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Surprising effects of minor viscosity gradients*

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Abstract

It is shown that a small stratification of viscosity in the flow through a channel, if carefully located to overlap the production layer of disturbance kinetic energy, can produce a large effect on flow stability. Some implications are discussed.

Keywords: Flow stabilization, viscosity stratification, flow control.

1. Introduction

The state of the flow of a fluid through a pipe or channel depends on its Reynolds number, *R*, which is the ratio of the inertial to the viscous forces in the flow. The flow is laminar and stable at low *R*. As the Reynolds number increases (Fig.1), the laminar flow goes through a series of instabilities, then small patches of turbulence appear in the flow which grow as *R* increases, until the entire flow is turbulent. The nature of the first instability is different in different flows: it may be caused by a single linear mode, or due to the nonlinear (modal or nonmodal) interaction of several perturbations. The precise values of *R* at which transition to turbulence begins and ends depends on many factors like the geometry of the flow, the roughness of the walls, the role of gravity and so on. We will discuss the effect of one property: transverse viscosity differences, on the linear instability of laminar flow through a two-dimensional channel.

Before we begin, let us review the current status of understanding in this area. The flow of a single fluid (of constant viscosity) has expectedly been studied for a long time and more is understood about it. Analytical studies of the initial stages of the instabilities (primary and secondary) are available; these are in agreement with experiment and direct numerical simulations. The later stages of disturbance growth and the breakdown into transitional flow have been obtained in direct numerical simulations of the Navier–Stokes equations: no theory exists which can predict transition. As for two-fluid flow, most work so far is on immiscible fluids. In the co-flow of oil and water, for example, the behaviour is qualitatively different from that of single-fluid flow. The earliest theory of instability in the co-flow of two immiscible fluids was by Yih [1], [2]. The literature on the subject is vast and continues to grow (see e.g. Timoshin and Hooper [3]). For a complete treatment of this subject, the reader is referred to Joseph and Renardy [4] and references, especially to the same authors, therein. We term these flows interface-dominated, because all the important dynamics take place close to the sharp boundary separating the two fluids. For these flows, stability analyses and direct simulations have been performed to obtain the behaviour of the early instability, but there is as yet a lot left to be understood in later stages on the route to turbulence. The effect on flow stability of a continuous viscosity variation (viscosity stratification) has again been investigated, though less extensively. Such viscosity

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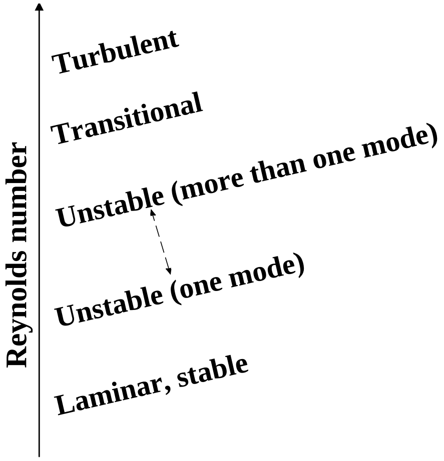

Fig. 1. Route to turbulence in a typical wall-bounded shear flow.

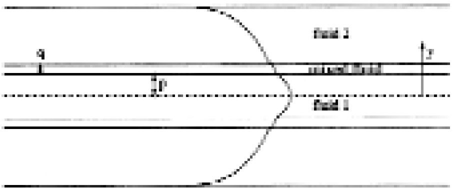

Fig. 2. Schematic of the flow.

variations could be created by transverse temperature variations, for example, as in Wazzan *et al.* [5] and Wall and Wilson [6], [7].

Global attention is increasingly being turned towards a more complicated but frequently occurring situation: miscible two-fluid flow. Several experiments in this area are proposed to be conducted in the International Space Station (e.g. Maxworthy *et al.* [8]): the purpose of conducting these experiments in space is to isolate the effect of viscosity stratification by eliminating errors introduced by the action of gravity on small density differences.

We study the stability of two miscible fluids of equal densities but different viscosities flowing through a channel (Fig. 2). The situation is in between that of a fully stratified flow and a sharp interface: we have a thin layer where the two fluids mix, and a local stratification of viscosity is created. We neglect here the effect on the stability of the downstream growth of the mixed layer thickness. This assumption is justified at high Peclet numbers, such as would occur in the co-flow of two alcohols, for example. The errors due to such an assumption at low Peclet numbers are estimated in Govindarajan [9]. A complete treatment of the problem including flow nonparallelism will be presented elsewhere.

Our main results are as follows. The stability of our flow is qualitatively different from both the interfacial instability of immiscible fluids and that of stratified fluids. A huge stabilization (or destabilization, if the sign of the viscosity-gradient is reversed) of about an order of magnitude is seen when the fluid at the wall is about 10% less viscous than that in the core for a carefully chosen location p of the mixed layer [10]. The apparently disproportionate effect of a minor change in flow conditions is due to the overlap of the stratification and the production layer (termed the critical layer) of the disturbance kinetic energy. A new mode of instability, called here the overlap mode, has been observed [9]. The behaviour persists over a wide range of fluid miscibilities (Peclet numbers going from 0 to over 10^6). At a higher Peclet number, the instability goes through a qualitative change and begins to resemble the well-known Yih mode of interfacial instability: the present work provides an extension into the miscible regime of the

Yih theory. The present results are additionally surprising given that the typical continuously stratified channel flow displays the opposite behaviour. There is, as is to be expected, no disproportionate effect observed in the continuously stratified case. We may argue that a mechanism derived from the overlap mode is in operation in turbulent drag reduction by polymers [11].

2. Stability analysis

The stability of this flow is described by the thermal Orr–Sommerfeld equations [6], which are derived from the 2D Navier–Stokes and scalar transport equations, respectively, by the standard procedure: the flow quantities (velocities and concentrations) are split into their respective means and perturbations. Since viscosity is a function of concentration alone, variations in concentration in the stability equation may be replaced by appropriate variations in viscosity. The perturbation stream function and viscosity respectively are written in the normal mode form as

$$[\hat{\phi}(x, y, t), \hat{\mu}(x, y, t)] = [\phi(y), \mu(y)]\exp[i\alpha(x - ct)], \tag{1}$$

where y is the normal distance from the center line, α, the disturbance wave number, c, the phase speed of the disturbance, x, the streamwise coordinate and t, the time. In the Navier–Stokes and scalar transport equations, all nonlinear terms in the perturbation are neglected, and the mean flow equations subtracted out, resulting in [9]

$$i\alpha\left[\left(\phi'' - \alpha^2\phi\right)(U - c) - U''\phi\right] = \frac{1}{R}\Big[\bar{\mu}\phi^{\mathrm{iv}} + 2\bar{\mu}'\phi''' + \left(\bar{\mu}'' - 2\alpha^2\bar{\mu}\right)\phi'' - 2\alpha^2\bar{\mu}'\phi' +$$

$$\left(\alpha^2\bar{\mu}'' + \alpha^4\bar{\mu}\right)\phi + U'\mu'' + 2U''\mu' + (U''' + \alpha^2 U')\mu\Big]. \tag{2}$$

and

$$i\alpha\left[(U - c)\mu - \bar{\mu}'\phi\right] = \frac{1}{Pe}\left(\mu'' - \alpha^2\mu\right) \tag{3}$$

with the boundary conditions

$$\phi(\pm 1) = \phi'(\pm 1) = \mu(\pm 1) = 0. \tag{4}$$

The Reynolds number and the Peclet number are defined, respectively, as $R \equiv \rho U_0 H/\bar{\mu}_1$ and $Pe \equiv U_0 H/\kappa$, where U_0 is the center line velocity, H, the channel half-width, and μ_1, the dimensional center line viscosity. $U(y)$ and $\bar{\mu}(y)$ are the nondimensional mean streamwise component of velocity and viscosity, respectively; primes denote differentiation with respect to y. The density is ρ and κ is the mass (or thermal, when appropriate) diffusivity. The equations describe an eigenvalue problem which is solved using a Chebychev spectral collocation method. Details of the computational method are available in Govindarajan [9]. The extent of the mixed layer q is taken equal to 0.1 unless otherwise specified, but our results are qualitatively similar for $0.01 \leq q \leq 0.2$.

3. Main results

In Fig. 3 is plotted the minimum instability Reynolds number R_i as a function of the extent p of fluid 1. The viscosity of fluid 2 differs from that of fluid 1 by 20%. One would expect the change in the stability to be of this order of magnitude too. At small p, this is indeed the case. However,

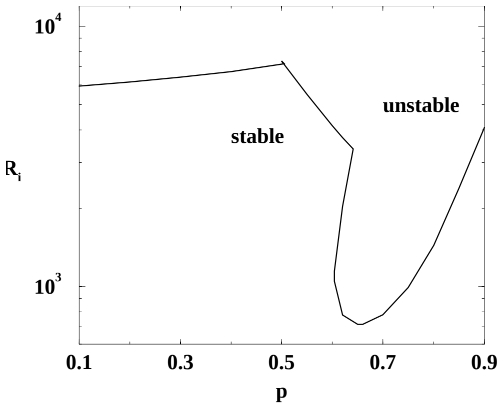

Fig. 3. The minimum instability Reynolds number as a function of the extent p of fluid 1. The Peclet number is zero. $\mu_2 = 1.2$.

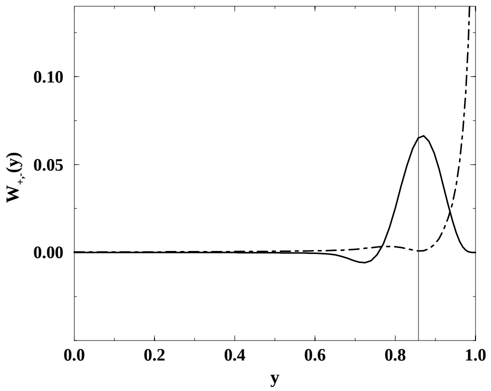

Fig. 4. Energy balance: production $W_+(y)$, solid line; dissipation $W_-(y)$, dot-dashed line, $R = 5772$. $\mu_2 = 1$ (single fluid). In this and the subsequent two figures the solid vertical lines show the location of the critical point, whereas the region between the dotted lines (in the next two figures) is the viscosity-stratified layer [11].

at some value of p the stability behaviour becomes qualitatively different, with a destabilization of an order of magnitude when fluid 2 is more viscous and a correspondingly dramatic stabilization (not shown, at a slightly different value of p) when it is less viscous (see [9], [10] for more results).

The reason for a strongly 'p-dependent' stability is that the effect of viscosity stratification is large only when the stratified layer coincides with the critical layer of the dominant disturbance. This will become evident upon examining the energy balance in the channel. The distribution of the production and dissipation of disturbance kinetic energy across the channel, in the absence of viscosity stratification (flow of a single fluid), is shown in Fig. 4 [11].

The integrals under both curves are equal here, since the Reynolds number is neutral. The manner in which these quantities are affected by a mixed layer created by fluid 2 being 10% less viscous is shown at the same Reynolds number in Figs 5 and 6 [11]. It is seen that when the mixed layer and the critical layer are well separated, as in Fig. 5, there is hardly any effect. However, when they are overlapping (Fig. 6), a large 'negative production' of disturbance kinetic energy is evident, which means that the disturbances are transferring energy to the mean flow (and therefore decaying). The dissipation, on the other hand, is affected very little. In other words, the flow remains near-neutral at $R = 5772$ when the mixed layer is separated from the production layer, while 5772 is a subcritical (very stable) Reynolds number for the same mode of instability when the two layers overlap.

At $\mu_2 > 1$, a new instability, called here the overlap (O) mode [9], is observed. The O mode is distinct from both the Tollmien-Schlichting (TS) and the inviscid (I) instabilities (while the I mode requires an inflexional velocity profile, the O mode does not). The overlap mode is always dominant (i.e. occurs at lower Reynolds numbers) than the other two. Figure 7 [9] shows the regimes of existence of the different modes of instability. The effect of reduced miscibility (increasing Peclet number) has been studied [9] and shows a crossover to immiscible flow

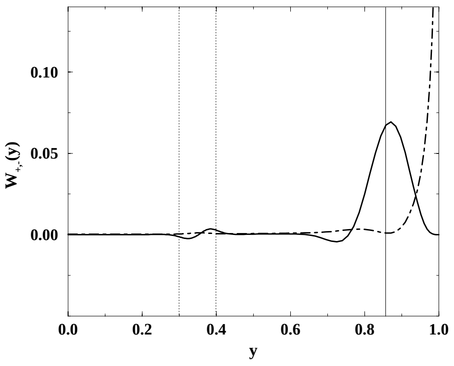

Fig. 5. The same as Fig. 4, but with $\mu_2 = 0.9$ and $p = 0.3$ [11].

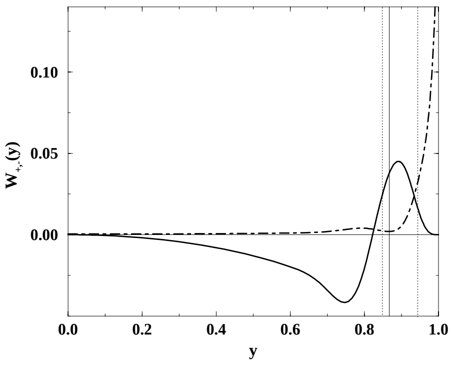

Fig. 6. The same as Fig. 4, but with $\mu_2 = 0.9$ and $p = 0.85$ [11].

behaviour at high Schmidt number (the Schmidt number is given by $Sc \equiv Pe/R$), as shown in Fig. 8. The fluid viscosities differ by only 5% here. At high diffusivity levels, i.e. when $Sc \sim 10$ or less, the TS and overlap mode are merged. Given that a single fluid ($\mu_2 = 1$) would become unstable only at $R = 5772$, it is remarkable how destabilizing a small difference in viscosity can be. The regimes of instability of the TS and O modes are coalesced at these Schmidt numbers. At intermediate levels of diffusivity ($Sc = 1000$ in the figure), the O mode is seen to be unstable in a domain distinct from that of the TS. For $Sc = 10^5$, i.e. when the fluids diffuse extremely slowly into each other, the behaviour is seen to be qualitatively different. The flow is unstable at very low Reynolds number and over a wide range of wave numbers, resembling the well-known Yih mode of instability for immiscible interfaces.

4. Discussion

In order to isolate the mechanism for the overlap mode, we must examine the lowest-order effects in the critical layer. The approach followed is similar to that described in Govindarajan and Narasimha [12]: we use asymptotic expansions, estimate the relevant small parameters (length scales), and transform the equations to the appropriate coordinates. Given below is the lowest-order equation valid for $Sc \leq O(1)$ (moderate to high diffusivity).

$$i\alpha\eta[U_c'\phi_0'' + \chi\alpha_3\phi_0] = \bar{\mu}\,\phi_0^{\mathrm{iv}} - 2\chi a_1\phi_0''' + \chi^2 a_2\phi_0''. \tag{5}$$

Here, x is the ratio of the thickness of the critical layer to that of the mixed layer, η, the critical layer coordinate, and a_1 and a_2 are $O(1)$ constants related to the gradients of the mean viscosity [9]. a_3 is an $O(1)$ constant directly related to U_c'', the second derivative of the mean velocity at the critical point. For purposes of demonstration, the above equation has been written for $\bar{\mu} \sim O(1)$, but corresponding equations may of course be written for any viscosity ratio. The key player in the critical layer is U_c'', which depends sensitively on the viscosity stratification, as discussed in Govindarajan *et al.* [11]. Deviations in this quantity are of opposite sign when the viscosity stratification switches sign; hence the stabilization in one case and destabilization in the other. When the flow is fully stratified, this surprising behaviour disappears, since $\chi \ll 1$, and the traditional balance [13], [14] prevails in the critical layer.

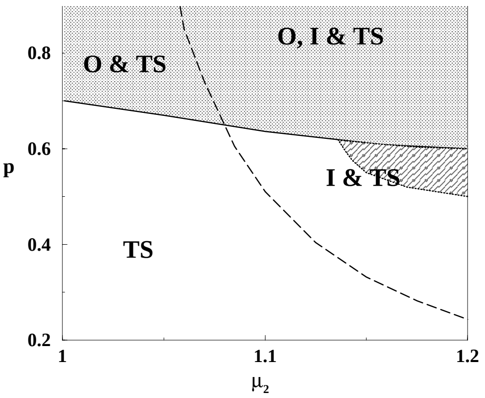

Fig. 7. Schematic of the regimes of existence and dominance of instabilities due to the O and I modes at high Reynolds number. Unstable TS modes exist everywhere in the regime. The O mode exists above the solid line and is dominant mode wherever it occurs (in the shaded region above the solid line), whereas the I mode exists to the right of the dashed line and is dominant over the TS only in the small shaded region shown.

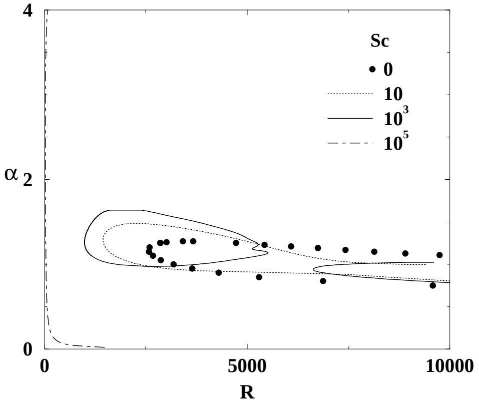

Fig. 8. Dependence of stability on diffusivity, $\mu_2 = 1.05$, $p = 0.75$.

The streamwise development of the flow due to the growth of the mixed layer has been neglected in this part of the study. A real flow would not be parallel, especially if the Peclet number were low, as it would be if the mixed layer were created by a temperature difference between the outer and inner layers: the effect of relaxing this assumption is currently under investigation. Preliminary work shows that the large effect on stability of the overlap mechanism persists through secondary disturbance growth as well, which gives rise to hope that a small layer of viscosity stratification, carefully placed so as to coincide with the critical layer of the dominant disturbance, may be used in flow control.

Another class of problems where the mechanism is possibly in operation is in the reduction of turbulent drag due to polymers. As has been well known for the last 50 years, a small amount of long-chain polymer (in concentrations of parts per million) introduced into turbulent flow has the effect of reducing the drag by up to 70% [15]–[18]. The large drag reduction obtained is not apparently commensurate with the small quantity of polymer added. Since polymeric solutions are often shear-thinning, a selective viscosity stratification in the annular region is generated naturally, with a region of constant viscosity extending through most of the core. A turbulent flow, especially one of a polymeric solution, is of course much more complicated than the simple flow considered here. The normal location of the production layer of turbulence, however, is similar to that for a disturbance mode in laminar flow. The production layer in this case too overlaps with the viscosity-stratified layer. We may at this point only suggest the present mechanism as one of the likely candidates for drag reduction. Further studies are needed to confirm our contention that it is the production of turbulent kinetic energy that is strongly suppressed by the action of the polymer, rather than a dramatic modification of the dissipation.

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