RADIATION FROM A SEMI-INFINITE DIELECTRIC-COATED SPHERICALLY TIPPED PERFECTLY CONDUCTING CONE PART I

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ABSTRACT

The solution to the problem of electromagnetic radiation from a semi-infinite dielectric-coated spherically tipped perfectly conducting cone excited by delta function sources has been obtained by using the orthogonal properties of Sommerfeld's complex-order spherical Hankel wave functions. The posibility of radiation of the symmetric as well as unsymmetric TM, TE *and hybrid modes from such a structure is discussed.*

1. INTRODUCTION

The problems of radiation and scattering of electromagnetic waves by a perfectly conducting cone has been studied by many authors[1-8, 12]. The exact solution for the problem of electromagnetic radiation from a circularly symmetric slot on the conducting surface of a semi-infinite dielectric-coated spherically tipped conducting cone has been obtained by Yeh[9] for the symmetric *TM* wave. It is the purpose of this paper to discuss the possibility of radiation of symmetric as well as unsymmetric *TM*, *TE*, and hybrid waves from such a structure.

2. THE STATEMENT OF THE PROBLEM

The geometry of the structure is given in Fig. 1. Spherical coordinates (r, θ, ϕ) are used. The vertex of the cone is taken at the origin of the coordinate system. To eliminate the singularity at the vertex, a small perfectly conducting spherical boss of radius a, with its centre at the origin is situated at the tip of the cone. The outer boundary of the perfectly conducting cone is $\theta = \theta_0$, and the outer boundary of the dielectric coating is $\theta = \theta_1$. The permittivity and permeability of the dielectric coating are ϵ_1 and μ_1 respectively. This radiating structure is embedded in a perfect dielectric medium of permittivity and permeability ϵ_2 and μ_2 respectively.

The excitation of the structure is by means of a radial electric field delta-function source for *TM* modes, a radial magnetic field delta-function source for *TE* modes, and a combination of these two sources for hybrid modes.

3. TM Modes

Assuming variation of all quantities with time as $\exp(-j\omega t)$, the field components inside the dielectric sheath is[10]

$$E_{ri} = \sum_{n=0}^{\infty} \frac{n(n+1)}{k_1} \frac{1}{r} h_n^{(1)}(k_1 r) \cos(m\phi) [A_n^m P_n^m(\cos\theta) + B_n^m Q_n^m \cos\theta)] \times \exp(-j\omega t) \quad [1]$$

$$E_{\theta i} = \sum_{n=0}^{\infty} \frac{1}{k_1} \frac{1}{r} \frac{d}{dr} [r h_n^{(1)}(k_1 r)] \cos(m\phi) \times \frac{d}{d\theta} [A_n^m P_n^m(\cos\theta) + B_n^m Q_n^m(\cos\theta)] \exp(-j\omega t) \quad [2]$$

$$E_{\phi i} = \sum_{n=0}^{\infty} \frac{1}{k_1} \frac{1}{r} \frac{d}{dr} [r h_n^{(1)}(k_1 r)] \frac{1}{\sin\theta} m \sin(m\phi) \times [A_n^m P_n^m(\cos\theta) + B_n^m Q_n^m(\cos\theta)] \exp(-j\omega t) \quad [3]$$

$$H_{ri} = 0 \quad [4]$$

$$H_{\theta i} = -\sum_{n=0}^{\infty} \frac{k_1}{j\omega\mu_1} h_n^{(1)}(k_1 r) \frac{1}{\sin\theta} m \sin(m\phi) \times [A_n^m P_n^m(\cos\theta) + B_n^m Q_n^m(\cos\theta)] \exp(-j\omega t) \quad [5]$$

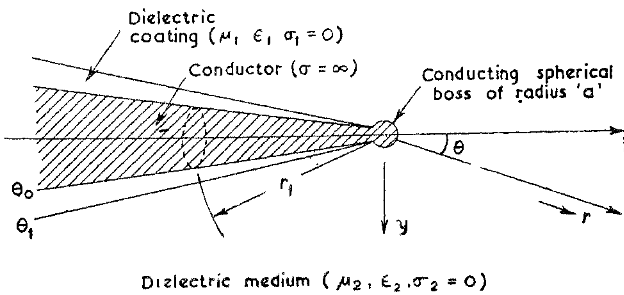

Fig. 1
The geometry of the structure

and

$$H_{\phi i} = -\sum_{n=0}^{\infty} \frac{k_1}{j\omega\mu_1} h_n^{(1)}(k_1 r)\cos(m\phi)\frac{d}{d\theta}\times$$

$$[A_n^m P_n^m(\cos\theta) + B_n^m Q_n^m(\cos\theta)]\exp(-j\omega t) \quad [6]$$

and the field components outside the dielectric sheath is

$$E_{re} = \sum_{n=0}^{\infty}\frac{n(n+1)}{k_2}\frac{1}{r}h_n^{(1)}(k_2 r)\cos(m\phi)[C_n^m P_n^m \cos\theta)]\exp(-j\omega t) \quad [7]$$

$$E_{\theta e} = \sum_{n=0}^{\infty}\frac{1}{k_2}\frac{1}{r}\frac{d}{dr}[r h_n^{(1)}(k_2 r)]\cos(m\phi)\frac{d}{d\theta}[C_n^m P_n^m(\cos\theta)]\exp(-j\omega t) \quad [8]$$

$$E_{\phi e} - \sum_{n=0}^{\infty}\frac{1}{k_2}\frac{1}{r}\frac{d}{dr}[r h_n^{(1)}(k_2 r)]\frac{m}{\sin\theta}\sin(m\phi)[C_n^m P_n^m \cos\theta)]\exp(-j\omega t) \quad [9]$$

$$H_{re} = 0 \quad [10]$$

$$H_{\theta e} = -\sum_{n=0}^{\infty}\frac{k_2}{j\omega\mu_2}h_u^{(1)}(k_2 r)\frac{1}{\sin\theta}m\sin(m\phi)[C_n^m P_n^m(\cos\theta)]\exp(-j\omega t) \quad [11]$$

and

$$H_{\phi e} = -\sum_{n=0}^{\infty}\frac{k_2}{j\omega\mu_2}h_n^{(1)}(k_2 r)\cos(m\phi)\frac{d}{d\theta}[C_n^m P_n^m(\cos\theta)]\exp(-j\omega t) \quad [12]$$

where $k_1 = \omega\sqrt{(\mu_1\epsilon_1)}$, $k_2 = \omega\sqrt{(\mu_2\epsilon_2)}$ and A_n^m, B_n^m, C_n^m are arbitrary constants to be determined by the boundary conditions and $m = 0, 1, 2, \cdots$ Applying the boundary condition that the tangential electric field components E_ϕ and E_θ must vanish for all ϕ and all θ on the conducting spherical boss of radius a, gives

$$\frac{d}{dr}[r h_n^{(1)}(k_1 r)]_{r=a} = 0 \qquad \text{inside the dielectric sheath} \quad [13]$$

and

$$\frac{d}{dr}[r h_n^{(1)}(k_2 r)]_{r=a} = 0 \qquad \text{outside the dielectric sheath} \quad [14]$$

The roots of Eqns. [13] and [14] are designated n_ν and n_μ respectively. n is put equal to n_ν in eqns. [1] to [6] and n is put equal to n_μ in eqns. [7] to [12], and the summation is taken over all values of n_ν and n_μ respectively.

For TM modes, let the excitation be

$$E_r^{app} = E_0\, d(r)\exp(-j\omega t)\cos(m\phi) \quad \text{for } \theta = \theta_0 \quad [15]$$

where

$$d(r) = 1 \ \text{for} \ r_0 < r < r_1,$$
$$= 0 \ \text{for} \ a < r < r_0, \ \text{and} \ r > r_1.$$

Expanding the applied field in terms of Sommerfeld's complex-order wave functions[11] gives

$$E_r^{app} = (1/k_1 r) \sum_{n_\nu} L_{n_\nu}^m n_\nu (n_\nu + 1) \, h_{n_\nu}^{(1)}(k_1 r) \, P_{n_\nu}^m (\cos\theta_0) \cos(m\phi) \exp(-j\omega t), \quad [16]$$

where $$L_{n_\nu}^m = \frac{1}{n_\nu(n_\nu+1) \, P_{n_\nu}^m(\cos\theta_0) \, N_{n_\nu}(k_1 a)} \int_{r_0}^{r_1} E_0 \, r \, h_{n_\nu}^{(1)}(k_1 r) \, d(k_1 r) \quad [17]$$

in which the normalising factor is

$$N_{n_\nu}(k_1 a) = \int_{k_1 a}^{\infty} [h_{n_\nu}^{(1)}(k_1 r)]^2 \, d(k_1 r) \quad [18]$$

By assuming $d(r)$ to be a delta-function source,

$$L_{n_\nu}^m = \frac{E_0 \, r_1 \, k_1 \, h_{n_\nu}^{(1)}(k_1 r_1)}{n_\nu(n_\nu+1) \, P_{n_\nu}^m(\cos\theta_0) \, N_{n_\nu}(k_1 a)} \quad [19]$$

Now applying the boundary conditions that the tangential electric and magnetic field components E_r, E_ϕ and H_ϕ are continuous at $\theta = \theta_1$, we obtain

$$\sum_{n_\mu} \frac{n_\mu(n_\mu+1)}{k_2 r} h_{n_\mu}^{(1)}(k_2 r) \, [C_{n_\mu}^m P_{n_\mu}^m(\cos\theta_1)] = \sum_{n_\nu} \frac{n_\nu(n_\nu+1)}{k_1 r} h_{n_\nu}^{(1)}(k_1 r) \times$$
$$[A_{n_\nu}^m P_{n_\nu}^m(\cos\theta_1) + B_{n_\nu}^m Q_{n_\nu}^m(\cos\theta_1)] \quad [20]$$

$$\sum_{n_\mu} \frac{1}{k_2 r} \frac{d}{dr} [r \, h_{n_\mu}^{(1)}(k_2 r)] \, [C_{n_\mu}^m P_{n_\mu}^m(\cos\theta_1)]$$
$$= \sum_{n_\nu} \frac{1}{k_1 r} \frac{d}{dr} [r \, h_{n_\nu}^{(1)}(k_1 r)] \, [A_{n_\nu}^m P_{n_\nu}^m(\cos\theta_1) + B_{n_\nu}^m Q_{n_\nu}^m(\cos\theta_1)] \quad [21]$$

$$\sum_{n_\mu} \frac{k_2}{j\omega\mu_2} h_{n_\mu}^{(1)}(k_2 r) \frac{d}{d\theta_1} [C_{n_\mu}^m P_{n_\mu}^m(\cos\theta_1)]$$
$$= \sum_{n_\nu} \frac{k_1}{j\omega\mu_1} h_{n_\nu}^{(1)}(k_1 r) \frac{d}{d\theta_1} [A_{n_\nu}^m P_{n_\nu}^m(\cos\theta_1) + B_{n_\nu}^m Q_{n_\nu}^m(\cos\theta_1)] \quad [22]$$

The boundary conditions [20], [21] and [22] cannot be satisfied by equating each term of the series expansion. These conditions have to be satisfied for all values of r from $r=a$ to $r=\infty$. Consequently, the orthogonal property of the spherical Bessel functions[11] is made use of. Therefore, substitute the following in the above three equations [20], [21] and [22]

$$h_{n_\nu}^{(1)}(k_1 r) = \sum_{n_\mu} \alpha_{n_\nu, n_\mu} h_{n_\mu}^{(1)}(k_2 r) \quad [23]$$

where,

$$\alpha_{n_\nu, n_\mu} = \frac{1}{M_{n_\mu}} \int_{k_2 a}^{\infty} h_{n_\nu}^{(1)}(k_1 r)\, h_{n_\mu}^{(1)}(k_2 r)\, d(k_2 r) \quad [24]$$

and

$$M_{n_\mu} = \int_{k_2 a}^{\infty} [h_{n_\nu}^{(1)}(k_2 r)]^2\, d(k_2 r) \quad [25]$$

The equations [20], [21] and [22] then become

$$\sum_{n_\mu} \frac{n_\mu (n_\mu + 1)}{k_2 r} h_{n_\mu}^{(1)}(k_2 r) [C_{n_\mu}^m P_{n_\mu}^m (\cos\theta_1)]$$

$$= \sum_{n_\nu} \frac{n_\nu (n_\nu + 1)}{k_1 r} [\sum_{n_\mu} \alpha_{n_\nu, n_\mu} h_{n_\mu}^{(1)}(k_2 r)] [A_{n_\nu}^m P_{n_\nu}^m (\cos\theta_1) + B_{n_\nu}^m Q_{n_\nu}^m (\cos\theta_1) \quad [26]$$

$$\sum_{n_\mu} \frac{1}{k_2 r} \frac{d}{dr} [r h_{n_\mu}^{(1)}(k_2 r)] [C_{n_\mu}^m P_{n_\mu}^m (\cos\theta_1)]$$

$$= \sum_{n_\nu} \frac{1}{k_1 r} \frac{d}{dr} [r \sum_{n_\mu} \alpha_{n_\nu, n_\mu} h_{n_\mu}^{(1)}(k_2 r)] [A_{n_\mu}^m P_{\nu_\nu}^m (\cos\theta_1) + B_{n_\nu}^m Q_{n_\nu}^m (\cos\theta_1)] \quad [27]$$

$$\sum_{n_\mu} \frac{k_2}{j\omega\mu_2} h_{n_\mu}^{(1)}(k_2 r) \frac{d}{d\theta_1} [C_{n_\mu}^m P_{n_\mu}^m (\cos\theta_1)]$$

$$= \sum_{n_\nu} \frac{k_1}{j\omega\mu_1} [\sum_{n_\nu} \alpha_{n_\nu, n_\mu} h_{n_\mu}^{(1)}(k_2 r)] \frac{d}{d\theta_1} [A_{n_\nu}^m P_{n_\nu}^m (\cos\theta_1) + B_{n_\nu}^m Q_{n_\nu}^m (\cos\theta_1)] \quad [28]$$

Also the applied electric field E_r^{app} must be equal to E_{ri} for all values of r between a and ∞ for $\theta = \theta_0$. Therefore,

$$E_r^{app} = \frac{1}{k_1 r} \sum_{n_\nu} L_{n_\nu}^m n_\nu (n_\nu + 1)\, h_{n_\nu}^{(1)}(k_1 r)\, P_{n_\nu}^m (\cos\theta_0) = E_{rt}\, (at = \theta = \theta_0)$$

$$= \sum_{n_\nu} \frac{n_\nu (n_\nu + 1)}{k_1 r} \; h_{n_\nu}^{(1)} (k_1 r) \, [A_{n_\nu}^m P_{n_\nu}^m (\cos \theta_0) + B_{n_\nu}^m Q_{n_\nu}^m (\cos \theta_0)] \qquad [29]$$

Equating coefficients of $h_{n_\mu}^{(1)} (k_2 r)$ in equations [26] and [28], those of $d\,[r\, h_{n_\mu}^{(1)} (k_2 r)]/dr$ in equation [27] and those of $h_{n_\nu}^{(1)} (k_1 r)$ in equation [29], we have

$$\frac{n_\mu (n_\mu + 1)}{k_2} [C_{n_\mu}^m P_{n_\mu}^m (\cos \theta_1)]$$

$$= \sum_{n_\nu} \alpha_{n_\nu, n_\mu} \frac{n_\nu (n_\nu + 1)}{k_1} [A_{n_\nu}^m P_{n_\nu}^m (\cos \theta_1) + B_{n_\nu}^m Q_{n_\nu}^m (\cos \theta_1)] \qquad [30]$$

$$\frac{1}{k_2} [C_{n_\mu}^m P_{n_\mu}^m (\cos \theta_1)]$$

$$= \sum_{n_\nu} \frac{1}{k_1} \alpha_{n_\nu, n_\mu} [A_{n_\nu}^m P_{n_\nu}^m (\cos \theta_1) + B_{n_\nu}^m Q_{n_\nu}^m (\cos \theta_1)] \qquad [31]$$

$$\frac{k_2}{j \omega \mu_2} \frac{d}{d \theta_1} [C_{n_\mu}^m P_{n_\mu}^m (\cos \theta_1)]$$

$$= \sum_{n_\nu} \frac{k_1}{j \omega \mu_1} \alpha_{n_\nu, n_\mu} \frac{d}{d \theta_1} [A_{n_\nu}^m P_{n_\nu}^m (\cos \theta_1) + B_{n_\nu}^m Q_{n_\nu}^m (\cos \theta_1)] \qquad [32]$$

$$L_{n_\nu}^m P_{n_\nu}^m (\cos \theta_0) = [A_{n_\nu}^m P_{n_\nu}^m (\cos \theta_0] + B_{n_\nu}^m Q_{n_\nu}^m (\cos \theta_0)] \qquad [33]$$

We have four sets of equations [30] to [33] and three sets of unknowns $A_{n\nu}^m$. $B_{n\nu}^m$ and $C_{n\mu}^m$ and these equations are independent of each other. Hence, there is no unique solution for these unknown coefficients. This shows that, in general, the unsymmetric *TM* modes cannot be excited on the dielectric coated spherically tipped conducting cone.

But for the special case $m = 0$, *i.e.*, for the TM symmetric mode, $E_\phi = H_r = H_\theta = 0$, and only E_r, E_θ and H_ϕ will exist. Proceeding as before and applying the boundary conditions for the tangential components E_r and H_ϕ at $\theta = \theta_1$, and equating the E_r component to E_r^{app} at $\theta = \theta_0$, the following three sets of equations are obtained for the three sets of unknowns $A_{n\nu}^o$. $B_{n_\nu}^o$ and $C_{n_\nu}^o$.

$$\frac{n_\mu (n_\mu + 1)}{k_2} [C_{n_\mu}^0 \; P_{n_\mu} \cos \theta_1)] = \sum_{n_\nu} \alpha_{n_\nu, n_\mu} \frac{n_\nu (n_\nu + 1)}{k_1} \times$$

$$[A_{n_\nu}^0 P_{n_\nu} (\cos \theta_1) + B_{n_\nu}^0 Q_{n_\nu} (\cos \theta_1)] \qquad [34]$$

$$\frac{k_2}{j\,\omega\mu_2}\,\frac{d}{a\,\theta_1}\,[C^0_{n_\mu}\,\mathrm{P}_{n_\mu}(\cos\theta_1)] = \sum_{n_\nu}\,\frac{k_1}{j\,\omega\mu_1}\,\alpha_{n_\nu},{}_{n_\mu}\,\frac{d}{d\,\theta_1}\times$$

$$[A^0_{r_\nu}\,P_{n_\nu}(\cos\theta_1) + B^0_{n_\nu}\,Q_{n_\nu}(\cos\theta_1)] \qquad [35]$$

and

$$L^0_{n_\nu}\,\mathrm{P}_{n_\nu}(\cos\theta_0) = [A^0_{n_\nu}\,P_{n_\nu}(\cos\theta_0) + B^0_{u_\nu}\,Q_{n_\nu}(\cos\theta_0)] \qquad [36]$$

There is a unique solution for the unknown coefficients $A^0_{n_\nu}$, $B^0_{n_\nu}$ and $C^0_{n_\mu}$ and hence the field components for the symmetric *TM* mode can be uniquely determined. This shows that the *TM* symmetric mode can exist on this type of structure.

4. *TE* MODES

The field inside the dielectric sheath is

$$E_{ri} = 0 \qquad [37]$$

$$E_{\theta i} = \sum_{n=0}^{\infty} h^{(1)}_n(k_1 r)\,\frac{1}{\sin\theta}\,m\cos(m\phi)\,[A'^m_n\,P^m_n(\cos\theta) + B'^m_n Q^m_n(\cos\theta)]\ \exp(-j\omega t) \qquad [38]$$

$$E_{\phi i} = -\sum_{n=0}^{\infty} h^{(1)}_n(k_1 r)\,\sin(m\phi)\,\frac{d}{d\theta}\,[A'^m_n\,P^m_n\cos\theta) + B'^m_n\,Q^m_n(\cos\theta)]\exp(-j\omega t) \qquad [39]$$

$$H_{ri} = \sum_{n=0}^{\infty}\,\frac{n(n+1)}{j\,\omega\mu_1}\,\frac{1}{r}\,h^{(1)}_n(k_1 r)\,\sin(m\phi)\,[A'^m_n\,P^m_n(\cos\theta)$$

$$+ B'^m_n\,Q^m_n(\cos\theta)]\exp(-j\omega t) \qquad [40]$$

$$H_{\theta i} = \sum_{n=0}^{\infty}\,\frac{1}{j\omega\mu_1}\,\frac{1}{r}\,\frac{d}{dr}\,[r\,h^{(1)}_n(k_1 r)]\sin(m\phi)\,\frac{d}{d\theta}\,[A'^m_n\,P^m_u(\cos\theta)$$

$$+ B'^m_n\,Q^m_n(\cos\theta)]\ \exp(-j\omega t) \qquad [41]$$

$$H_{\phi i} = \sum_{n=0}^{\infty}\,\frac{1}{j\omega\mu_1}\,\frac{1}{r}\,\frac{d}{dr}\,[r\,h^{(1)}_n(k_1\,r)]\,\frac{m}{\sin\theta}\,\cos(m\phi)\,[A'^m_n\,P^m_n(\cos\theta)$$

$$+ B'^m_n\,Q^m_n\cos\theta)]\exp(-j\omega t) \qquad [42]$$

The field outside the dielectric sheath is

$$E_{re}=0 \quad [43]$$

$$E_{\theta e}=\sum_{n=0}^{\infty} h_n^{(1)}(k_2 r)\,\frac{1}{\sin\theta}\,m\cos(m\phi)\,[C_n'^m P_n^m(\cos\theta)]\exp(-j\omega t) \quad [44]$$

$$E_{\phi e}=-\sum_{n=0}^{\infty} h_n^{(1)}(k_2 r)\sin(m\phi)\,\frac{d}{d\theta}[C_n'^m P_n^m(\cos\theta)]\exp(-j\omega t) \quad [45]$$

$$H_{re}=\sum_{n=0}^{\infty}\frac{n(n+1)}{j\omega\mu_2}\,\frac{1}{r}\,h_n^{(1)}(k_2 r)\sin(m\phi)\,[C_n'^m P_n^m(\cos\theta)]\exp(-j\omega t) \quad [46]$$

$$H_{\phi e}=\sum_{n=0}^{\infty}\frac{1}{j\omega\mu_2}\,\frac{1}{r}\,\frac{d}{dr}[r h_n^{(1)}(k_2 r)]\sin(m\phi)\,\frac{d}{d\theta}[C_n'^m P_n^m(\cos\theta)]\exp(-j\omega t) \quad [47]$$

$$H_{\phi e}=\sum_{n=0}^{\infty}\frac{1}{j\omega\mu_2}\,\frac{1}{r}\,\frac{d}{dr}\,r h_n^{(1)}(k_2 r)]\,\frac{m}{\sin\theta}\cos(m\phi)\,[C_n'^m P_n^m(\cos\theta)]\exp(-j\omega t) \quad [48]$$

Applying the boundary condition that the tangential components E_ϕ and E_θ must vanish for all ϕ and for all θ on the spherical boss of radius a, gives

$$h_n^{(1)}(k_1 a)=0 \qquad \text{inside the dielectric sheath, } i.e.\ n=n_\nu' \quad [49]$$

and

$$h_n^{(1)}(k_2 a)=0 \qquad \text{outside the dielectric sheath, } i.e.\ n=n_\mu' \quad [50]$$

Let the excitation be

$$H_r^{\text{app}}=H_0\,d(r)\exp(-j\omega t)\sin(m\phi) \qquad \text{at } \theta=\theta_0 \quad [57]$$

where
$$d(r)=1 \ \text{ for } r_0<r<r_1,$$
$$=0 \ \text{ for } a<r<r_0 \text{ and } r>r_1$$

In terms of Sommerfeld's complex-order wave functions,

$$H_r^{\text{app}}=\frac{1}{r}\sum_{n_{\nu'}} L_{n_{\nu'}}'^m\, n_{\nu'}(n_{\nu'}+1)\,h_{n_{\nu'}}^{(1)}(k_1 r)\,P_{n_{\mu'}}^m(\cos\theta_0)\sin(m\phi)\exp(-j\omega t) \quad [52]$$

where,

$$L_{n_{\nu'}}'^m=\frac{1}{n_{\nu'}(n_{\nu'}+1)\,P_{n_{\nu'}}^m(\cos\theta_0)\,N_{n_{\nu'}}(k_1 a)}\int_{r_0}^{r_1} H_0\,r h_{n_{\nu'}}^{(1)}(k_1 r)\,d(k_1 r) \quad [53]$$

and $$N_{n_{\nu'}}(k_1 a) = \int_{k_1 a}^{\infty} [h_{n_{\nu'}}^{(1)}(k_1 r)]^2 \, d(k_1 r) \qquad [54]$$

By assuming $d(r)$ to be a delta-function source,

$$L_{n_{\nu'}}^{\prime m} = \frac{H_0 r_1 k_1 h_{n_{\nu'}}^{(1)}(k_1 r_1)}{n_{\nu'}(n_{\nu'}+1) P_{n_{\nu'}}^m(\cos\theta_0) N_{n_{\nu'}}(k_1 a)} \qquad [55]$$

Proceeding in a similar manner for the *TM* modes and applying the boundary conditions that the tangential electric and magnetic field components E_ϕ, H_r and H_ϕ are continuous at $\theta = \theta_1$, also equating the applied magnetic field H_r^{app} to H_{ri} for all values of r between a and ∞ at $\theta = \theta_0$ and making use of the orthogonal properties of the spherical Bessel functions, it can be shown that the following four sets of equations [56] to [59] result for the three sets of unknowns $A_{n_{\nu'}}^{\prime m}$, $B_{n_{\nu'}}^{\prime m}$ and $C_{n_{\nu'}}^{\prime m}$.

$$\frac{d}{d\theta_1}[C_{n_{\mu'}}^{\prime m} P_{n_{\mu'}}^m(\cos\theta_1)] = \sum_{n_{\nu'}} \alpha_{n_{\nu'}, n_{\mu'}} \frac{d}{d\theta_1}[A_{n_{\nu'}}^{\prime m} P_{n_{\nu'}}^m(\cos\theta_1) + B_{n_{\nu'}}^{\prime m} Q_{n_{\nu'}}^m(\cos\theta_1)] \qquad [56]$$

$$\frac{n_{\mu'}(n_{\mu'}+1)}{j\omega\mu_2}[C_{n_{\mu'}}^{\prime m} P_{n_{\mu'}}^m(\cos\theta_1)] = \sum_{n_{\nu'}} \frac{n_{\nu'}(n_{\nu'}+1)}{j\omega\mu_1} \alpha_{n_{\nu'}, n_{\tau'}} [A_{n_{\nu'}}^{\prime m} P_{n_{\nu'}}^m(\cos\theta) + B_{n_{\nu'}}^{\prime m} Q_{n_{\nu'}}^m(\cos\theta_1)] \qquad [57]$$

$$\frac{1}{j\omega\mu_2}[C_{n_{\mu'}}^{\prime m} P_{n_{\mu'}}^m(\cos\theta_1)] = \frac{1}{j\omega\mu_1} \sum_{n_{\nu_1}} \alpha_{n_{\nu'}, n_{\mu'}} [A_{n_{\nu'}}^{\prime m} P_{n_{\nu'}}^m(\cos\theta_1) + B_{n_{\nu'}}^{\prime m} Q_{n_{\nu'}}^m(\cos\theta_1)] \qquad [58]$$

$$L_{n_{\nu'}}^{\prime m} P_{n_{\nu'}}^m(\cos\theta_0) = \frac{1}{j\omega\mu_1} A_{n_{\nu'}}^{\prime m} P_{n_{\nu'}}^m(\cos\theta_0) + B_{n_{\nu'}}^{\prime m} Q_{n_{\nu'}}^m(\cos\theta_0)] \qquad [59]$$

This shows that there is no unique solution for the unsymmetric *TE* modes.

But for the symmetric case, $m = 0$, $E_r = E_\theta = H_\phi = 0$ and only E_ϕ H_r and H_θ exist. By applying the boundary conditions for the tangential components E_ϕ and H_r at $\theta = \theta_1$ and equating the H_r component to H_r^{app} at $\theta = \theta_0$ will result in the following three sets of equations [60] to [62] for three sets of unknowns $A_{n_{\nu'}}^{\prime o}$, $B_{n_{\nu'}}^{\prime o}$ and $C_{n_{\mu'}}^{\prime o}$.

$$\frac{d}{d\theta_1}[C_{n_{\mu'}}^{\prime o} P_{n_{\mu'}} \cos\theta_1)] = \sum_{n_{\nu'}} \alpha_{n_{\mu'}, n_{\tau'}} \frac{d}{d\theta_1}[A_{n_{\nu'}}^{\prime o} P_{n_{\nu'}}(\cos\theta_1) + B_{n_{\nu'}}^{\prime o} Q_{n_{\nu'}}(\cos\theta_1)] \qquad [60]$$

$$\frac{n_{\mu'}(n_{\mu'}+1)}{j\omega\mu_2} C'^{o}_{n_{\mu'}} P_{n_{\mu'}}(\cos\theta_1)] = \sum_{n_{\nu'}} \frac{n_{\nu'}(n_{\nu'}+1)}{j\omega\mu_1} \alpha_{n_{\nu'}, n_{\mu'}} [A'^{o}_{n_{\nu'}} P_{n_{\nu'}}(\cos\theta_1) + B'^{o}_{n_{\nu'}} Q_{n_{\nu'}}(\cos\theta_1)] \qquad [61]$$

$$L'^{o}_{n_{\nu'}} P_{n_{\nu'}}(\cos\theta_0) = \frac{1}{j\omega\mu_1} [A'^{o}_{n_{\nu'}} P_{n_{\nu'}}(\cos\theta_0) + B'^{o}_{n_{\nu'}} Q_{n_{\nu'}}(\cos\theta)] \qquad [62]$$

This shows that the *TE* symmetric mode can exist on this type of structure

5. Hybrid Modes

A superposition of the field components of the *TM* mode as given by eqns. [1] to [6] and those of the *TE* mode as given by eqns. [37] to [42] gives a hybrid mode for each particular value of m. To excite such a mode a superposition of the excitation eqn. [15] for a *TM* mode and excitation eqn. [51] for a *TE* mode is necessary. Applying the boundary conditions that the tangential components E_ϕ and E_θ must vanish at $r=a$, and that the tangential components E_r, E_ϕ, H_r and H_ϕ are continuous at $\theta=\theta_1$, by equating the applied electric and magnetic fields E_r^{app} and H_r^{app} to E_r and H_r respectively at $\theta=\theta_0$, and by making use of the orthogonal properties of the spherical Bessel functions, the following six equations [63]–[68] are obtained.

$$\sum_{n_\mu} \frac{n_\mu(n_\mu+1)}{k_2} \frac{1}{r} h^{(1)}_{n_\mu}(k_2 r) [C^m_{n_\mu} P^m_{n_\mu}(\cos\theta_1)]$$

$$= \sum_{n_\nu} \frac{n_\nu(n_\nu+1)}{k_1} \frac{1}{r} [\sum_{n_\mu} \alpha_{n_\nu, n_\mu} h^{(1)}_{n_\mu}(k_2 r)] [A^m_{n_\nu} P^m_{n_\nu}(\cos\theta_1) + B^m_{n_\nu} Q^m_{n_\nu}(\cos\theta_1)] \qquad [63]$$

$$\sum_{n_\mu} \frac{1}{k_2 r} \frac{d}{dr} [r h^{(1)}_{n_\mu}(k_2 r)] [C^m_{n_\mu} P^m_{n_\mu}(\cos\theta_1)] \frac{m}{\sin\theta_1}$$

$$+ \sum_{n_{\mu'}} h^{(1)}_{n_{\mu'}}(k_2 r) \frac{d}{d\theta_1} [C'^m_{n_{\mu'}} P^m_{n_{\mu'}}(\cos\theta_1)]$$

$$= \sum_{n_\nu} \frac{1}{k_1 r} \frac{d}{dr} [r \sum_{n_\mu} \alpha_{n_\nu, n_\mu} h^{(1)}_{n_\mu}(k_2 r)] [A^m_{n_\nu} P^m_{n_\nu}(\cos\theta_1) + B^m_{n_\nu} Q^m_{n_\nu}(\cos\theta_1)] \frac{m}{\sin\theta_1}$$

$$+ \sum_{n_{\nu'}} \sum_{n_{\mu'}} \alpha_{n_{\nu'}, n_{\mu'}} h^{(1)}_{n_{\mu'}}(k_2 r) \frac{d}{d\theta_1} [A'^m_{n_{\nu'}} P'^m_{n_{\nu'}}(\cos\theta_1) + B'^m_{n_{\nu'}} Q^m_{n_{\nu'}}(\cos\theta_1) \qquad [64]$$

$$\sum_{n_{\mu'}} \frac{n_{\mu'}(n_{\mu'}+1)}{j\omega\mu_2 r} h^{(1)}_{n_{\mu'}}(k_2 r)\,[C'^m_{n_{\mu'}} P^m_{n_{\mu'}}(\cos\theta_1)]$$

$$= \sum_{n_{\nu'}} \frac{n_{\nu'}(n_{\nu'}+1)}{j\omega\mu_1 r} \sum_{n_{\mu'}} \alpha_{n_{\nu'},\,n_{\mu'}}\, h^{(1)}_{n_{\mu'}}(k_2 r)\,[A'^m_{n_{\nu'}} P^m_{n_{\nu'}}(\cos\theta_1) + B'^m_{n_{\nu'}} Q^m_{n_{\nu'}}(\cos\theta_1)] \quad [65]$$

$$-\sum_{n_\mu} \frac{k_2}{j\omega\mu_2} h^{(1)}_{n_\mu}(k_2 r)\frac{d}{d\theta_1}[C^m_{n_\mu} P^m_{n_\mu}(\cos\theta_1)]$$

$$+\sum_{n_{\mu'}} \frac{1}{j\omega\mu_2}\,\frac{1}{r}\,\frac{d}{dr}[r h^{(1)}_{n_{\mu'}}(k_2 r)]\,[C'^m_{n_{\mu'}} P^m_{n_{\mu'}}(\cos\theta_1)]\,\frac{m}{\sin\theta_1}$$

$$= -\sum_{n_\nu} \frac{k_1}{j\omega\mu_1} \sum_{n_\mu} \alpha_{n_\nu,\,n_\mu}\, h^{(1)}_{n_\mu}(k_2 r)]\,\frac{d}{d\theta_1}[A^m_{n_\nu} P^m_{n_\nu}(\cos\theta_1) + B^m_{n_\nu} Q^m_{n_\nu}(\cos\theta_1)]$$

$$+\sum_{n_{\nu'}} \frac{k_1}{j\omega\mu_1 r}\,\frac{d}{dr}[r\sum_{n_{\mu'}} \alpha_{n_{\nu'},\,n_{\mu'}}\, h^{(1)}_{n_{\mu'}}(k_2 r)]\,[A'^m_{n_{\nu'}} P^m_{n_{\nu'}}(\cos\theta_1)$$

$$+ B^m_{n_{\nu'}} Q^m_{n_\nu}(\cos\theta_1)]\; m/(\sin\theta_1) \quad [66]$$

$$(1/k_1 r)\sum_{n_\nu} L^m_{n_\nu} n_\nu(n_\nu+1)\, h^{(1)}_{n_\nu}(k_1 r)\, P^m_{n_\nu}(\cos\theta_0)$$

$$= \sum_{n_\nu} \frac{n_\nu(n_\nu+1)}{k_1 r} h^{(1)}_{n_\nu}(k_1 r)\,[A^m_{n_\nu} P^m_{n_\nu}(\cos\theta_0) + B^m_{n_\nu} Q^m_{n_\nu}\cos\theta_0)] \quad [67]$$

$$(1/r)\sum L'^m_{n_{\nu'}}\, n_{\nu'}(n_{\nu'}+1)\, h^{(1)}_{n_{\nu'}}(k_1 r)\, P^m_{n_{\nu'}}(\cos\theta_0)$$

$$= \sum_{n_{\nu'}} \frac{n_{\nu'}(n_{\nu'}+1)}{j\omega\mu_1 r} h^{(1)}_{n_{\nu'}}(k_1 r)\,[A'^m_{n_{\nu'}} P^m_{n_{\nu'}}(\cos\theta_0) + B'^m_{n_{\nu'}} Q^m_{n_{\nu'}}(\cos\theta_0)] \quad [68]$$

For $k_1 a << 1$, the roots of equation $\frac{d}{dr}[r\,h^{(1)}_n(k_1 r)]_{r=a} = 0$ approaches the roots of $[H^{(1)}_\xi(k_1 r)]_{r=a} = 0$, where $\xi = n + \frac{1}{2}$. The roots of $[h^{(1)}_n(k_1 r)]_{r=a} = 0$ are the roots of $[\{\sqrt{(\pi/(2k_1 r)}\ H^{(1)}_{n+\frac{1}{2}}(k_1 r)]_{r=a} = 0$ *i.e.* those of $[H^{(1)}_\xi(k_1 r)]_{r=a} = 0$. Hence for small values of $k_1 a$ or of $k_2 a$, we may approximately take $n_\mu = n_{\mu'}$, and $n_\nu = n_{\nu'}$ in equations [63]—[68].

Using the asymptotic expression for $h^{(1)}_{n_\mu}(k_2 r)$ for $k_2 r >> 1$ as given by

$$h^{(1)}_{n_\mu}(k_2 r) \sim (1/k_2 r)\,(-j)^{n_\mu+1}\exp(jk_2 r) \quad [69]$$

in equations [64] and [66] and putting $n_\mu = n_{\mu'}$ and $n_\nu = n_{\nu'}$, we obtain

$$\sum_{n_\mu} \frac{1}{r}(-j)^{n_\mu+1} \frac{j}{k_2} \exp(jk_2 r) [C^m_{n_\mu} P^m_{n_\mu} (\cos\theta_1)] \frac{m}{\sin\theta_1}$$

$$+ \sum_{n_\mu} \frac{1}{k_2 r} (-j)^{n_\mu+1} \exp(jk_2 r) \frac{d}{d\theta_1} [C'^m_{n_\mu} P^m_{n_\mu} (\cos\theta_1)]$$

$$= \sum_{n_\nu} \frac{1}{k_1 r} \sum_{n_\mu} (-j)^{n_\mu+1} j \exp((jk_2 r) \alpha_{n_\nu, n_\mu} [A^m_{n_\nu} P^m_{n_\nu} (\cos\theta_1)$$

$$+ B^m_{n_\nu} Q^m_{n_\nu} (\cos\theta_1)] \frac{m}{\sin\theta_1} + \sum_{n_\nu} \sum_{n_\mu} \frac{1}{k_2 r} (-j)^{n_\mu+1} j \exp(jk_2 r) \alpha_{n_\nu, n_\mu} \times$$

$$d/d\theta [A^m_{n_\nu} P^m_{n_\nu} (\cos\theta_1) + B^m_{n_\nu} Q^m_{n_\nu} (\cos\theta_1)] \qquad [70]$$

$$- \sum_{n_\mu} \frac{1}{\mu_2 r} (-j)^{n_\mu+1} \exp(jk_2 r) \frac{d}{d\theta_1} [C'^m_{n_\mu} P^m_{n_\mu} (\cos\theta_1)]$$

$$+ \sum_{n_\mu} \frac{1}{\mu_2 r} (-j)^{n_\mu+1} j \exp(jk_2 r) [C'^m_{n_\mu} P^m_{n_\mu} (\cos\theta_1)] \frac{m}{\sin\theta_1}$$

$$= - \sum_{n_\nu} \sum_{n_\mu} \frac{k_1}{\mu_1} \alpha_{n_\nu, n_\mu} \frac{1}{k_2 r} (-j)^{n_\mu+1} \exp(jk_2 r) \frac{d}{d\theta_1} [A^m_{n_\nu} P^m_{n_\nu} (\cos\theta_1)$$

$$+ B^m_{n_\nu} Q^m_{n_\nu} (\cos\theta_1)] + \sum_{n_\nu} \sum_{n_\mu} \frac{1}{\mu_1 r} \alpha_{n_\nu, n_\mu} (-j)^{n_\mu+1} j \exp(j_2 kr) A'^m_{n_\nu} P^m_{n_\nu} (\cos\theta_1)$$

$$+ B'^m_{n_\nu} Q^m_{n_\nu} (\cos\theta_1)] \frac{m}{\sin\theta_1} \qquad [71]$$

Equating coefficients of $h^{(1)}_{n_\mu}(k_2 r)$ in equations [63] and [65], those of $h^{(1)}_{n_\nu}(k_1 r)$ in equations [67] and [68], and those of $(1/r)(-j)^{n_\mu+1} \exp(jk_2 r)$ in equations [70] and [71], we obtain

$$\frac{n_\mu(n_\mu+1)}{k_2} [C^m_{n_\mu} P^m_{n_\mu} (\cos\theta_1)] = \sum_{n_\nu} \frac{n_\nu(n_\nu+1)}{k_1} \alpha_{n_\nu, n_\mu} [A^m_{n_\nu} P^m_{\nu} (\cos\theta_1)$$

$$+ B'^m_{n_\nu} Q^m_{n_\nu} (\cos\theta_1)] \qquad [72]$$

$$\frac{n_\mu(n_\mu+1)}{\mu_2} [C'^m_{n_\mu} P^m_{n_\mu} (\cos\theta_1) = \sum_{n_\nu} \frac{n_\nu(n_\nu+1)}{\mu_1} \alpha_{n_\nu, n_\mu} [A'^m_{n_\nu} P^m_{n_\nu} (\cos\theta_1)$$

$$+ B'^m_{n_\nu} Q^m_{n_\nu} (\cos\theta_1)] \qquad [73]$$

$$L_{n_\tau}^{m} P_{n_\nu}^{m} (\cos \theta_0) = \sum_{n_\nu} [A_{n_\nu}^{m} P_{n_\nu}^{m} (\cos \theta_0) + B_{n_\nu}^{m} Q_{n_\nu}^{m} (\cos \theta_0)] \qquad [74]$$

$$L_{n_\nu}^{\prime m} P_{n_\nu}^{m} (\cos \theta_0) = \sum_{n_\nu} (1/j\omega\mu_1) [A_{n_\nu}^{\prime m} P_{n_\nu}^{m} (\cos \theta_0) + B_{n_\nu}^{\prime m} Q_{n_\nu}^{m} (\cos \theta_0)] \qquad [75]$$

$$j [C_{n_\mu}^{m} P_{n_\mu}^{m} (\cos \theta_1)] \, m/(\sin \theta_1) + d [C_{n_\mu}^{\prime m} P_{n_\mu}^{m} (\cos \theta_1)]/d\theta_1$$
$$= j (k_2/k_1) \sum_{n_\nu} \alpha_{n_\nu, n_\mu} [A_{a\nu}^{m} P_{n_\nu}^{m} (\cos \theta_1) + B_{n_\nu}^{m} Q_{n_\nu}^{m} (\cos \theta_1)] \, m/\sin \theta_1$$
$$+ \sum_{n_\nu} \alpha_{n_\nu, n_\mu} \, d \, [A_{n_\nu}^{\prime m} P_{n_\nu}^{m} (\cos \theta_1) + B_{n_\nu}^{\prime m} Q_{n_\nu}^{m} [\cos \theta_1)]/a\theta_1 \qquad [76]$$

$$- \frac{1}{\mu_2} \frac{d}{d\theta_1} [C_{n_\mu}^{m} P_{n_\mu}^{m} (\cos \theta_1)] + \frac{j}{\mu_2} [C_{n_\mu}^{\prime m} P_{n_\mu}^{m} (\cos \theta_1)] \, (m/\sin \theta_1)$$
$$= - ((k_1/\mu_1 k_2) \sum_{n_\nu} \alpha_{n_\nu, n_\mu} \, d \, [A_{n_\nu}^{m} P_{n_\nu}^{m} (\cos \theta_1) + B_{n_\nu}^{m} Q_{n_\nu}^{m} (\cos \theta_1)]/d\theta_1$$
$$+ (j/\mu_1) \sum_{n_\nu} \alpha_{n_\nu, n_\mu} [A_{n_\nu}^{\prime m} P_{n_\nu}^{m} (\cos \theta_1) + B_{n_\nu}^{\prime m} Q_{n_\nu}^{m} (\cos \theta_1)] \, m/(\sin \theta_1) \qquad [77]$$

Equations [72]−[77] are six sets of independent equations for the six sets of unknown coefficients $A_{n_\nu}^{m}$, $B_{n_\nu}^{m}$, $C_{n_\nu}^{m}$, $A_{n_\nu}^{\prime m}$, $B_{n_\nu}^{\prime m}$, $C_{n_\mu}^{\prime m}$. Hence there can exist a unique solution for the six sets of unknown coefficients for all values of m. This proves the existence of the hybrid symmetric as well as unsymmetric modes on the structure.

6. Conclusions

It has been shown that a semi-infinite dielectric-coated spherically tipped perfectly conducting cone can support the symmetric TE, TM and hybrid modes as well as the unsymmetric hybrid modes. The unsymmetric TE and TM modes cannot be supported by this type of structures.

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