THE STABILITY OF AN INFINITELY CONDUCTING CYLINDER IN THE PRESENCE OF A MONOTONIC *MAGNETIC FIELD*

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ABSTRACT

The stability of an infinitely long gravitating cylinder of incompressible, inviscid fluid of infinite electrical conductivity in the presence of a magnetic field having toroidal as well as poloidal components is discussed. The toroidal component is a monotonic function of the radial coordinate. By employing the energy principle and normal mode analysis, we have compared the results The results seem to be in good agreement. The general conclusion is that in every case the magnetic field increases the stability of the cylinder.

1. INTRODUCTION

The stability of an infinitely long cylinder of incompressible inviscid fluid of infinite electrical conductivity in the presence of different types of magnetic field configurations has been investigated by many authors[1,2,3]. They showed that the magnetic field has a stabilising effect on the cylinder. Recently Auluck and Nayyar[4] have investigated the stability of such a system in the presence of magnetic field having both toroidal and Poloidal components. Their conclusion is that the field has a stabilising influence on the system. In the present paper we have investigated the stability of an infinitely long cylinder in the presence of a magnetic field having a toroidal as well as poloidal component. The toroidal component is a monotonic function of the radial coordinate and the poloidal component is uniform. The field we have postulated satisfies the well known equilibrium equations. Recently Bobeldijk[5] has discussed the equilibrium of a plasma in a field configuration of similar type with a constant pitch of the field.

We consider as an idealised model any cylindrically symmetrical and infinitely long configuration of plasma and magnetic field with the following properties : (1) The plasma is infinitely conducting, incompressible and inviscid. (2) The magnetic field has a toroidal and Poloidal Components but no radial component. (3) The plasma tensor is isotropic. (4) In the steady state the plasma pressure gradient is balanced by the electromagnetic force and gravitational force. (5) The radius of the cylinder is R and is surrounded by a vacuum region in which there is a uniform axial magnetic field. In section 2, we have described the equilibrium state of the system. In section 3, we have discussed the stability of the system by energy method[2,6] and in section 4. we have applied the normal mode method in order to make a comparitive study of the stability of the system.

2. Equilibrium State

The magnetic field configuration in the various regions of the system in dimentionless form is

$$\left.\begin{array}{ll} \vec{B}^{(i)}_{(0)} = [0, \ \lambda r/(1+\mu_p r^2), \ 1] & 0 \leq r < 1 \\ \vec{B}^{(o)}_{(0)} = (0, \ \ 0, \ \ H_2) \ , & r > 1 \end{array}\right\} \qquad [2.1]$$

where we have taken R as the characteristic length and the uniform axial magnetic field in the plasma as the characteristic magnetic field.

The pressure inside the plasma is given by

$$P_0^{(i)} = -\int \frac{\lambda}{1+\mu_p r^2} \cdot \frac{\partial}{\partial r}\left(\frac{\lambda r^2}{1+\mu_p r^2}\right) dr + \gamma \phi + A, \qquad [2.2]$$

where A is a constant of integration.

The gravitational potential is

$$\left.\begin{array}{ll} \phi = \frac{1}{2}(1-r^2) & o \leq r < 1 \\ \phi = -\ln r & r > 1 \end{array}\right\} \qquad [2.3]$$

3. Energy Principle

To investigate whether the above configuration is stable or unstable we deform the cylinder in such a way that the boundary becomes

$$r = 1 + a \cos kz, \qquad [3.1]$$

where $a << 1$. Since the plasma is incompressible, the volume per unit length of the cylinder does not change and hence

$$R_0^2 = 1 + \tfrac{1}{2} a^2, \qquad [3.2]$$

where R_0 is the radius after the deformation.

Any arbitrary deformation of an incompressible fluid body can be realised by applying at each point of the body a displacement $\vec{\xi}=(\xi_r, \xi_\theta, \xi_z)$ which is such that $\vec{v} = (\partial\vec{\xi}/\partial t)$. Assuming the motion to be irrotational and since the plasma is incompressible, we have

$$\vec{\xi}=\text{grad } \psi, \qquad [3.3]$$

$$\text{div } \vec{\xi}=0. \qquad [3.4]$$

From [3.3] and [3.4] we obtain

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z^2}=0. \qquad [3.5]$$

Let

$$\psi=\psi(r)\cos kz, \qquad [3\ 6]$$

where $\psi(r)$ is the amplitude of the axisymmetric disturbance.

A solution of [3.5] which is regular at the axis of the deformed cylinder is

$$\psi=A\, I_0(kr)\cos kz \qquad [3.7]$$

Since at $r=1$, $\xi_r=a\cos kz$. [3.8]

Thus $A=a/[k\, I_1(kr)]$, [3.9]

where I_0 and I_1 are modified Bessel functions of first kind.

Hence the displacement $\delta\vec{\xi}$, which must be applied to increase the amplitude a to $a+\delta a$ is

$$\left.\begin{aligned}\delta\xi_r&=\delta a\,[I_1(kr)/I_1(k)]\cos kz,\\ \delta\xi_\theta&=0,\\ \delta\xi_z&=-[\delta a/I_1(k)]\,I_0(kr)\sin kz\end{aligned}\right\} \qquad [3.10]$$

The change in the potential energy $\delta(\Delta\Omega)$ per unit length of the cylinder involved in the infinitesimal deformation [3.10] can be obtained by integrating over the whole cylinder the work done by the displacement $\delta\vec{\xi}$ in the force field specified by gravitational potential. Following Chandrasekhar and Fermi[1] we find that this is given by

$$\delta(\Delta\Omega)=+2\pi\gamma\left[\tfrac{1}{2}-I_0(k)\,K_0(k)\right]a\,\delta a, \qquad [3.11]$$

where $K_0(k)$ is a modified Bessel function of second kind. Integrating [3.11] from $a=0$ to a the change in potential energy is

$$\Delta\,\Omega = 2\,\pi\,\gamma\,[\tfrac{1}{2} - I_0(k)\,K_0(k)]\,a^2, \qquad [3.12]$$

where

$$\gamma = \frac{2\,\pi\,G\,\rho^2\,R^2}{\mu_e\,H_1^2}.$$

The change in magnetic energy per unit length of the cylinder can be calculated as follows :

From magnetic induction equation

$$\vec{h} = \text{curl}\,(\vec{\xi} \times \vec{H}^{(0)}), \qquad [3.13]$$

where $\vec{h}$ is the induced field and $\vec{H}^{(0)}$ is the equilibrium field inside the plasma. From [3.1], [3.10] and [3.13] the total magnetic field after deformation is

$$\left.\begin{aligned} H_r &= -\frac{a\,k\,I_1(kr)}{I_1(k)}\sin k\,z\,, \\ H_\theta &= \frac{\lambda\,r}{1+\mu_p r^2}\left[1 + \frac{2\,a\,\cos kz}{(1+\mu_p\,r^2)}\,\frac{I_1(kr)}{I_1(k)}\,\mu_p\,r\right], \\ H_z &= 1 - a\,k\frac{I_0(kr)}{I_1(kr)}\cos kz\,. \end{aligned}\right\} \qquad [3.14]$$

The total current density after deformation is

$$\left.\begin{aligned} J_r &= \frac{2\,\lambda a\,k\,\mu_p\,r^2}{(1+\mu_p\,r^2)^2}\,\frac{I_1(kr)}{I_1(k)}\sin kz, \\ J_\theta &= 0\,. \\ J_z &= \frac{2\lambda}{(1+\mu_p\,r^2)} + \frac{2\,\mu_p\,\lambda\,a\cos kz}{r\,I_1(k)}\,\frac{\partial}{\partial r}\left(\frac{I_1(kr)\,r^3}{(1+\mu_p r^2)^2}\right) \end{aligned}\right\} \qquad [3.15]$$

The work done by the electromagnetic force $\vec{J} \times \vec{H}$

$$\delta\vec{\xi} \cdot (\vec{J} \times \vec{H}) \qquad [3.16]$$

The change in the magnetic energy per unit length of the cylinder is

$$\delta \triangle M_1 = -2\pi \left[\int_0^{1+a\cos kz} \vec{\delta\xi} \cdot (\vec{J} \times \vec{H})\, r\, dr \right]_{\text{Average}}, \qquad [3.17]$$

where the averaging is done with respect to z. From [3.10], [3.14], [3.15] and [3.17] we obtain

$$\triangle M_1 = \frac{\pi \lambda^2 a^2}{I_1(k)} \int_0^1 \left\{ 3 I_1(kr) + kr\, I_1'(kr) \right.$$

$$\left. - \frac{6 \mu_p r^2}{(1+\mu_p r^2)} I_1(kr) \right\} \frac{r^2}{(1+\mu_p r^2)^3}\, dr +$$

$$+ \frac{\pi \lambda^2 \mu_p a^2}{I_1^2(k)} \left[k \int_0^1 \frac{r^4}{(1+\mu_p r^2)^3} I_0(kr)\, I_1(kr)\, dr \right.$$

$$\left. + \int_0^1 \frac{r\, I_1(kr)}{(1+\mu_p r^2)} \frac{\partial}{\partial r} \left(\frac{r^3 I_1(kr)}{(1+\mu_p r^2)^2} \right) dr + 2 \int_0^1 \frac{r^3 I_1^2(kr)}{(1+\mu_p r^2)^4}\, dr \right] \qquad [3.18]$$

In order to find the work done by surface currents induced by the deformation of the cylinder, we have to calculate the field outside the cylinder, after deformation. Thetotal field in the vacuum

$$\left.\begin{aligned} H_r &= -\, ka\, H_2 \frac{K_1(kr)}{K_1(k)} \sin kz, \\ H_\theta &= 0 \\ H_z &= H_2 \left[1 + k a \frac{K_0(kr)}{K_1(k)} \cos kz \right] \end{aligned}\right\} \qquad [5.19]$$

and the surface current

$$\left.\begin{aligned} J_k^* &= \frac{\lambda}{(1+\mu p)} ak \sin kz \\ J_\theta^* &= -\left[H_2 - 1 + ak \cos kz \left\{ \frac{I_0(k)}{I_1(k)} + H_2 \frac{K_0(k)}{K_1(k)} \right\} \right] \\ J_z^* &= -\frac{\lambda}{1+\mu_p} \left[1 + \frac{2\mu_p a \cos kz}{1+\mu_p} \right] + \frac{\lambda(1-\mu_p)}{(1+\mu_p)^2} a \cos kz \end{aligned}\right\} \qquad [3.20]$$

The work done per unit length of the cylinder against the surface forces $\vec{J} \times \vec{\mathrm{H}}$ when the amplitude is changed from a to $a+\delta a$ is

$$\delta \triangle M_2 = -2\pi [\vec{J} \times \bar{\vec{\mathrm{H}}}_1 \quad \delta \vec{\xi}\, r]_{\text{Average}}, \tag{3.21}$$

where $\bar{\mathrm{H}}$ is the average field on the surface. Integrating [3.21] from $a=0$ to a, we obtain the change in the surface energy as

$$\triangle M_2 = \pi a^2 \left[H_2^2 k \frac{K_0(k)}{K_1(k)} + k \frac{I_0(k)}{2 I_1(k)} - \frac{3}{4} \cdot \frac{\lambda^2}{(1+\mu_p)^2} + \left(\frac{H_2^2 - 1}{4}\right) k \frac{I_1'(k)}{I_1(k)} \right.$$
$$\left. - \frac{\lambda^2 k}{4(1+\mu_p)^2} \cdot \frac{I_1'(k)}{I_1(k)} - \frac{k I_0(k)}{4 I_1(k)} \left\{ (H_2^2 - 1) - \frac{\lambda^2}{(1+\mu_p)^2} \right\} + \frac{H_2^2 - 1}{4} \right]. \tag{3.22}$$

If we include the surface tension T, the work done by this surface force is

$$\triangle M_3 = \frac{\pi a^2 T k}{2} \left[\frac{I_1'(k)}{I_1(k)} + k - \frac{I_0(k)}{I_1(k)} \right]. \tag{3.23}$$

Thus the total change in the energy, per unit length of the cylinder is

$$\triangle W = \triangle \Omega + \triangle M_1 + \triangle M_2 + \triangle M_3$$

$$\triangle' W = \frac{\triangle W}{\pi a^2} = \left[\frac{\lambda^2}{I_1(k)} \int_0^1 r^2 \left\{ \frac{2 I_1(kr) + kr I_0(kr)}{(1+\mu_p r^2)^3} \right\} dr \right.$$
$$+ \frac{\mu_p \lambda^2}{I_1^2(k)} \left\{ 2k \int_0^1 \frac{r^4 I_0(kr) I_1(kr)}{(1+\mu_p r^2)^3} dr + 2 \int_0^1 \frac{r^3 I_1^2(kr)}{(1+\mu_p r^2)^4} dr \right.$$
$$\left. \left. + 2 \int_0^1 \frac{r^3 I_1^2(kr)}{(1+\mu_p r^2)^3} dr \right\} - \frac{1}{2} \cdot \frac{\lambda^2}{(1+\mu_p)^2} \right]$$
$$- \left[\frac{6 \mu_p \lambda^2}{I_1(k)} \int_0^1 \frac{r^4 I_1(kr)}{(1+\mu_p r^2)^4} dr + \frac{4 \mu_p^2 - \lambda^2}{I_1^2(k)} \int_0^1 \frac{r^5 I_1^2(kr)}{(1+\mu_p r^2)^4} dr \right.$$
$$- \frac{k}{2} \left\{ H_2^2 \frac{K_0(k)}{K_1(k)} + \frac{I_0(k)}{I_1(k)} \right\} - \frac{T}{2}(k^2 - 1)$$
$$\left. - \gamma \left\{ \tfrac{1}{2} - I_0(k) K_0(k) \right\} \right]. \tag{3.24}$$

3. DISCUSSION OF RESULTS

The criterion that the system is stable or unstable depends on the sign of $\Delta' W$ for various wave numbers and steady state parameters. We shall discuss the following cases:

(i) $\lambda = 0, \quad \gamma = 0$: when magnetic field and the gravitational force are absent. In this case

$$\Delta' W = \frac{k}{2} \left\{ H_2^2 \frac{K_0(k)}{K_1(k)} + \frac{I_0(k)}{I_1(k)} \right\} + \frac{T}{2}(k^2 - 1).$$

If T is small $\Delta' W > 0$ for all $k > 0$. This means that a sharp pinch is stable.

In case $0 < k < 1$, then

$$\Delta' W = \frac{k}{2} \left\{ H_2^2 \frac{K_0(k)}{K_1(k)} + \frac{I_0(k)}{I_1(k)} \right\} - \frac{T}{2}(1 - k^2),$$

which implies that the system is stable or unstable according as

$$\frac{k}{2} \left\{ H_2^2 \frac{K_0(k)}{K_1(k)} + \frac{I_0(k)}{I_1(k)} \right\} > \text{ or } < \frac{T}{2}(1 - k^2).$$

Hence for $k = k^*$, $\Delta' W = 0$ for a given H_2 and for large T, then the system is stable for $k < k^*$ and unstable for $k > k^*$ where $0 < k^* < 1$.

(ii) $\mu_p = 0, \quad \gamma = 0$. The change in energy is

$$\Delta' W = \left[\frac{\lambda^2}{I_1(k)} S_1 - \frac{\lambda^2}{2} + \frac{k}{2} \left\{ H_2^2 \frac{K_0(k)}{K_1(k)} + \frac{I_0(k)}{I_1(k)} \right\} + \frac{T}{2}(k^2 - 1) \right];$$

where $S_1 = \int_0^1 r^3 \{ 2\, I_1(kr) + kr\, I_0(kr) \}\, dr > 0.$

Thus a diffuse linear pinch is stable or unstable according as

$$\frac{\lambda^2}{I_1(k)} S_1 + \frac{k}{2} \left\{ H_2^2 \frac{K_0(k)}{K_1(k)} + \frac{I_0(k)}{I_1(k)} \right\} + \frac{T}{2}(k^2 - 1) > \text{ or } < (\lambda^2/2),$$

for $k \geq 1$.

In case $0 < k < 1$, the system is stable or unstable according as

$$\frac{\lambda^2}{I_1(k)} S_1 + \frac{k}{2} \left\{ H_2^2 \frac{K_0(k)}{K_1(k)} + \frac{I_0(k)}{I_1(k)} \right\} > \text{ or } < (\lambda^2/2) + (T/2)(1 - k^2).$$

We note that the magnetic field as well as surface tension add to the stability of the system for $k > 1$, whereas for $0 < k < 1$ the surface tension has a destabilising influence on the system, but the magnetic field tends to stabilise the system.

(iii) $\lambda = 0$, $T = 0$, $\gamma = 0$. Then

$$\Delta' W = \frac{k}{2} \left\{ H_2^2 \frac{K_0(k)}{K_1(k)} + \frac{I_0(k)}{I_1(k)} \right\},$$

which is positive for all $k > 0$. Thus the system is stable for all $k > 0$. We conclude that the ring surface currents have a stabilising influence on the system.

(iv) When $\mu_p \neq 0$.

From the numerical computations done on Elliot 803, digital computer for the expression $\Delta' W$ we arrive at the following conclusions.

For choosen values of H_2, λ, T, γ, k and for any value of μ_p whether small or large it is seen that the system is always stable. Further it is seen from results that as the wave number k increases the value of $\Delta' W$ also increases and the effect of surface tension T and γ is to support for the stability of the system. The numerical values choosen and the value obtained are not reproduced here.

4. Normal Mode Analysis

In this section we shall discuss the stability of the system against axi-symmetric disturbances by the normal mode analysis. Thus we shall take all perturbed quantities to vary as

$$X = X(r) \exp(i k z + \omega t),$$

where ω is the frequency, k is the wave number and $X(r)$ is the amplitude of the perturbation. Using cylinderical co-ordinates r, θ, z^7, the linearised set of equations is

$$u_r \omega - \left(H_r ik + \frac{2 \lambda r}{r(1 + \mu_p r^2)} H_\theta \right)$$

$$= - \left[\frac{dP}{dr} + \frac{\lambda r}{(1 + \mu_p r^2)} \frac{dH_\theta}{dr} + H_\theta \frac{d}{dr} \left(\frac{\lambda r}{1 + \mu_p r^2} \right) + \frac{dH_z}{dr} \right] + \gamma \frac{d\phi}{dr} \quad [4.1]$$

$$u_\theta \omega - H_r \frac{d}{dr} \left(\frac{\lambda r}{1 + \mu_p r^2} \right) - H_\theta ik - \frac{H_r \lambda}{(1 + \mu_p r^2)} = 0, \quad [4.2]$$

$$u_z \omega = -ik\left[P(r) + \frac{\lambda r}{1+\mu_p r^2} H_\theta\right] + ik\,\gamma\,\phi, \tag{4.3}$$

$$\frac{du_r}{dr} + \frac{u_r}{r} + u_z(ik) = 0, \tag{4.4}$$

$$H_r\,\omega - u_r(ik) = 0, \tag{4.5}$$

$$H_\theta\,\omega - u_\theta(ik) + u_r\left[\frac{d}{dr}\left(\frac{\lambda r}{1+\mu_p r^2}\right) - \frac{\lambda}{1+\mu_p r^2}\right] = 0, \tag{4.6}$$

$$H_z\,\omega - u_z\,ik = 0, \tag{4.7}$$

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - k^2\phi = 0, \tag{4.8}$$

where velocity $= (u_r, u_\theta, u_z)$, Pressure $= P$, gravitational potential $= \phi$, induced magnetic field $= (H_r, H_\theta, H_z)$. In Passing we note that we have used the following characteristic quantities.

Lenth $= R$, Magnetic field $= H_1$, Velocity = Alfven Velocity (V_A)

Pressure $= \dfrac{\mu_e H_1^2}{4\pi\rho}$, Surface current density $= \dfrac{H_1}{4\pi R}$.

Solving [4.1]–[4.7] for u_θ, u_z, H_r, H_θ, H_z, P in terms of u_r, we have

$$u_\theta = \frac{2ik\lambda}{(1+\mu_p r^2)(\omega^2+k^2)} u_r, \tag{4.9}$$

$$u_z = \frac{i}{k}\left[\frac{du_r}{dr} + \frac{u_r}{r}\right], \tag{4.10}$$

$$H_r = \frac{ik}{\omega} u_r, \tag{4.11}$$

$$H_\theta = -\frac{u_r}{\omega(1+\mu_p r^2)}\left[\frac{\lambda^2(k^2-\omega^2)}{(\omega^2+k^2)} + \frac{\lambda(1-\mu_p r^2)}{(1+\mu_p r^2)}\right] \tag{4.12}$$

$$H_z = -\frac{1}{\omega}\left[\frac{du_r}{dr} + \frac{u_r}{r}\right], \tag{4.13}$$

$$P(r) = -\frac{\omega}{k^2}\left[\frac{du_r}{dr} + \frac{u_r}{r}\right] + \frac{\lambda r\,u_r}{\omega(1+\mu_p r^2)^2}\left[\frac{\lambda(k^2-\omega^2)}{(\omega^2+k^2)} + \frac{\lambda(1-\mu_p r^2)}{(1+\mu_p r^2)}\right] + \gamma\phi. \tag{4.14}$$

From [4.8] we obtain

$$\phi = C\, I_0 (kr)\,, \qquad [4.15]$$

where $I_0 (kr)$ is a modified bessel function of first kind and C is an arbitrary constant of integration.

Substituting [4.9]—[4.15] in [4.1], we obtain

$$\frac{d^2 u_r}{dr} + \frac{1}{r}\ \frac{d u_r}{dr} - \left[k^2 + \frac{1}{r^2} - \frac{4\, k^2\, \lambda^2\, (k^2 - \mu_p\, r^2\, \omega^2)}{(\omega^2 + k^2)^2\, (1 + \mu_r\, r^2)^3}\right] u_r = 0. \qquad [4.16]$$

We seek a perturbation solution for [4.16] owing to its complexity in solving. Let us assume μ_p as small and expand u_r and ω in powers of μ_p as

$$u_r = u_{r0} + \mu_p\, u_{r1} + 0\, (\mu_p^2), \qquad [4.17]$$

$$\omega = \omega_0 + \mu_p\, \omega_1 + 0\, (\mu_p^2)\,. \qquad [4.18]$$

Substituting [4.17] and [4.18] in [4.16], the zeroth and first order equations for u_{r0} and u_{r1} are

$$\frac{d^2 u_{r0}}{dr^2} + \frac{1}{r} \cdot \frac{du_{r0}}{dr} - \left[k^2 + \frac{1}{r^2} - \frac{4\, k^4\, \lambda^2}{(\omega_0 + k^2)^2}\right] u_{r0} = 0, \qquad [4.19]$$

and

$$\frac{d^2 u_{r1}}{dr^2} + \frac{1}{r} \cdot \frac{du_{r1}}{dr} - \left[k^2 + \frac{1}{r^2} - \frac{4\, k^4 \lambda^2}{(w_0^2 + k^2)^2}\right] u_{r1}$$

$$= \frac{4\, k^2\, \lambda^2}{(\omega_0^2 + k^2)^2} \left[\frac{4\, \omega_1\, \omega_0}{(\omega_0^2 + k^2)}\, k^2 + 3\, r^2 k^2 + r^2 \omega_0^2\right] u_{r0}\,. \qquad [4.20]$$

Putting $\alpha = k^2 - [4\, k^4\, \lambda^2 / (\omega_0^2 + k^2)^2]$ in [4.19] and solving we have its regular solution as

$$u_{r0} = A\ \mathrm{I}_1\ (\alpha\ r), \qquad [4.21]$$

where A is an arbitrary constant of integration and $I_1\ (\alpha r)$ is a modified Bessel function of first kind.

Solving [4.20] on using [4.21] by the method of variation of Parameters we have

$$u_{r1} = \beta_1'\, A\ \left[\frac{\alpha\ r^3}{6}\, I_0\, (\alpha r) - \frac{r^2}{3}\, I_1\, (\alpha r)\right] + \frac{\beta_2\, A}{\alpha^2} \left[\frac{\alpha\ r}{2}\, I_0\, (\alpha r) - I_1 (\alpha\, r)\right], \qquad [4.22]$$

where

$$\beta_1' = \frac{\beta_1}{\alpha^2}\,; \quad \beta_1 = \frac{4\,k^2\,\lambda^2}{(\omega_0^2+k^2)^2}\,(3\,k^2+\omega_0^2)\;; \quad \beta_2 = \frac{16\,\omega_1\,\omega_0\,k^4\,\lambda^2}{(\omega_0^2+k^2)^3}. \qquad [4.23]$$

From [4.13] and [4.21] and [4.23] we obtain

$$H_z = -(A/\omega_0)\left\langle \alpha\, I_0(\alpha r) + \mu_p \left\{ -(\omega_1/\omega_0)\;\alpha I_0(\alpha r) + \beta_1' \left[\tfrac{1}{3}\,\alpha r^2 I_0(\alpha r) \right.\right.\right.$$
$$\left.\left.\left. -(2r/3)\, I_1(\alpha r) + (\alpha^2 r^3/6)\, I_1(\alpha r)\right] + (\beta_2 r/2)\, I_1(\alpha r)\right\}\right\rangle. \qquad [4.24]$$

Simillarly we can find u_θ, u_z, H_r, H_θ and P from [4.9]—[4.12], [4.14] upto first order in μ_p.

5. SOLUTIONS IN VACUUM

These are

$$H_r = -Bk\,K_1(kr), \qquad [5.1]$$

$$H_\theta = 0, \qquad [5.2]$$

$$H_z = i\,k\,B\,K_0(kr), \qquad [5.3]$$

$$\phi = D\,K_0(kr), \qquad [5.4]$$

where B and D are constants. The linearised set of boundary conditions is

$$u_r = \omega\,\delta\,r, \qquad [5.5]$$

$$P = H_2\,H_{z0} - H_{zi} - H_{\theta i}^{(0)}\;H_{\theta i} + T\,\delta r\,(k^2-1), \qquad [5.6]$$

$$H_{r0} = H_2\,i\,k\,\delta\,r, \qquad [5.7]$$

$$\phi_i = \phi_0\,;\;\; (\partial\,\phi_{(i)}/\partial r) = (\partial\,\phi_{(0)}/\partial r)\;+\;2\,\delta\,r, \qquad [5.8]$$

where the subscripts i and o stand for inside and outside quantities. The boundary conditions [5.5]—[5.7] have to be applied at $r=1$.

6. DISPERSION RELATIONS AND DISCUSSION

We shall now obtain the zeroth and first order dispersion relations for ω_0 and ω_1 as follows:

From boundary conditions [5.7], [5.8] we obtain

$$B = -i\,H_2\,\delta r/K_1(k), \qquad [6.1]$$

$$C = 2\,\delta\,r\,K_0(k). \qquad [6.2]$$

Applying the boundary condition [5 6] we obtain

$$[(\omega^2/k^2)+1]\ H_{z_1}+\gamma\,\phi_1 = H_2\,H_{z0} + T\,\delta r\,(k^2-1), \qquad [6\ 3]$$

which gives us with [6.1] and [6 2] as

$$-(A/\omega_0)\,[1+\omega^2/k^2]\,\langle \alpha\, I_0(\alpha) + \mu_p\{-(\omega_1/\omega_0)\,\alpha\, I_0(\alpha)+\beta_1'[\tfrac{1}{3}\,\alpha\, I_0(\alpha)$$
$$-\tfrac{2}{3} I_1(\alpha)+(\alpha^2/6)\, I_1(\alpha)]+(\beta_2/2)\, I_1(\alpha)\}\rangle+[2\,\gamma\, I_0(k)\, K_0(k)$$
$$-k H_2^2[K_0(k)/K_1(k)]-T\,(k^2-1)]\;\delta r=0 \qquad [6.4]$$

From [5.5] we obtain

$$A\,\langle I_1(\alpha)+\mu_p[\beta_1'\{(\alpha/6)\, I_0(\alpha)-\tfrac{1}{3} I_1(\alpha)\}$$
$$+(\beta_2/\alpha_2)\{(\alpha^2/2)\, I_0(\alpha)-I_1(\alpha)\}]\rangle-(\omega_0+\mu_p\,\omega_1)\,\delta r=0 \qquad [6.5]$$

Equations [6.4] and [6.5] are two simultaneous homogeneous equations in A and δr. The condition for the existence of a non-trivial solution gives us the following zeroth and first order dispersion relations respectively:

$$I_1(\alpha)\,\{2\,\gamma\, K_0(k)\, I_0(k)-k\, H_2^2[K_0(k)/K_1(k)]-T\,(k^2-1)\}$$
$$-\alpha\, I_0(\alpha)\,[1+\omega_0^2/k^2]=0, \qquad [6.6]$$

and

$$\omega_1\left[\alpha\, I_0(\alpha)\,\frac{2\,\omega_0}{k^2}+\frac{8\,\omega_0 k^2\lambda^2\, I_1(\alpha)}{(\omega_0^2+k^2)}-\frac{16\,\omega_0\,\lambda^2 k^2}{\alpha\,(\omega_0^2+k^2)^3}\left\{\tfrac{1}{2} I_0(\alpha)-\frac{I_1(\alpha)}{\alpha}\right\}T'\right]$$
$$=-\frac{4\,\lambda^2}{\alpha\,(\omega_0^2+k^2)}\,(3\,k^2+\omega_0^2)\left\{\tfrac{1}{3}\, I_0(\alpha)-\tfrac{2}{3}\,\frac{I_1(\alpha)}{\alpha}+\frac{\alpha}{6}\, I_1(\alpha)\right\}$$
$$+\frac{4\,k^2\lambda^2}{\alpha\,(\omega_0^2+k^2)^2}\,(3\,k^2+\omega_0^2)\left\{\tfrac{1}{6}\, I_0(\alpha)-\tfrac{1}{3}\,\frac{I_1(\alpha)}{\alpha}\right\}T', \qquad [6.7]$$

where

$$T'=\{2\,\gamma\, I_0(k)\, K_0(k)-k H_2^2[K_0(k)/K_1(k)]-T\,(k^2-1)\}.$$

We shall write [6.6] in the form

$$\alpha\,\phi\,(\omega_0,\ \alpha,\ k,\ \lambda)=0 \qquad [6.8]$$

This implies that either $\alpha=0$ or $\phi=0$. In case $\alpha=0$, $\phi\neq 0$ the system is stable or unstable according as

$$k >\ \text{or}\ < 2\,\lambda. \qquad [6.9]$$

In case $k < 2\lambda$ we can write [6.7] in the form

$$\omega_1 = \frac{(\lambda + k)\,\lambda k^2\,(7k - 12\,\lambda)}{3\,[\sqrt{(2\lambda k - k^2)}]\,(16\lambda - 16\lambda^2 k^3 - 37k)}, \tag{6.10}$$

which implies that the corresponding first order effect is to damp or enhance the disturbance according as $\omega_1 <$ or > 0. However, the system as a whole is unstable.

In case $\alpha \neq 0$, $\phi = 0$, we find that the frequency of stable oscillations is increased due to the form of the field we have chosen[8]. Further for such modes we find that

(i) For fixed values of 'λ' and 'γ' as the wave number 'k' increases

$$\omega = \omega_0 + \mu_p\,\omega_1 \quad \text{increases}$$

(ii) For fixed 'λ' and 'k' the frequency increases as 'γ' increases and for fixed 'γ' and 'k' the frequency increases as 'λ' increases[8].

Also we note that the system is stable in the absence of the gravitational force and the azimuthal magnetic field. This result is in agreement with our conclusions drawn in section (3) of the present paper Further from the conclusion drawn in section (3) for small 'μ_p' we note that our results obtained by applying normal mode analysis are in good agreement.

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