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Short Communication

Some results on the independence number of a graph

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Abstract

In this paper, we give new lower bounds for the independence number $\alpha(G)$ of a finite and simple graph G.

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Graphs, considered here, are finite and simple (without loops or multiple edges), and [1], [2] are followed for terminology and notation. Let $G = (V, E)$ be an undirected graph, with the set of vertices $V = \{v_1, v_2, \ldots, v_n\}$ and the set of edges E, such that $|E| = m$.

We denote by $d(v)$ the degree of a vertex v in G. It is well known (e.g. see [2]) that $\sigma(G) = d(v_1) + d(v_2) + \ldots + d(v_n) = 2m$.

Let $\delta_i(v)$ be the number of vertices having the distance i from a vertex v of G and let $\alpha(G)$ be the independence number of G.

LEMMA 1. *If G is a triangle-free graph, then*

$$\alpha(G) \geq \alpha^*(G) = \sum_{v\in V} \delta_1(v)/(1+\delta_1(v)+\delta_2(v)).$$

Proof. We randomly label the vertices of G with a permutation of the integers from 1 to n. Let $S \subseteq V$ be the set of vertices v for which the minimum label on vertices at distance 0, 1 or 2 from v is on a vertex at distance 1. Obviously, the probability that S contains a vertex v is given by $\delta_1(v)/(1+\delta_1(v)+\delta_2(v))$ and, therefore, the expected size of S is equal to $\alpha^*(G)$. Moreover, S must be an independent set of G, since, otherwise, if S contains an edge it is easy to see that it must lie in a triangle of G, contradicting the hypothesis. Thus, the lemma is proved.

THEOREM 1. *If G is a triangle- and pentagon-free graph with m edges, then $\alpha(G) \geq \sqrt{m}$.*

Proof. Let $d(G)$ be the average degree of vertices of G. Since G is a triangle- and pentagon-free graph, then we have $\alpha(G) \geq \delta_1(v)$, by considering the neighbours of v, and $\alpha(G) \geq 1+\delta_2(v)$, by considering v and the vertices at distance 2 from v, for any vertex v of G. Thus, by the above lemma, $\alpha(G) \geq \alpha^*(G) \geq \sum_{v\in V} \delta_1(v)/2\alpha(G)$, that is,

$$\alpha(G)^2 \geq nd(G)/2 \text{ or } \alpha(G) \geq \sqrt{nd(G)/2}.$$

But, $d(G) \geq \sigma(G)/n = 2m/n$ and, therefore, $\alpha(G) \geq \sqrt{m}$, the theorem being proved.

LEMMA 2. *If G is a graph with an odd girth $2k+3$ ($k \geq 2$) or greater, then*

$$\alpha(G) \geq \sum_{v\in V} \left(\tfrac{1}{2}(1+\delta_1(v)+\ldots+\delta_{k-1}(v))\right)/(1+\delta_1(v)+\ldots+\delta_k(v)).$$

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Proof. We randomly label the vertices of G with a permutation of the integers from 1 to n. Let $S_1 \subseteq V$ (respectively $S_2 \subseteq V$) be the set of vertices v for which the minimum label on vertices at distance k or less from v is at even (respectively odd) distance $k-1$ or less. It is easy to see that S_1 and S_2 are independent sets and that the expected size of $S_1 \cup S_2$ is given by

$$\sum_{v\in V} (1+\delta_1(v) + \ldots + \delta_{k-1}(v))/(1+\delta_1(v) + \ldots + \delta_k(v)),$$

the lemma being proved.

THEOREM 2. *If G is a graph with an odd girth $2k+3$ $(k \geq 2)$ or greater, then* $\alpha(G) \geq 2^{-(k-1)/k}(\sum_{v\in V} \delta_1(v)^{1/(k-1)})^{(k-1)/k}$.

Proof. By Lemma 1 and applying Lemma 2 for all the values between 3 and k, we have,

$$\alpha(G) \geq \sum_{v\in V}\{\delta_1(v)/(1+\delta_1(v) + \delta_2(v)) + \tfrac{1}{2}((1+\delta_1(v)+\delta_2(v))/(1+\delta_1(v)+\delta_2(v)+\delta_3(v))) + \ldots +$$

$$\tfrac{1}{2}((1+\delta_1(v) + \ldots + \delta_{k-1}(v))/(1+\delta_1(v) + \ldots + \delta_k(v)))\}/(k-1).$$

Since the arithmetic mean is greater than the geometric mean, then

$$\alpha(G) \geq \sum_{v\in V} ((\delta_1(v)\, 2^{-(k-2)})/(1+\delta_1(v) + \ldots + \delta_k(v)))^{1/(k-1)}.$$

Since the vertices at even (odd) distance less than or equal to k from any vertex v of G form independent sets, then

$$2\alpha(G) \geq 1 + \delta_1(v) + \ldots + \delta_k(v).$$

Thus,

$$\alpha(G) \geq \sum_{v\in V}(\delta_1(v)/2^{k-1}\alpha(G))^{1/(k-1)}$$

or

$$\alpha(G)^{k/(k-1)} \geq \tfrac{1}{2}(\sum_{v\in V}\delta_1(v)^{1/(k-1)})$$

or

$$\alpha(G) \geq 2^{-(k-1)/k}(\sum_{v\in V}\delta_1(v)^{1/(k-1)})^{(k-1)/k},$$

the theorem being proved.

COROLLARY. *If G is a regular graph of the degree $r(G)$ and with an odd girth $2k+3(k \geq 2)$ or greater, then*

$$\alpha(G) \geq 2^{-(k-1)/k} n^{(k-1)/k} r(G)^{1/k}.$$

Proof. It follows, immediately, from Theorem 2.

Remark. Marcu [3] presents an algorithm with a computer program which for a given graph G finds all its maximal independent sets and the exact value of $\alpha(G)$.

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References

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