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PROPAGATION OF LONG WAVES OF FINITE AMPLITUDE

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ABSTRACT

We consider the radiation problem for long waves of small amplitude, caused by an instantaneous disturbance of unit height at the origin. The equations governing this phenomenon were derived by Long (1964). The asymptotic expressions for the wave front and for large times are obtained. The initial value problem for the non-linear system of equations is also solved, using a perturbation scheme based on the small parameter α, *the non-dimensional amplitude of the disturbance. The solution holds only for* $t << 1/\alpha$ *as a result of the appearance of a secular term in the first order solution.*

1. INTRODUCTION

Long (1964) derived a set of equations governing the development of arbitrary, small but finite amplitude long waves. These waves are, therefore, characterised by the inequalities

$$(a/h) << 1, \quad (\lambda/h) >> 1 \qquad [1.1]$$

where h is the uniform depth of the water, a is a length representative of the amplitude and λ is a length representative of the wavelength of the disturbance. In contrast to the theory of Airy[1] which imposes the additional restriction that

$$(a/h)\,(\lambda^2/h^2) >> 1 \qquad [1.2]$$

and of Jeffereys and Jeffereys (1946) which requires

$$(a/h)\,(\lambda^2/h^2) << 1 \qquad [1.3]$$

Boussinesq[2] derived an equation which governs the propagation of long waves when

$$(a/h)\,(\lambda^2/h^2) \sim 1. \qquad [1.4]$$

The initial-value problems for the Boussinesq equation which were discussed by Korteweg and de Vries[7], for example, were for waves for which the wave heights travel only in one direction with not an arbitrary speed but one nearly equal to $(gh)^{1/2}$. Long (1964), following Rayeigh[11], expanded the complex potential for the unsteady motion and derived a set of equations which govern arbitrary long waves of small but finite amplitude without any restriction on their speed to $(gh)^{1/2}$ or the direction of their propagation. With these restrictions these equations properly reduce to Boussinesq equations, Long has shown that these equations also yield the solitary wave which is a long wave that propagates without change of form. He has also considered numerical solutions of his equations for some symmetric initial values of elevation and zero initial velocity. After some time, the wave profile in either of the two directions is very nearly that corresponding to a solitary wave.

The purpose of this paper is to study the non-linear hyperbolic system of equations derived by Long (1964). These equations govern the development of an arbitrary, small (but finite) amplitude long wave disturbances and also yield the solitary wave when they are suitably approximated. Our treatment follows the well-known approach of Lighthill and Whitham[9] and Whitham[13], particularly the latter, to the system which is obtained by linearising the non-linear system. Whitham showed that the highest order derivative in a partial and differential equation governing wave propagation, yield the phenomenon in the earlier stages of propagation, coupled with a damping caused by lower order terms, while it is the lowest order terms which finally govern the phenomenon, these being accompanied by a diffusion due to the higher order terms. The characteristics of differential equations that we consider have constant slopes $\pm\sqrt{3}$, 0, 0, which, however, do not introduce any simplicity in the analysis of the equations. First we consider the radiation problem for the linearised form of these equations and derive the form of the wave for small and large times respectively. The solution for the initial boundary conditions $t=0$, $\eta=\eta_t=\eta_{tt}=\eta_{ttt}=0$, $x>0$ and $\eta=\delta(t)$ at $x=0$, is expressed in terms of Bessel function of first order for small time, that is, when the high frequency waves dominate or in the region where the discontinuities in the wave form appear. The solution is expressed in terms of Airy function when we consider the wave form after a large time. We also consider an initial value problem for the non-linear system in a power series in the small parameter α, characterising the non-dimensional amplitude of the disturbance. The first order term in the solution contains a secular term, that is, one containing the independent variable t, so that the soluton is valid only for $\alpha t << 1$.

2. Differential Equations and their Characteristic Form

The differential equations describing the two dimensional long gravity waves and satisfying the kinematic and dynamic conditions on the surface of water were written by Long in terms of the non-dimensional height $\eta = (\eta'/h)$ of the disturbance above the undisturbed level and a non-dimensional velocity $U = -F_x = -[F'_{x'}(x', t')/\sqrt{(gh)}]$. Here the primed quantities denote the dimensional variables so that $y' = 0$ is the x'-axis along the bottom of the channel and $y' = h$ is the vertical undisturbed height as shown in the figure. t' is time. The function F' is a function of x' and t' in terms of which the real part of the complex velocity potential ϕ' is expanded about $y' = 0$, that is,

$$\phi'(x', y', t') = F'(x', t') - (y'^2/2)\, F'_{x'x'}(x', t') + \ldots \qquad [2.1]$$

The dimensionless quantities are expressed as

$$x = (x'/h),\ y = (y'/h),\ t = t'/\sqrt{(g/h)},$$

$$\phi(x, y, t) = \frac{\phi'(x', y', t')}{h\sqrt{(gh)}},\ \eta = (\eta'/h),\ F(x, t) = \frac{F'(x', t')}{h\sqrt{(gh)}}. \qquad [2.2]$$

U, in fact, is the velocity at the bottom of the channel. Long made certain assumptions as to the order of different terms, which correspond to those employed in the derivation of the solitary wave. Thus, he assumed that if the non-dimensional amplitude of the disturbance is of the order α, a small quantity, that is, if

$$\eta \sim \alpha, \qquad [2.3]$$

then

$$(\partial/\partial x) \sim \alpha^{1/2}. \qquad [2.4]$$

This, in fact, expresses [1.4]. Besides, he assumed that

$$U \equiv -F_x \sim \alpha,\ (\partial/\partial t) \sim (\partial/\partial x). \qquad [2.5]$$

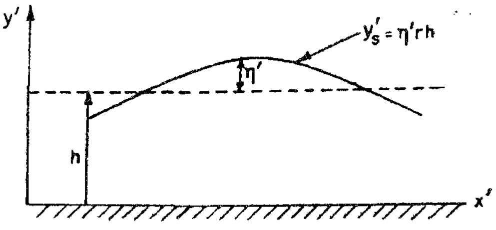

FIG. 1
Long-wave Propagation

By substituting the non-dimensional velocity potential in the non-dimensional form of surface condition, Long obtained the following equations for η and U with an error of $0(\alpha^3)$ in η,

$$\eta_t + U\eta_x + U_x - \eta\eta_t + \tfrac{1}{6}\Omega_t = 0, \qquad [2.6]$$

$$\eta_x + U_t - U\eta_t + \tfrac{1}{2}\Omega_x = 0, \qquad [2.7]$$

$$\eta_t - \omega = 0, \qquad [2.8]$$

$$\omega_t - \Omega = 0. \qquad [2.9]$$

The equations [2 8] and [2 9] are already in characteristic form. The former two equations, [2.6] aud [2.7], can be suitably combined to give the characteristic form:

$$\partial(\eta + \sqrt{3}\,U + \tfrac{1}{2}\Omega)/\partial z + \sqrt{3}\,U\,(\partial\eta/\partial z)$$
$$= \tfrac{1}{2} w[\sqrt{3}\,\eta + 2\,U - (2/\sqrt{3})] \qquad [2\,6]'$$

$$\partial(-\eta + \sqrt{3}\,U - \tfrac{1}{2}\Omega)/\partial\bar{z} + \sqrt{3}\,U\,(\partial\eta/\partial\bar{z})$$
$$= \tfrac{1}{2} w[\sqrt{3}\,\eta - 2\,U - (2/\sqrt{3})] \qquad [2.7]'$$

where the independent variables are the characteristics $z = x + \sqrt{3}\,t$, $z = x - 3\sqrt{t}$. Thus, we have a hyperbolic system of equations with explicit characteristics, having constant slopes $\pm\sqrt{3}$, 0, 0.

3. SOLUTION OF THE LINEARISED EQUATIONS

If we put $U = U_0 + U$, $\eta = \eta_0 + \eta$, where U_0 and η_0 (in particular $\eta_0 = 0$) are the solutions of equations [2 6]' and [2.7]' giving uniform flow and U and η are of order $0(\alpha^2)$, and linearise (thus omitting terms of $0(\alpha^{1/2})$, we obtain, after elimination of U, a linear equation in η.

$$\left[\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial t^2} - 3\,\frac{\partial^2}{\partial x^2}\right) + 6\,(1-\eta_0)\frac{\partial^2}{\partial t^2} + 12\,U_0\,\frac{\partial^2}{\partial t\,\partial x} - 6\,\frac{\partial^2}{\partial x^2}\right]\eta = 0. \qquad [3.1]$$

Thus the second order operator gives the lower order waves with speeds $[U_0 \pm \sqrt{(U_0^2 + 1 - \eta_0)}/1 - \eta_0]$ and the fourth order operator gives the higher order waves with speeds $o, o, \pm\sqrt{(3)}$. Unlike the differential operators considered by Whitham (1959), here the order of the adjacent differential operators differs by two. If we substitute

$$\eta \propto \exp\,(ikx - \alpha t) \qquad [3.2]$$

in equation [3.1] where k is real and consider long waves so that $k << 1$, we easily verify that all roots α of the dispersion relation are pure imaginary. This shows that we have a stable situation, with progressive waves as solutions. Similarly, if we consider a periodic wave maintained at $x=o$ so that

$$\eta \;\alpha\; \exp.(\beta x - iwt) \tag{3.3}$$

with ω (real) $<< 1$, we again get all four roots corresponding to β pure imaginary, leading to the same result as noted above. Thus, for large times, we again have a stable situation. However, to be able to study the general wave motion, we consider the following signalling problem for the differential equation [3.1], an unsteady wave phenomenon on a running stream.

Initial conditions : $\eta = \eta_t = \eta_{tt} = \eta_{ttt} = 0$ at $t=o,\ x>0$

Boundary conditions : $\eta = f(t)$ at $x=0$ [3.4]

In the above we consider waves propagating in the $x > 0$ direction due to the signal at $x=0$, but we could also consider the waves in the opposite direction. We find the Laplace transform of the equation [3.1] with the initial conditions [3.4]. We have

$$(3p^2+6)\ \bar{\eta}_{xx} - 12\ U_0 p\ \bar{\eta}_x - [6\,(1-\eta_0)\,p^2 + p^4]\ \bar{\eta} = 0. \tag{3.5}$$

The solution of this equation is

$$\bar{\eta} = A_1\,(p)\ \exp.[\gamma_1\,(p)\ x] + A_2\,(p)\ \exp.[\gamma_2(p)x] \tag{3.6}$$

where A_1 and A_2 are some functions of p and

$$\gamma_{1,2} = \frac{6\,U_0 p \pm p\,\{36\,U_0^2 + (3\,p^2+6)[\,p^2 + 6\,(1-\eta_0)]\}^{1/2}}{3\,(p^2+2)}. \tag{3,7}$$

This expression is rather complicated and therefore we approximate this for the following two situations (a) when p is very large *i.e.*, t is small, this approximation is valid when the high frequency waves dominate or near discontinuities in the wave form. (b) when p is very small so that we consider the solution for large times.

(a) When p is large. In this case,

$$\gamma_{1,2} \approx \frac{\pm p}{\sqrt{(3)}} + \frac{1}{3p}\,[6\,U_0 \pm \sqrt{(3)}\,(2 \mp 3\,\eta_0)]. \tag{3.8}$$

For the forward moving wave we take negative sign, with $A_1(p)=0$, and inverting equation [3.6],

$$\eta=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}A_2(p)\,\text{exp.}\left\{p\left(t-\frac{x}{\sqrt{(3)}}\right)+[6U_0-\sqrt{(3)}(2-3\eta_0)]\frac{x}{3p}\right\}dp \quad [3.9]$$

where $\gamma=\text{Re}p$ is such that all singularities of the integrand are to the left of $\text{Re}p=\gamma$. It is obvious that

$$A_2(p)=\int_0^\infty \text{exp.}(-pt)\,f(t)\,dt. \quad [3.10]$$

If we take $f(t)=\delta(t)$, the Dirac delta function, so that $A_2(p)=1$, the above integral is easily evaluated, Roberts and Kaufmann[12], the solution $\hat{\eta}$ is

$$\hat{\eta}(x,t)=0 \qquad 0<t<x/\sqrt{3} \quad [3.11]$$

$$=\delta\left(t-\frac{x}{\sqrt{(3)}}\right)-\left(\frac{x}{3}\right)^{1/2}\left[\frac{\sqrt{(3)}\,(2-3\,\eta_0)-6\,U_0}{t-[x/\sqrt{(3)}]}\right]^{1/2}\times$$

$$\times J_1\left\{2\left(\frac{x}{3}\right)^{1/2}\left(t-\frac{x}{\sqrt{(3)}}\right)^{1/2}[\sqrt{(3)}\,(2-3\eta_0)-6\,U_0]^{1/2}\right\}$$

$$t\geq[x/\sqrt{(3)}.$$

For any other $f(t)$, we can use the faltung theorem to obtain

$$\eta=\int_0^t\hat{\eta}(x,u)\,f(t-u)\,du. \quad [3.12]$$

If we were to consider wave propagation in the negative direction, we would have

$$\hat{\eta}=\delta\left(t+\frac{x}{\sqrt{(3)}}\right)-(-x)^{1/2}\frac{[\sqrt{(3)}\,(2-3\eta_0)+6U_0]^{1/2}}{\sqrt{3}\,[\,t+x/\sqrt{3}]^{\frac{1}{2}}}\times$$

$$J_1\left\{\frac{2}{\sqrt{(3)}}(-x)^{1/2}[\sqrt{(3)}\,(2-3\eta_0)+6U_0]^{1/2}\left(t+\frac{x}{\sqrt{(3)}}\right)^{1/2}\right\}.$$

$$t\geq-[x/\sqrt{(3)}]$$

$$=0. \qquad t<-[x/\sqrt{(3)}] \quad [3.13]$$

The above solution represents higher order progressive waves with speeds $+\sqrt{(3)}$ and $-\sqrt{(3)}$ respectively. The wave height near the front initially decreases from 1 as x increases. We also note that if $\eta_0=0$, $U_0=[1/\sqrt{3}]$, $\eta(x,t)=\delta[t-x/\sqrt{(3)}]$ in equation [3.11] which gives $\eta(x,t)=1$ on $t=[x/\sqrt{(3)}]$ and $\eta=0$ elsewhere. Of course these results are based on linear theory, the non-linear effects will alter the situation considerably. Unlike the exponential damping of dynamic waves by the kinematic waves, in the flood wave problem of Lighthill and Whitham, we have near the wave front a diminishing of the amplitude from unity and an oscillatory character, given by Bessel function of order one.

(*b*) In this case p is small, we approximate $\gamma_{1,2}$ to

$$\gamma_{1,2}=p\,[U_0^2\pm(U_0^2+1-\eta_0)^{1/2}]+p^3\left[\pm\frac{4-3\eta_0}{12(U_0^2+1-\eta_0)}-\frac{1}{2}U_0\mp\frac{1}{2}(U_0^2+1-\eta_0)^{1/2}\right]$$

$$\equiv B_{1,2}\,p+C_{1,2}\,p^3. \qquad [3.14]$$

Again if we consider wave propagation in the positive direction only, then taking the lower sign,

$$\eta=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\bar{f}(p)\ \exp.\,[p(t+B_2x)+C_2xp^3]\,dp \qquad [3.15]$$

where $Rl\,\gamma$ is defined in the usual way. Again if we choose $f(t)=\delta\,(t)$ so that $\bar{f}(p)=1$, the above integral can be easily integrated, Magnus *et al*[10]. We can transform this integral into the form

$$\hat{\eta}=(1/\pi)\int_0^\infty\cos\,[z\,(t+B_2x)-c_2\,xz^3]\,dz \qquad [3.16]$$

which is expressible in terms of Airy functions

$$\hat{\eta}=\frac{1}{(3\,C_2\,x)^{1/3}}\,A_i\left(-3^{-1/3}\frac{B_2x+t}{(C_2x)^{1/3}}\right)\quad \cdots\cdots$$

$$\approx\frac{1}{\sqrt{\pi}\,[3\,C_2\,x\,(B_2x+t)]^{1/4}}\ \cos\left[\tfrac{2}{3}\,\frac{(B_2x+t)^{3/2}}{(3\,C_2x)^{1/2}}-\pi/4\right], \qquad [3.16]'$$

the asymptotic expression of Ai for large value of t when x/t is kept fixed. We briefly verify this result by the method of saddle points. The exponential term in the integral [3.15] can be written as

$$\exp\ t\,\{p\,[1+B_2\,(x/t)\,]+C_2\,(x/t)\,p^3\}.$$

For fixed value of x/t, we evaluate this integral for large t. The saddle points of $F(p)=p[1+B_2(x/t)]+C_2(x/t)]p^3$ are

$$p_{0,1}=\pm i[(t/x+B_2)(1/3C_2)]^{1/2}, \qquad [3.16]''$$

both of these being equally important. Therefore, by the usual method of saddle points,

$$\eta \sim \frac{1}{[48\pi^2 C_2 x(t+B_2 x)]^{\frac{1}{4}}}[\bar{\eta}(p_0)\exp i\{\tfrac{2}{3}(t+B_2 x)\sqrt{[(t/x+B_2)(1/3C_2)]}-\pi/4\}$$

$$+\bar{\eta}(p_1)\exp -i\{\tfrac{2}{3}(t+B_2 x)\sqrt{[(t/x+B_2)(1/3C_2)]}-\pi/4\}]$$

$$=\frac{1}{\sqrt{\pi}[3C_2 x(B_2 x+t)]^{1/4}}\cos\left[\frac{2}{3}\frac{(B_2 x+t)^{\frac{3}{2}}}{(3C_2 x)^{1/2}}-\pi/4\right] \qquad [3.17]$$

when $\bar{\eta}(p)=1$. This is the same as in [3.16]. This represents essentially the lower order waves. We find that the solution does not hold at the observation point $x=0$ and the front $B_2 x+t=0$, Lighthill and Whitham[9]. We also note that $\eta \propto (1/\sqrt{x})$ or $\eta \propto (1/\sqrt{t})$ for fixed x/t, showing diffusion of the lower order wave by the higher order ones.

The solution in the negative x direction can be easily obtained by changing B_2 and C_2 to B_1 and C_1 respectively.

Before we consider the non-linear wave propagation, we briefly indicate the results as obtained by the quick method, Whitham[13]. For example, for the wave corresponding to $(\partial x/\partial t)=\sqrt{3}$ we put $(\partial/\partial t)=-\sqrt{3}\,(\partial/\partial x)$ in equation [3.1] and introduce the variable $\xi=x-\sqrt{3}\,t$, we get the equation

$$(\partial^2\eta/\partial x\,\partial\xi)=-\tfrac{1}{3}[2-3\eta_0-2\sqrt{3}\,U_0]\eta. \qquad [3.18]$$

With the conditions that $\eta=0$ on the front $x-\sqrt{3}\,t=\xi=0$ and $\eta=f(t)$ on $x=0$, the solution of equation [3.18] can be written in the form, Garabedian[5],

$$\eta(x,\xi)=\frac{f(0)J_0(2\sqrt{\lambda x\xi})}{2}+\int_0^{\xi} f'(\eta_1)J_0[2\sqrt{\lambda(\xi-\eta_1)}]\,d\eta_1. \qquad [3.19]$$

Similarly, the solution near the wave front $x=-\sqrt{3}\,t$ is obtained by simply changing ξ to $\xi'=x+\sqrt{3}\,t$. For the lower order waves, if we put $(\partial/\partial t)=-C_{1,2}(\partial/\partial x)$ in equation where $C_{1,2}=[U_0\pm\sqrt{(U_0^2+1-\eta_0)}]/1-\eta_0$, we get

$$\frac{\partial\eta}{\partial t}+C_i\frac{\partial\eta}{\partial x}-\frac{C_i^2(C_i^2-3)}{6(1-\eta_0)}\frac{\partial^3\eta}{\partial x^3}=0,\ i=1,\ 2 \qquad [3.20]$$

after integrating out once with respect to x. The solution of this equation can be expressed in terms of Bessel functions.

4. Initial Value Problem

Now we consider the initial value problem for the hyperbolic system $(2.6)'-(2.9)'$ where equations [2.8] and [2.9] are expressed in terms of the characteristic variables z and $\bar{z}$ as

$$(\partial\eta/\partial z)-(\partial\eta/\partial\bar{z})=(w/\sqrt{3}), \qquad [2.8]'$$

$$(\partial w/\partial z)-(\partial w/\partial\bar{z})=(\Omega/\sqrt{3}). \qquad [2.9]'$$

We assume that $\eta(x, 0)$ and $U(x, 0)$ are given. We seek the solution in the form

$$\eta=\alpha(\eta_0+\alpha\eta_1),\quad U=\alpha(U_0+\alpha U_1),\quad \omega=\alpha^{3/2}(\omega_0+\alpha\omega_1),$$
$$\Omega=\alpha^2(\Omega_0+\alpha\Omega_1), \qquad [4.1]$$

since the equations of Long give η with an error of $0(\alpha^3)$. Here η and U are functions of z and $\bar{z}$. We substitute the expressions [4.1] in equations $[2.6]'-[2.9]'$, taking note of the assumptions [2.4] and [2.5].

After some calculation, we get the following equations. Zero order system :

$$\frac{\partial(\eta_0+\sqrt{3}U_0)}{\partial z}=-(1/\sqrt{3})\,\omega_0, \qquad [4.2]$$

$$(\partial/\partial\bar{z})(-\eta_0+\sqrt{3}U_0)=(-\omega_0/\sqrt{3}), \qquad [4.3]$$

$$(\partial\eta_0/\partial z)-(\partial\eta_0/\partial\bar{z})=(\omega^0/\sqrt{3}), \qquad [4.4]$$

$$(\partial\omega_0/\partial z)-(\partial\omega_0/\partial\bar{z})=(\Omega_0/\sqrt{3}). \qquad [4.5]$$

First order system :

$$\frac{\partial[\eta_1+\sqrt{3}U_1+(\Omega_0/2)]}{\partial z}+\sqrt{3}U_0(\partial\eta_0/\partial z)$$
$$\cdot(\omega_0/2)(\sqrt{3}\eta_0+2U_0)-(\omega_1/\sqrt{3}), \qquad [4.6]$$

$$\frac{\partial[-\eta_1+\sqrt{3}\,U_1-(\Omega_0/2)]}{\partial\bar{z}}+\sqrt{3}\,U_0\,(\partial\eta_0/\partial\bar{z})$$

$$=(\omega_0/2)\,(\sqrt{3}\,\eta_0-2\,U_0)-(\omega_1/\sqrt{3}), \qquad [4.7]$$

$$(\partial\eta_1/\partial z)-(\partial\eta_1/\partial\bar{z})=(\omega_1/\sqrt{3}), \qquad [4.8]$$

$$(\partial\omega_1/\partial z)-(\partial\omega_1/\partial\bar{z})=(\Omega_1/\sqrt{3}). \qquad [4.9]$$

The equations satisfied by η_0 and η_1 are found to be

$$\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial\bar{z}^2}-4\,\frac{\partial^2}{\partial z\,\partial\bar{z}}\right)\eta_0=0, \qquad [4.10]$$

$$\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial\bar{z}^2}-4\,\frac{\partial^2}{\partial z\,\partial\bar{z}}\right)\eta_1=F\,(z,\,\bar{z}) \qquad [4.11]$$

Where $F\,(z,\,\bar{z})$

$$=\frac{\partial^2\Omega_0}{\partial z\,\partial\bar{z}}+\sqrt{3}\left[\frac{\partial}{\partial\bar{z}}\left(U_0\frac{\partial\eta_0}{\partial z}\right)-\frac{\partial}{\partial z}\left(U_0\frac{\partial\eta_0}{\partial\bar{z}}\right)\right]$$

$$-\frac{\sqrt{3}}{2}\,\frac{\partial}{\partial\bar{z}}\left(\frac{\partial\eta_0}{\partial z}-\frac{\partial\eta_0}{\partial\bar{z}}\right)(\sqrt{3}\,\eta_0+2\,U_0)$$

$$+(\sqrt{3}/2)\,(\partial/\partial z)\,[(\partial\eta_0/\partial z)-(\partial\eta_0/\partial z)]\,(\sqrt{3}\,\eta_0-2\,U_0). \qquad [4.12]$$

We easily verify that the characteristics of the differential operator on the left hand side equations [4.1] and [4.2] are $\bar{z}+(2\pm\sqrt{3})\,z=(3\pm\sqrt{3})\,(x\mp t)=$ const, agreeing with the linearised equation [29] of Long (1964). Thus the characteristic slopes of the zero order solutions are ± 1, while those of the non-linear system are $\pm\sqrt{3}$. We consider, in particular, the initial value problem $\eta\,(x,\,0)=2\,\alpha\cos x$, $U\,(x,\,0)=0$, so that we find from equations [4.10], [4.2] and [4.3] that

$$\eta_0\,(x,\,t)=\cos\,(x+t)+\cos\,(x-t),\quad U_0\,(x,\,t)=\cos\,(x-t)-\cos\,(x+t). \qquad [4.13]$$

Thus, it is more convenient to introduce the characteristic variables

$$\alpha_1=x-t,\quad \beta_1=x+t \qquad [4.13]$$

so that η_1 can be shown to satisfy the equation

$$(\partial^2\eta_1/\partial\alpha_1\partial\beta_1) = -\tfrac{1}{2}\cos(\alpha_1+\beta_1)+\tfrac{3}{4}[\cos 2\,\alpha_1+\cos 2\,\beta_1]$$
$$-\tfrac{1}{12}(\cos\alpha_1+\cos\beta_1). \qquad [4.14]$$

The solution of this hyperbolic differential equation with **the initial** condition $\eta_1=(\partial\eta_1/\partial\alpha_1)=(\partial\eta_1/\partial\beta_1)=0$ on the initial line $\alpha_1=\beta_1$ **from** equations [4.6] and [4.7] is

$$\eta_1=(\beta_1-\alpha_1)\,[\tfrac{3}{8}(\sin 2\,\alpha_1-\sin 2\,\beta_1)+\tfrac{1}{12}\sin\beta_1-\sin\alpha_1)]$$
$$+\tfrac{1}{2}\cos(\alpha_1+\beta_1)-\cos 2\,\alpha_1]+\tfrac{1}{4}(\cos 2\,\alpha_1-\cos 2\,\beta_1)$$

$$=t\,[\tfrac{3}{4}\{\sin 2\,(x-t)-\sin 2\,(x+t)\}+\tfrac{1}{6}\{\sin(x+t)-\sin(x-t)\}]$$
$$+\tfrac{1}{2}[\cos 2x-\cos 2\,(x-t)]+\tfrac{1}{4}[\cos 2\,(x-t)-\cos 2\,(x+t)]. \qquad [4.15]$$

We find that a secular term in the first order term of the solution appears so that the solution is valid only for $\alpha t << 1$. While the secular terms in ordinary differential equations have been treated quite successfully, there does not seem to be any general way of tackling them for partial differential equations. For example, Broer (1965) has considered some simple cases when a transformation of the time variable can be guessed from the solution. The term $(\alpha t/6)$ $[\sin(x+t)-\sin(x-t)]$ in [4.15] can be easily combined with the zero order term by the transformation $t'=1+(\alpha t/6)$ but the other secular terms cannot be removed. The divergence of the solution for large t is not due to the linearising of the characteristics since we can easily fit the exact characteristics by stretching the x-co-ordinate by $1/\sqrt{3}$, but does not remove the singularity for large t. This perturbation scheme is not suited to give solution for the far field for which a different procedure similar to that given by Cole (1968) would lead to the Korteweg equation which provides the solitary wave and other periodic solutions, Kruskal and Zabusky[8]. In any case, the above solution for $t<(1/\alpha)$ shows that in the first order solution we get zero order solution and its double harmonic out of phase with the zero order solution by $\pi/2$ and these together have their amplitude increasing linearly with time while the other double harmonic in η_1 remains bounded.

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