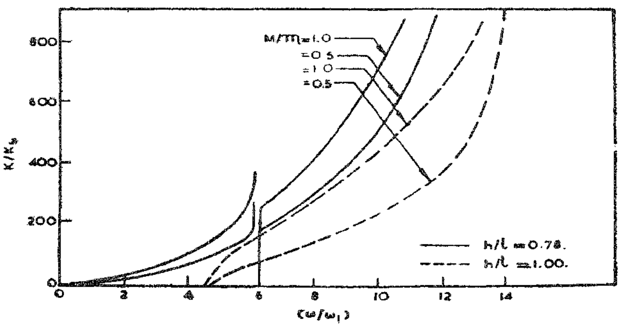

FIG. 11.10
Dynamic vibration absorber for cantilever, curve for choosing the absorber spring (K)

The maximum values of x_1 will be $(x_1)_s = (2F_{11}/p_{11}^2) = 2\,\delta$, where

$$\delta = F_{11}/p_{11}^2 \qquad [12.4]$$

Next consider the two degree of freedom nonlinear system shown in fig. (12.2).

We shall denote m_2 = absorber mass

k_4 = absorber spring.

The two masses are coupled by a nonlinear spring whose force displacement characteristic is given by $f(x) = K_2\, x + K_3\, x^3$.

The equation of motion of the two masses become

$$m_1\,(d^2x_1/dt^2) + K_1 x_1 + K_2\,(x_1 - x_2) + K_3\,(x_1 - x_2)^3 = F_1 \qquad [12.5]$$

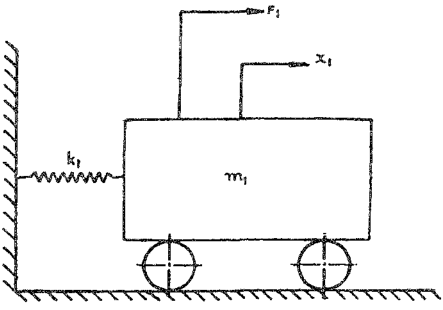

FIG. 12.1

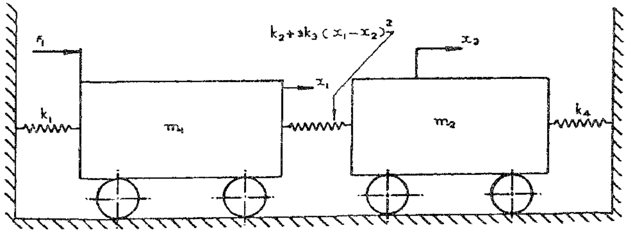

FIG. 12.2

$$m_2 (d^2x_2/dt^2) + K_2 (x_2 - x_1) + K_3 (x_2 - x_1)^3 + k_4 x_2 = 0 \quad [12.6]$$

Let $(K_1/m_1) = p_{11}^2, \ (K_2/m_1) = p_{21}^2, \ (K_3/m_1) = \mu_{31}, \ (F_1/m_1) = F_{11}$
$(K_2/m_2) = p_{22}^2, \ (K_3/m_2) = \mu_{32}, \ (K_4/m_2) = p_{42}^2$

Then $$(d^2x_1/dt^2) + (p_{11}^2 + p_{21}^2) x_1 - p_{21}^2 x_2 + \mu_{31} (x_1 - x_2)^3 = F_{11} \quad [12.7]$$

$$(d^2x_2/dt^2) + (p_{22}^2 + p_{42}^2) x_2 - p_{22}^2 x_1 + \mu_{32} (x_2 - x_1)^3 = 0 \quad [12.8]$$

Let $m_2/m_1 = K_4/K_1 = \lambda = \text{constant}$

Then $p_{22}^2 = K_2/m_2 = p_{21}^2/\lambda, \ p_{42}^2 = K_4/m_2 = p_{11}^2$
$\mu_{32} = K_3/m_2 = \mu_{31}/\lambda$

$\therefore$ [12.7] becomes

$$d^2x_1/dt^2 + (p_{11}^2 + p_{21}^2) x_1 - p_{21}^2 x_2 + \mu_{31} (x_1 - x_2)^3 = F_{11} \quad [12.9]$$

$$\lambda (d^2x_2/dt^2) + (\lambda p_{11}^2 + p_{21}^2) x_2 - p_{21}^2 x_1 + \mu_{31} (x_2 - x_1)^3 = 0 \quad [12.10]$$

Adding $(d^2x_1/dt^2 + \lambda d^2x_2/dt^2) + p_{11}^2 (x_1 + \lambda x_2) = F_{11}$

Setting $x_1 + \lambda x_2 = \xi, \quad \ddot{\xi} + p_{11}^2 \xi = F_{11}$ [12.11]

[12.9] – [12.8] give

$$(d^2x_1/dt^2 - d^2x_2/dt^2) + p_{11}^2 (x_1 - x_2) + p_{21}^2 [(\lambda + 1)/\lambda] (x_1 - x_2)$$
$$+ \mu_{31} [(\lambda + 1)/\lambda] (x_1 - x_2)^3 = 0 \quad [12.12]$$

Let $(x_1 - x_2) = \eta$

$$\therefore \frac{d^2 \eta}{dt^2} + \left[p_{11}^2 + p_{21}^2 \left(\frac{\lambda + 1}{\lambda} \right) \eta \right] + \mu_{31} \left(\lambda + 1 \right) \eta^3 = 0 \quad [12.13]$$

$$x_1 = \frac{\xi + \lambda \eta}{(\lambda + 1)}, \quad x_2 = \frac{\xi - \eta}{(\lambda + 1)} \quad [12.14]$$

We shall assume the initial conditions to be

$$x_1 (0) = 0, \quad (dx_1/dt)(0) = 0, \quad x_2 = (0), \quad (dx_2/at) = 0$$

$\therefore \xi (0) = 0 = \dot{\xi} (0)$

$\eta (0) = 0 = \dot{\eta} (0)$

Consider Equation [12.13]

Let $\sqrt{\{p_{11}^2 + p_{21}^2 [(\lambda + 1)/\lambda]\}} \cdot t = t_1$ [12.15]

$$\therefore \frac{d^2\eta}{dt_1^2} + \eta + \frac{\mu_{31}(\lambda+1)/\lambda}{p_{11}^2[1+(p_{21}^2/p_{11}^2)(\lambda+1)/\lambda]}\eta^3 = \frac{F_1}{p_{11}^2[1+(p_{21}^2/p_{11}^2)(\lambda+1)/\lambda]} \quad [12\ 16]$$

Put

$$p_{21}^2/p_{11}^2 = K_2/K_1 = \alpha, \quad \mu_{31}/p_{11}^2 = K_3/K_1 = \beta,$$

$$\therefore \frac{d^2\eta}{dt_1^2} + \eta + \frac{\beta(\lambda+1)/\lambda}{1+\alpha(\lambda+1)/\lambda}\eta^3 = \frac{\delta}{1+\alpha(\lambda+1)/\lambda}$$

Let $\eta = \{\delta\lambda/[\lambda+\alpha(\lambda+1)]\}\eta_1$

$$\therefore \frac{d^2\eta_1}{dt_1^2} + \eta_1 + \frac{\beta(\lambda+1)\delta^2\lambda^2}{[\lambda+\alpha(\lambda+1)]^3}\eta_1^3 = 1 \quad [12.17]$$

Let

$$[\beta(\lambda+1)/\lambda]\delta^2 = B; \quad 1+\alpha(\lambda+1)/\lambda = A \quad [12.18]$$

Consider Eqn. [12.11]. Substituting [12 15]

$$\frac{d^2\xi}{dt_1^2} + \frac{\lambda}{[\lambda+\alpha(\lambda+1)]}\xi = \frac{\delta\lambda}{[\lambda+\alpha(\lambda+1)]}$$

with $\xi = \dfrac{\delta\lambda}{\lambda+\alpha(1+\lambda)}\xi_1$ [12.19]

$$\frac{d^2\xi_1}{dt_1^2} + \left[\frac{\lambda}{\lambda+\alpha(\lambda+1)}\right]\xi_1 = 1$$

Substituting [12.17]

$$\frac{d^2\xi_1}{dt_1^2} + \frac{\xi_1}{A} = 1 \quad [12.20]$$

The maximum value of ξ_1, from [12.20] is ξ_1 (max) $= 2A$. Integrating [12.17], putting $\eta_1 = \eta_1$ (max), $d\eta_1/dt = 0$, and dividing throughout by η_1 max

$$\eta_{max}^3 + (2A/B)\eta_{max} - 4(A/B) = 0$$

or $\quad 2(A/B) = \eta_{max}^3/(2-\eta_{max})$ [12.21]

$\therefore$ From (12.14)

$$(x_1)_{max} = (\xi_{1max} + \lambda\eta_{1max})/(\lambda+1)$$

$$= [\delta/A(\lambda+1)]\ \{\xi_{1max} + \lambda\eta_{1max}\}$$

$$= [\delta/A(\lambda+1)]\ \{2A + \lambda\eta_{1max}\} \quad [12.22]$$

$$\therefore (X_{1max})_N/(X_{1max})_S = [(2A + \lambda\,\eta_{1max})/2A\,(\lambda + 1)] = \Omega \qquad [12.23]$$

From (12.17) as $\alpha > 0$ and $\lambda > 0$, and $\delta^2 > 0$. A is always > 0.

If we admit softening type of nonlinear spring then $\beta < 0$ and B can assume negative values.

Consider equations [12.23].

We are interested in values of $\Omega < 1$ for absorber purposes [12.23] can be written as

$$2A\,[\Omega\,(\lambda + 1) - 1] = \lambda\,\eta_{1max} \qquad [12.24a]$$

Now ξ_{1max} is always > 0.

$(x_{1max})_N$ will be greater than zero if η_{1max} is > 0. We shall therefore use $\eta_{1max} > 0$ for a conservative estimate.

Then $\Omega\,(\lambda + 1) - 1 > 0$, from (12.24a)

or $\lambda > (1 - \Omega)/\Omega$ (12.24b)

Further we are led to restrict λ to $0 < \lambda < 1$ to have an absorber unit not bulkier than the main unit.

$$\therefore 1 > (1 - \Omega)/\Omega \text{ or } \Omega > 0.5 \qquad [12\ 25]$$

Consistent with all the above requirements we can provide a reduction in amplitude not greater than 50%.

Even this reduction would be very significant for practical purposes. Eliminating η_{1max} between [12.21] and [12.24]

$$2\left(\frac{A}{B}\right) = \frac{(2A/\lambda)\,[\Omega\,(\lambda + 1) - 1]^3}{[2 - (2A/\lambda)\,\{\Omega\,(\lambda + 1) - 1\}]} \qquad [12.26]$$

If the coupling spring is linear from [12.17] $B = 0$

$\therefore$ From (12.18) $\eta_{1max} = 2$

$$\therefore \frac{(x_{1max})_N}{(x_{1max})_S} = \frac{\delta}{A\,(\lambda + 1)}\;\frac{2A + 2\lambda}{2\delta} = (\Omega)_{\text{Linear}} \qquad [12.27]$$

For the nonlinear spring $\eta_{1max} < 2$ for $\beta > 0$ $(\Omega)_{\text{Linear}} > (\Omega)_{\text{Non-linear}}$ and definite advantage results by using a "hardening" type of nonlinear spring.