Wavelet transforms and localisation operators by M. W. Wong, Birkhauser Verlag AG, Klösterberg 23, CH-4010, Basel, Switzerland, 2002, pp.156. sFr.144.

This beautiful book deals with various aspects of wavelet transforms and localisation operators defined in terms of square integrable representations of locally compact groups. The central theme of the book is a study of Schatten-von Neumann properties of localisation operators. An application of this book is not possible without knowing the definitions of wavelet transforms and localisation operators which we briefly recall now.

A remarkable discovery of recent times is that certain functions can be translated and dilated to form orthonormal bases for $L^2(\mathbb{R})$. To be more precise, there are functions $\varphi \in L^2(\mathbb{R})$ oscillating on an interval and rapidly decaying outside the same such that the family $\{\varphi_{j,k} : j, k \in \mathbb{Z}\}$ where

$$\varphi_{j,k}(t) = 2^{-j/2}\varphi(2^{-j}(t-k)), \ t \in \mathbb{R}$$

is an orthonormal basis for $L^2(\mathbb{R})$. This is not at all obvious as one has to prove the existence of such φ, which Y. Meyer did around 1986. On the other hand, if we consider the continuous family $\{\varphi_{a,b}(t) = a^{-\frac{1}{2}}\varphi(a^{-1}(t-b))\}$ allowing translations by all real b and dilations by all $a > 0$ then the story is different. For any function $\varphi \in L^2(\mathbb{R})$ with the property that

$$c_\varphi = \int_0^\infty t^{-1}|\hat{\varphi}(t)|^2\,dt < \infty,$$

one has the resolution of identity

$$= c_\varphi^{-1} \int\int (f, \varphi_{a,b})\,\varphi_{a,b}\,\frac{da\,db}{a^2}$$

for any $f \in L^2(\mathbb{R})$. This was proved by Calderon long ago.

The above transform taking $f \to (f, \varphi_{a,b})$ is called the wavelet transform associated to the affine group since $(a, b) \to \pi(a, b)$, $\pi(a, b)\varphi = \varphi_{a,b}$ is a unitary representation of the affine group $G = \mathbb{R}^+ \times \mathbb{R}$ equipped with the law $(a, b)(a', b') = (aa', b + ab')$. This transform immediately generalises to locally compact groups G admitting square integrable representations. If π is such a representation of G acting on a Hilbert space H and if $\varphi \in H$ for which $(\pi(g)\varphi, \varphi) \in L^2(G)$ then there is a resolution of identity

$$c(\varphi)\int(x, \pi(g)\varphi)\pi(g)\varphi\,dg, \quad x \in H.$$

The transform taking x into $(x, \pi(g)\varphi)$ is called a wavelet transform

Another important group on which wavelet transform has been studied is the Weyl-Heisenberg group which is $\mathbb{R}^n \times \mathbb{R}^n \times [0, 2\pi]$ with the law

$$(q, p, t)(q', p', t') = (q + q', p + p', t + t' + q.p').$$

There is a unitary representation of this group on $L^2(\mathbb{R}^n)$ given by

$$\pi(q, p, t)\,\varphi(x) = e^{i(t+p\cdot x-q\cdot p)}\,\varphi(x - q)$$

which leads to the resolution of identity

$$f = (2\pi)^{-n}\int\int(f, \varphi_{q,p})\,\varphi_{q,p}\,dq\,dp$$

where $\varphi_{q,p}(x) = e^{ip\cdot x}\varphi(x - q)$. The transform $f \to (f, \varphi_{a,b})$ is known under several names (Gabor transform, coherent states) and is actually older than the wavelet transform.

In the context of signal analysis, Daubechies introduced an operator $D_{F,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined by the rule

$$(D_{F,\varphi}f, g) = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F(q, p)\,(f, \varphi_{q,p})\,(\varphi_{q,p}, g)\,dq\,dp$$

where F is a function on $\boldsymbol{R}^n \times \boldsymbol{R}^n$ called the symbol. This is reminiscent of pseudodifferential operators and is called the Daubechies operators. Now, there is no sanctity about the Weyl–Heisenberg group and whenever we have a resolution of identity we can define an operator

$$(L_{F,\varphi}x, y) = c_\varphi^{-1} \int_G F(g)(x, \pi(g)\varphi)(\pi(g)\varphi, y)\, dg.$$

This operator $L_{F,\varphi}$ is called the localisation operator with symbol F.

The main aim of this book is to study the spectral properties of the localisation operators $L_{F,\varphi}$ in the general context of locally compact groups and square integrable representations. Schatten-von Neumann properties of these operators are investigated in terms of the symbols F and product formulas have been proved for $L_{F,\varphi}$. Most of the spectral results for the localisation operators are from the recent works of the author and his students and as such they appear for the first time in a book.

The book is divided into 26 small chapters, which makes it quite readable. The author develops the necessary background material in the first six chapters. Besides, the author has the habit of recalling important theorems when and where they are required. For example, when Riesz–Thorin interpolation theorem is needed to prove a result in the book, the author states the theorem for the convenience of the readers who are not familiar with interpolation. Like his earlier books on pseudodifferential operators and Weyl transforms, this book too makes a delightful reading. The author is known for his clarity of exposition. The author leaves nothing to the reader to check-his proofs are complete in every detail. This makes the book quite readable even for an average student.

This book is invaluable for any graduate student or a research mathematician interested in studying wavelet transforms from the point of view of operator theory. It is certainly a welcome addition to any library.

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