The Navier-Stokes equations, **An elementary functional analytic approach** by Herman Sohr, Birkhauser Verlag, Klösterberg, 23, CH-4010, Basel, Switzerland, 2001, pp. 367, 104.

The Clay Mathematics Institute of Cambridge, Massachusetts (CMI) had announced in 2000 seven Millennium Prize Problems to celebrate mathematics in the new millennium, each one carrying a prize of one million US dollars. One of the seven, million dollar problems concerns with the regularity of weak solutions equivalently, the existence of classical solutions of Navier- Stokes (N-S) equations in three-space dimensions.

This beautiful book under review by Hermann Sohr was published in the following year (2001). The N-S equations are well known and famous in the literature because of their importance in practical applications and more so after carrying a prize tag of one million dollars. This makes the reviewer's job easier as there is no need to explain the importance of the equation. The N-S equations represent the motion of an incompressible fluid given by

$$\left.\begin{aligned} ut - v\mathrm{D}u + (u.\tilde{\mathrm{N}})u + \tilde{\mathrm{N}}p &= f \\ div\ u &= 0 \end{aligned}\right\} \qquad (1)$$

for $x \in \Omega$, $t \in [0, T)$ and with appropriate boundary and initial conditions. Here Ω is a general domain in R^n, $0 < T \leq \infty$, the velocity $u = u(x, t) = (u_1(x, t), \ldots, u_n(x, t))$ and the pressure $p = p(x, t)$ are unknowns. The first equation is nothing but Newton's law for a fluid element subject to the external force $f = f(x, t) = (f_1(x, t), \ldots f_n(x, t))$ and to the forces arising from pressure and friction, whereas the second represents the incompressiblity of the fluid.

The modern treatment of the above equation involves the study of existence and uniqueness of weak solutions in appropriate Sobolev spaces using integral formulation. After obtaining weak solutions, one proves regularity of weak solutions, that is, that the solutions belong to higherorder Sobolev spaces which eventually lead to the smoothness of the solutions when the given data are smooth. This establishes the existence of classical solutions.

The interesting cases are in dimensions two and three ($n = 2$ and $n = 3$). In the case of two dimensions, the existence of classical solutions has been known for a long time. This gives no hint about the three-dimensional case. In other words, regularity and uniqueness of weak solutions are unsolved, when $n = 3$. Of course, under an additional assumption known as Serrin's condition, that is

$$\|u\|_{q,s,T} = \left(\int_0^T \|u\ (s)\|sq\right)^{1/s} < \infty,$$

with $n < q < \infty$, $2 < s < \infty$, $n/q + 2/s \leq 1$, one can prove such properties, that is in the class of weak solutions called Serrin's class. It is to be noted that Serrin's condition is always satisfied if $n = 2$. On the other hand, if $n = 3$, there is no general existence result for weak solutions within the Serrin's class. Till now, the existence in this class can be shown for $n = 3$ only under an additional smallness assumption on the data or in a small time interval $(0, T)$. Thus the open problem can be rephrased as: *prove uniqueness and regularity* *(under smoothness assumption on the given data) of a given weak solution of the N-S equation in three-space dimension or prove the existence of at least one weak solution in Serrin's class.*

A systematic functional analytic approach is developed in the book under review. The author has taken immense pain and care to write this book in a well-ordered and self-contained manner. Any researcher who is familiar with basic functional analysis (in particular spectral theory), distribution theory and Sobolev spaces should be able to read it. One does not require the knowledge of fluid flow to read through it. Though the results presented are known to specialists, in author's own words "its systematic treatment is not available and the diverse aspects are spread out in the literature". Another important aspect of the book is that the theory is completely formulated in the general domain, both bounded and unbounded giving particular results as and when available.

Let me briefly go through the contents. As remarked earlier, this is a self-contained book. The basic function spaces are introduced in Chapter I. Other preliminary results like embedding properties are presented in Chapter II. In fact, the solvability of $div\ v = g$ and $\nabla p = f$ is also studied in this chapter. The study of N-S equation (1) is done in four steps. In Chapter III, the stationary Stokes (time independent linear case) and stationary N-S (time independent) equation are studied. The Stokes operator A and A^a are well presented here which are at the heart of the entire study of N-S equations. The existence, uniqueness and regularity results are presented for a general dimension in the linear case. However, due to the structure of the nonlinear term $(u.\nabla)u$, the results are established only for $n = 2, 3$ in the stationary nonlinear case.

The nonstationary linear theory is considered in Chapter IV for the general dimension $n \leq 2$, whereas Chapter V is devoted to the full nonlinear N-S equations (1). The existence of weak solutions are proved in general, but the uniqueness and regularity are proved under the Serrin's condition for $n = 2, 3$. As remarked earlier, Serrin's condition is always true when $n = 2$ and thus the two-space dimensional theory is complete. This leads the mathematicians to search for one million dollars in the present millennium.

I would like to congratulate the author for this excellent book. It is suitable to all working on theoretical or computational aspects of N-S equations. I enjoyed reading it and wholeheartedly recommend it to both libraries as well as for other individuals working in PDEs. It can also be used as textbook at the graduate level.

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