

# COMPUTATION OF MATRIX INVERSE BY A POWER SERIES METHOD

By SYAMAL KUMAR SEN

(Central Instruments and Services Laboratory, Indian Institute of Science, Bangalore-12, India)

[Received: August 28, 1970]

## ABSTRACT

*Discussed here is a computational procedure for the inverse of a square matrix by using a power series method<sup>1</sup> that first transforms a matrix into one whose inverse can be equated to a convergent power series and then finds the inverse by a procedure reverse to the aforesaid one that rests on only matrix addition, subtraction and multiplication but no inversion.*

## 1. DISCUSSION

Cofactor method or triangular decomposition methods<sup>2</sup> obtain the inverse of a matrix directly. Almost all these methods are variants of the Gauss's method and they are very susceptible for the ill-conditioned (with respect to inverse) matrices. A suggested method of Maulik<sup>3</sup>, since it is independent of the spacing of the characteristic roots of the matrix and since it does not demand division except at the last step and avoids redundant multiplications, is much more rapid as also more accurate than the co-factor method. This novel method, though not a variant of Gaussian type, can be applied only to matrices of order  $2^n$ ,  $n$  being a positive integer. The present method cannot be classified in either of the aforesaid two categories. The method, in the first phase, converts any arbitrary square matrix into one whose inverse is replaced by a convergent power series. This inversion then allows the method, in the second phase, to obtain the inverse of the original matrix, that requires matrix addition, subtraction and multiplication but no inversion.

The general form for the conversion of the original matrix  $A (= a_{ij})$  of order  $n$  into one whose inverse is approximated by a convergent power series is

$$A_{p-1} = A_p + B_p, \quad p = 1, 2, \dots, n \quad [1.1]$$

where  $A_p$  is identically equal to  $A_{p-1}$  excepting the  $p$ -th diagonal element of  $A_{p-1}$  and  $A_0 \equiv A$  and

$$B_p = u_p v_p \quad [1.2]$$

where the column vector  $u_p$  and row vector  $v_p$  are

$$u_p = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ a_{pp} - a'_{pp} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad v_p = [0 \ 0 \ \dots \ 1 \ 0 \ \dots \ 0] \quad [1.3]$$

$a'_{pp}$  is the  $p$ -th diagonal element of  $A_p$  and

$$a'_{pp} > n \left( \sum_{q=1}^n a_{pq} \right), \quad q \neq p \quad [1.4]$$

$a_{pp} - a'_{pp}$  in  $u_p$  and 1 in  $v_p$  are the  $p$ -th elements of  $u_p$  and  $v_p$ , respectively. We take  $a'_{pp}$  such that  $A_n$  possesses only non-zero diagonal elements. This is very easy since we are at liberty to choose  $a'_{pp}$  satisfying condition [1.4]. The matrix  $A_n$ , thus obtained, is the final transformed matrix. For the above procedure of conversion the following theorem will be true.

*Theorem:* In an arbitrary row, if the diagonal element of a matrix is greater than  $n$  times the sum of the moduli of the off-diagonal elements, the inverse of the matrix can be equated to a convergent power series.

We write

$$A_n = P + Q$$

$$i.e. \quad A_n^{-1} = (I - P^{-1}Q + (P^{-1}Q)^2 - (P^{-1}Q)^3 + (P^{-1}Q)^4 - \dots) P^{-1} \quad [2.1]$$

where  $P$  is a diagonal matrix having the diagonal elements identical to those of  $A_n$  whose diagonal elements are already non-zero as  $a'_{pp}$  has been chosen in the manner where no zero can appear on the diagonal of  $A_n$ .  $Q$  is a matrix identical to  $A_n$  excepting its diagonal elements which are exactly zeros.

In Newton-Horner's scheme<sup>4</sup>

$$A_n^{-1} = \{I - P^{-1}Q \langle I + P^{-1}Q \langle I + P^{-1}Q \langle I + P^{-1}Q + \dots \rangle \rangle \rangle\} P^{-1} \quad [2.1a]$$

We determine first  $P^{-1}Q \langle I + P^{-1}Q \rangle$  and call it  $X$ . We then obtain  $P^{-1}Q \langle I + X \rangle$  and call it  $X_1$ . Next we find  $P^{-1}Q \langle I + X_1 \rangle$  and call it  $X_2$  and so on. We stop this procedure when the Erhard-Schmidt's norm<sup>5</sup> of  $X_i - X_{i-1}$  i.e.,

$$\left\| X_i - X_{i-1} \right\|_{E.S.} < \text{a pre-assigned accuracy, say, } 10^{-10}$$

$i = 2, 3, \dots$

$$\text{Hence } A_n^{-1} = (I - P^{-1} Q X_i) P^{-1} \text{ for sufficiently large } i \quad [2.1b]$$

The evaluation of  $A_n^{-1}$  from [2.1a] is preferred to that of  $A_n^{-1}$  from [2.1] as the Newton-Horner's scheme requires less arithmetic operations ( $n$  matrix multiplications and  $n$  additions) and consequently results in less rounding errors.

It is easy to see that  $|P^{-1}Q|_{E.S.} < 1$ . It can, furthermore, be noted that we can increase the speed of convergence indefinitely by taking the diagonal elements of  $A_n$  sufficiently large. We should, however, refrain from taking too large diagonal elements for too rapid convergence, since these introduce rounding errors due to matrix addition operations the effect of which, however, is very much dependent on the precision of the computer employed. This fact is illustrated through numerical examples that find description in subsequent pages.

The method, in the second phase, obtains  $A^{-1}$  using the knowledge of  $A_n^{-1}$ . The general form of the recurrence relations for obtaining  $A^{-1}$  is

$$\gamma_p = 1 + v_p A_p^{-1} u_p \quad [3.1]$$

$$B_p = u_p v_p \quad [3.2]$$

$$A_{p-1}^{-1} = A_p^{-1} - (1/\gamma_p) A_p^{-1} B_p A_p^{-1} \quad [3.3]$$

$$p = n, n-1, \dots, 1$$

The above recurrence relations demand only simple matrix multiplications and additions and no inversion. We can, moreover, see that number of multiplications and additions are only a few because, all elements are zeros except one in  $u_p$  and one  $v_p$ . An efficient computer program is easy to prepare for the aforesaid procedure.

## 2. PROGRAMMING ASPECT

$$(I) \text{ Store } A = A_0 = (a_{ij}) \quad i=1,2, \dots, n; j=1,2, \dots, n$$

$$(II) c_p = a_{pp} \quad p=1,2, \dots, n$$

$c_p$  ( $p=1,2, \dots, n$ ) are the diagonal elements  $a_{ii}$  ( $i=1, 2, \dots, n$ ) of  $A$ .

$$(III) \left\{ \begin{array}{l} d_p = m.n \sum_{q=1}^n |a_{pq}| \quad p=1,2, \dots, n, \quad q \neq p \\ m > 1, \quad m=5, \text{ say} \\ \text{If } d_p = 0 \text{ for any } p, \text{ put } d_p = \text{any non-zero number, say, } 1. \end{array} \right.$$

$d_p$  ( $p=1,2, \dots, n$ ) are the diagonal elements  $a'_{ii}$  ( $i=1,2, \dots, n$ ) of  $P$  which is a diagonal matrix.

$$(IV) \quad a_{pp} = 0 \quad p = 1, 2, \dots, n$$

The matrix  $A$  is now our  $Q$ . The problem is to find  $A_n^{-1}$ .

$$(V) \quad \begin{cases} a_{ii} = 0 & i = 1, 2, \dots, n \\ a_{ij} = b_{ij} = -\frac{a_{ij}}{d_{ij}} & i = 1, 2, \dots, n; j = 1, 2, \dots, n \end{cases}$$

$(a_{ij})$  and  $(b_{ij})$  are the elements of matrix  $-P^{-1}Q$ .

$$(VI) \quad k = 1$$

$$(VII) \quad b_{ij} = 1 + b_{ij} \quad i = 1, 2, \dots, n$$

Now  $(b_{ij})$  are the elements of the matrix  $I + (-P^{-1}Q)$ . The following two steps, namely, steps VIII and IX obtain the value of  $(I + P^{-1}Q)^{-1}$  using Newton-Horner's scheme.

$$(VIII) \quad e_{ij} = \sum_{p=1}^n a_{ip} b_{pj} \quad i = 1, 2, \dots, n; j = 1, 2, \dots, n$$

$$(IX) \quad b_{ij} = e_{ij} \quad i = 1, 2, \dots, n; j = 1, 2, \dots, n$$

$$(X) \quad \begin{cases} f_k = (\sum_{i=1}^n \sum_{j=1}^n b_{ij}^2)^{1/2} \\ \text{if } k > 1 \text{ go to step (XI), if } k \geq 1, \text{ replace } k \text{ by } k+1 \text{ and go to} \\ \text{step (VII).} \end{cases}$$

$$(XI) \quad \begin{cases} f_3 = \text{mod}(f_1 - f_2) \\ \text{if } f_3 < 10^{-8} \text{ say, go to step (XII) otherwise go to step (VI).} \end{cases}$$

$$(XII) \quad b_{ij} = 1 + b_{ij} \quad i = 1, 2, \dots, n$$

Now  $(b_{ij})$  produce the  $A_n^{-1}P$  matrix.

$$(XIII) \quad b_{ij} = \frac{b_{ij}}{d_j} \quad i = 1, 2, \dots, n; j = 1, 2, \dots, n$$

$(b_{ij})$  are now the elements of  $A_n^{-1}$  matrix.

$$(XIV) \left\{ \begin{array}{l} k = n, n-1, \dots, 1 \\ \gamma_k = 1 + b_{kk} (c_k - d_k) \\ e_{ij} = \frac{b_{ik} (c_k - d_k)}{\gamma_k} b_{kj}, \quad i=1, 2, \dots, n; j=1, 2, \dots, n. \\ b_{ij} = b_{ij} - e_{ij} \quad i=1, 2, \dots, n; j=1, 2, \dots, n \end{array} \right.$$

Relations [3.1], [3.2] and [3.3] are computationally represented in step (XIV). ( $b_{ij}$ ) thus obtained are the elements of  $A_0^{-1}$  or  $A^{-1}$ . For  $k=n$ , we determine  $\gamma_n$ , all  $e_{ij}$ 's and all  $b_{ij}$ 's. We then take  $k=n-1$ , and obtain  $\gamma_{n-1}$ , all  $e_{ij}$ 's and then all  $b_{ij}$ 's and so on. Thus for  $k=1$ , we calculate  $\gamma_1$ , all  $e_{ij}$ 's and subsequently  $b_{ij}$ 's. The latest  $b_{ij}$ 's are the elements of  $A^{-1}$  matrix. The '=' sign in all the aforesaid computational steps has the identical meaning as that in Fortran.

*Numerical Results:* 8 dit floating point arithmetic has been employed for all the calculations.

*Example 1.* A matrix that does not satisfy row (or column)-sum criterion.

$$\begin{pmatrix} 1 & 4 & 3 \\ 4 & 2 & 1 \\ 3 & 2 & 2 \end{pmatrix}$$

Three times the sum of the off-diagonal elements in the first, second and third rows are 21, 15 and 15 respectively. If we choose their multiplying factor 10, the number of effective terms in the power series becomes 6 and the final inverse ( $A^{-1}$ ) is correct up to 6 significant figures.

Any additional terms in the power series will not contribute anything towards improving or diminishing the accuracy of  $A_n^{-1}$ . An extra term in the series does, however, improve the accuracy when the precision of calculations is considerably increased.

If the multiplying factor (m. f.) is  $10^2$ , the  $A^{-1}$  is correct up to 5 significant figures. The number of effective terms in this power series for  $A_n^{-1}$  is 4. For the m. f.  $10^3$ ,  $A^{-1}$  becomes less accurate and the accuracy is up to 4 significant figures. When the m. f. is  $10^4$ ,  $A^{-1}$  is correct up to 3 significant figures, the number of terms in the power series for  $A_n^{-1}$  being 3. The inverse is, for m. f. = 1.1,

$$\begin{pmatrix} -.1666667 \times 10^0 & .1666667 \times 10^0 & .1666667 \times 10^0 \\ .4166667 \times 10^0 & .5833333 \times 10^0 & -.9166667 \times 10^0 \\ -.1666667 \times 10^0 & -.8333333 \times 10^0 & .1166667 \times 10^1 \end{pmatrix}$$

and  $AA^{-1}$  is

$$\begin{pmatrix} .1000000 \times 10^1 & .1862645 \times 10^{-8} & .3725290 \times 10^{-8} \\ -.2980232 \times 10^{-7} & .1000000 \times 10^1 & -.1490116 \times 10^{-7} \\ .1490116 \times 10^{-7} & .0000000 \times 10^0 & .1000000 \times 10^1 \end{pmatrix}$$

The number of terms in the series for the aforesaid m. f. is 16 and the result is correct up to all significant figures noted. When m. f. is  $10^5$ , the accuracy of  $A^{-1}$  comes down still further and it is correct up to 2 significant figures. When it is  $10^6$ , the  $A^{-1}$  is correct up to 1 significant figure.

The m. f., when increased, reduces the number of terms (in the power series) and consequently the computing time at the cost of introducing more error due to matrix addition. The other examples which we have attempted produce good results for m. f. lying between 1.1 and 10.

*Example 2.* A matrix satisfying row (or column)-sum criterion

$$\begin{pmatrix} 10 & 5 & 3 & 1 \\ 2 & 8 & 2 & -3 \\ 3 & 2 & 19 & 7 \\ 5 & 2 & 1 & 15 \end{pmatrix}$$

When m. f. = 10, number of terms in the power series is 5 and the inverse is correct up to 7 significant figures. When it is  $10^2$ , the result is correct up to 6 significant figures with effective number of terms in the series = 4. The inverse, for m. f. 1: 1, is

$$\begin{pmatrix} .1265405 \times 10^0 & -.7185299 \times 10^{-1} & -.1149868 \times 10^{-1} & -.1744058 \times 10^{-1} \\ -.4467430 \times 10^{-1} & .1462368 \times 10^0 & -.1028829 \times 10^{-1} & .3702685 \times 10^{-1} \\ -.1980634 \times 10^{-2} & -.5831866 \times 10^{-2} & .5496259 \times 10^{-1} & -.2668354 \times 10^{-1} \\ -.3609155 \times 10^{-1} & .4841549 \times 10^{-2} & .1540493 \times 10^{-2} & .6932218 \times 10^{-1} \end{pmatrix}$$

and  $AA^{-1}$  is

$$\begin{pmatrix} .1000000 \times 10^1 & .3259629 \times 10^{-8} & .1804437 \times 10^{-8} & .1862645 \times 10^{-8} \\ .9313226 \times 10^{-9} & .1000000 \times 10^1 & -.1746230 \times 10^{-8} & .0000000 \times 10^0 \\ .0000000 \times 10^0 & -.4656613 \times 10^{-9} & .1000000 \times 10^1 & -.1862645 \times 10^{-8} \\ .1862645 \times 10^{-8} & -.1862645 \times 10^{-8} & -.4656613 \times 10^{-9} & .1000000 \times 10^1 \end{pmatrix}$$

The number of effective terms here is 12 and the result is correct in all the significant figures shown. For m. f. = 1.5, the number of terms in the series is 10 and the result is identical to the above result up to all the significant figures retained. When m. f. =  $10^5$ , number of terms in the power series is 2 and  $A^{-1}$  is correct up to 3 significant figures.

*Example 3.* A near-singular matrix

$$\begin{pmatrix} 1 & 5 & 3 & 7 \\ 2 & 4 & 1 & 6 \\ 3 & 1 & -2 & 3 \\ 2 & 9.90 & 6 & 14 \end{pmatrix}$$

The (4,2)-th element is made 9.9 instead of 10 to make it slightly near-singular.

When the multiplying factor = 10, number of terms in the power series is 6 and the inverse is correct up to 4 significant figures. The inverse is, for m. f. 1.5 with number of terms in the series 12,

$$\begin{pmatrix} .6750068 \times 10^2 & -.1150010 \times 10^2 & .5500045 \times 10^1 & -.3000031 \times 10^2 \\ .2000022 \times 10^2 & -.3179908 \times 10^{-4} & .1478940 \times 10^{-4} & -.1000010 \times 10^2 \\ .4600046 \times 10^2 & -.9000064 \times 10^1 & .4000030 \times 10^1 & -.2000021 \times 10^2 \\ -.4350045 \times 10^2 & .5500064 \times 10^1 & -.2500030 \times 10^1 & .2000021 \times 10^2 \end{pmatrix}$$

and  $AA^{-1}$  is

$$\begin{pmatrix} 1000003 \times 10^1 & -.3814697 \times 10^{-5} & -.1907349 \times 10^{-5} & -.1335144 \times 10^{-4} \\ .1072884 \times 10^{-5} & .9999990 \times 10^0 & -.4768372 \times 10^{-6} & -.3814697 \times 10^{-5} \\ .2145767 \times 10^{-5} & -.1907349 \times 10^{-5} & .9999981 + 10^0 & -.1144409 \times 10^{-4} \\ -.2145767 \times 10^{-5} & .9536743 \times 10^{-6} & .1907349 \times 10^{-5} & .1000010 \times 10^1 \end{pmatrix}$$

When the (4,2)-th element is made 9.99, the inverse becomes, for m. f. 10 with number of terms in the series 6,

$$\begin{pmatrix} .6076449 \times 10^3 & -.1150218 \times 10^2 & .5501015 \times 10^1 & -.3000717 \times 10^3 \\ .2000490 \times 10^3 & -.7378161 \times 10^{-3} & .3428161 \times 10^{-3} & -.1000242 \times 10^3 \\ .4060965 \times 10^3 & -.9001453 \times 10^1 & .4000676 \times 10^1 & -.2000478 \times 10^3 \\ -.4035970 \times 10^3 & .5501461 \times 10^1 & -.2500679 \times 10^1 & .2000480 \times 10^3 \end{pmatrix}$$

and  $AA^{-1}$  is

$$\begin{bmatrix} .1000088 \times 10^1 & .4577637 \times 10^{-4} & .7629394 \times 10^{-5} & -.1831055 \times 10^{-3} \\ .2670288 \times 10^{-4} & .1000004 \times 10^1 & .0000000 \times 10^0 & -.7629395 \times 10^{-4} \\ .5531311 \times 10^{-4} & .1525879 \times 10^{-4} & .1000008 \times 10^1 & -.1525879 \times 10^{-3} \\ -.5722046 \times 10^{-4} & -.2288818 \times 10^{-4} & .7629394 \times 10^{-5} & .1000137 \times 10^1 \end{bmatrix}$$

When the m. f. = 1.5, the accuracy of  $A^{-1}$  is nearly the same as above; the number of effective terms, however, is doubled.

When (4,2)-th element is made 9.999, the inverse becomes, for m. f. 1.5 with number of effective terms 14,

$$\begin{bmatrix} .6006932 \times 10^4 & -.1150117 \times 10^2 & .5500659 \times 10^1 & -.2999716 \times 10^4 \\ .1999808 \times 10^4 & -.3841519 \times 10^{-3} & .2170578 \times 10^{-3} & -.9999039 \times 10^3 \\ .4005622 \times 10^4 & -.9000781 \times 10^1 & .400044 \times 10^1 & -.1999811 \times 10^4 \\ -.4003120 \times 10^4 & .5500776 \times 10^1 & -.2500438 \times 10^1 & .1999810 \times 10^4 \end{bmatrix}$$

and  $AA^{-1}$  is

$$\begin{bmatrix} .1000488 \times 10^1 & .0000000 \times 10^0 & .0000000 \times 10^0 & .9765625 \times 10^{-3} \\ .1678467 \times 10^{-3} & .1000061 \times 10^1 & .3051758 \times 10^{-4} & .3662109 \times 10^{-3} \\ .3204346 \times 10^{-3} & .0000000 \times 10^0 & .1000061 \times 10^1 & .6103516 \times 10^{-3} \\ -.3356934 \times 10^{-3} & -.1220703 \times 10^{-3} & -.6103516 \times 10^{-4} & .9992676 \times 10^0 \end{bmatrix}$$

Any result better than above can only be achieved by using higher precision arithmetic.

We have, in all the aforesaid examples, used Newton-Horner's<sup>5</sup> scheme for the evaluation of the power series  $(I+P^{-1}Q)^{-1}$ .

#### ACKNOWLEDGMENT

The author wishes to express his sincere gratitude to Prof. P. L. Bhatnagar and Prof. S. Dhawan for their constant encouragement.

#### REFERENCES

1. Sen, S. K. .. .. A Power Series Method for the Inverse of Matrices, communicated to Computer J.
2. Sen, S. K. .. .. Proc. 1st Mastech Conference on Matrix Analysis held at NAL, Bangalore, (Sept. 25-Oct. 3, 1969) pp 113-123.
3. Maulik, T. N. .. .. *Ibid*, 1969, pp 149-160.
4. Eve, J. .. .. *Num. Math.*, 1964, 6, 17.
5. Collatz, L. .. .. *Functional Analysis and Numerical Maths.*, Academic Press (1966), 174.