NONLINEAR TRANSVERSE OSCILLATIONS IN TRAVELLING STRINGS BY A DIRECT LINEARISATION METHOD

By: V. Achyuta Rao Bapat and P. Srinivasan

(*Department of Mechanical Engineering, Indian Institute of Science, Bangalore 12, India*)

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Abstract

Applying Panavko's direct linearisation method the frequency of transverse vibrations of a non-linear travelling string has been obtanied which is more accurate than that obtanied by the author's by the method of harmonic balance earlier. The method in extended to the curve of moving thread line, considered earlier by Ames

1. Introduction

The problem of nonlinear transverse oscillations of travelling strings has been approximately solved by Mote[1] using the method of characteristics. It has been shown that the fundamental frequency of transverse oscillations depends not only on the initial tension, but also on the amplitude of transverse oscillation. The same feature has been demonstrated by the method of harmonic balance[2] and an approximate expression for frequency obtained. The present paper is concerned with the solution of this problem by a direct linearisation procedure. The frequency as obtained by the direct linearisation procedure is found to be more accurate than that obtained by the method of harmonic balance.

2. Equation of Motion

The transverse oscillations of a travelling string are governed by the following nonlinear partial differential equation[1].

$$w_{\eta\eta} + 2\beta\, w_{\xi\eta} - [\,(1-\beta^2) + (3\beta^2/2\lambda^2\, w_{\xi}^{2})\,]\, w_{\xi\xi} = 0 \qquad [1]$$

where

w = nondimensional transverse displacement of the string

η = nondimensional time

ξ = nondimensional axial coordinate

λ = nondimensional velocity

$\beta = \lambda/\alpha$, α = nondimensional initial tension

The boundary conditions are,

$$w(0,\eta) = 0, \qquad w(1,\eta) = 0 \tag{2}$$

The initial conditions are

$$w(\xi, 0) = w^* \sin(\pi\xi), \qquad w_\eta(\xi, 0) = 0 \tag{3}$$

In the harmonic balance method[2], the approximate solution is assumed in the separable form,

$$\tilde{w}(\xi,\eta) = X(\xi) \,.\, T(\eta) \tag{4}$$

As a first approximation, $X(\xi)$ is assumed to be of the form

$$X(\xi) = X_0 \,.\, \sin(\pi\xi) \tag{5}$$

so that the boundary conditions in eq. [2] are satisfied. Substituting eq. [5] into eq. [1] and equating the coefficient of $(\sin\pi\xi)$ to zero, an ordinary differential equations in $T(\eta)$ is obtained.

$$\ddot{T} + T\left\{\pi^2(1-\beta^2)\right\} + T^3\left\{\frac{3}{8}\,\frac{\pi^4\beta^2}{\lambda^2}X_0^2\right\} = 0 \tag{6}$$

Assuming an approximate solution of eq. [6] in the form $\tilde{T}(\eta) = T_0\cos(\omega\eta)$ and equating the coefficient of $(\cos\omega\eta)$ to zero, the approximate frequency will be,

$$\omega = \pi\sqrt{[(1-\beta^2) + (9/32)\,\pi^2\,(\beta^2/\lambda^2)\,\omega^{*2}]} \tag{7}$$

where $w^* = X_0 T_0$ = amplitude of transverse oscillation of the string. Eq. [6] can be solved exactly in terms of elliptic functions. The exact solution is

$$T(\eta) = T_0\, Cn\left[\pi\sqrt{\left((1-\beta^2) + \frac{3}{8}\pi^2\frac{\beta^2}{\lambda^2}w^{*2},\right)}\sqrt{\left(2\left\{\frac{8\lambda^2(1-\beta^2)}{3\pi^2\beta^2 w^{*2}}\right\}+1\right)}\right]^{-1} \quad [8$$

It can be seen that the frequency as obtained by the harmonic balance method is very close to the exact frequency (at least, for small values of the transverse oscillation amplitude w^*).

3. DIRECT LINEARISATION METHOD

Equation [1] can be rewritten in the form

$$w_{\eta\eta} + 2\beta\, w_{\xi\eta} - (1-\beta^2)\, w_{\xi\xi} + (\beta^2/2\lambda^2)\,(\delta/\delta\xi)\,(w_\xi^3) \qquad [9]$$

In the direct linearisation method[3], the term w_ξ^3 in eq [9] is replaced by an equivalent linear term $p^2 w_\xi$ where the constant p^2 is chosen in such a way as to make the mean square of the moment of the difference between the w_ξ^3 and $p^2 w_\xi$ curves a minimum, over the range $-H \leq 0 \leq H$, H, being the amplitude of w_ξ. The minimising condition yields

$$(d/dp^2) \int_0^H [(w_\xi^3 - p^2 w_\xi)\, .\, w_\xi]^2\, d\, w_\xi = 0 \qquad [10]$$

$$\text{or } p^2 = \tfrac{5}{7} H^2 \qquad [11]$$

where H is the maximum value of w_ξ. The approximation for eq [9] will then be

$$\tilde{w}_{\eta\eta} + 2\beta\, \tilde{w}_{\xi\eta} = \tilde{w}_{\xi\xi}\{(1-\beta^2) + (5/14)\,(\beta^2/\lambda^2)\ \ H^2\} \qquad [12]$$

As in the harmonic balance method, an approximate solution of the same form as in eq. [4] is assumed. $X(\xi)$ is further assumed to be of the same form as in eq. [5]. Equating the coefficient of $(\sin \pi\, \xi)$ to zero, the differential equation for $T(\eta)$ becomes,

$$\ddot{T} + \pi^2\,[(1-\beta^2) + (5/14)\,(\beta^2/\lambda 2)\ H^2]\, T = 0 \qquad [13]$$

The amplitude of w_ξ will be $H = \pi\, w^*$ [14]

The approximete frequency is then,

$$\omega = \sqrt{[\,(1-\beta^2) + (5/14)\,(\beta^2/\lambda^2)\,\pi^2\, w^{*2}\,]} \qquad [15]$$

It is evident that eq. [15] represents a closer approximation to the exact frequency of eq. [6] than the approximate frequency as obtained by harmonic balance method. eq. [7]. The error in the quantity

$$\left\{\frac{[\omega^2 - \pi^2\,(1-\beta^2)]\,\lambda^2}{\beta^2\,\pi^4\, w^{*2}}\right\}$$

is 25% by the harmonic balance method and 4.76% by the direct linearisation method.

Ames[4] has considered the nonlinear oscillations of a moving thread line. The approximate fundamental frequency, in two of the three cases considered by him, has been obtained by the method of harmonic balance[2]. The direct linearisation procedure can be applied to these two cases.

Case 1

$$\frac{\alpha^2}{4} U_{xx} + \alpha\, U_{tt} \cong + U_{tt} \frac{U_{xx}}{1+U_x^2}$$

with auxiliary conditions

$$U(0,t) = 0 = U(1,t)$$
$$U(x,0) = U \sin x, \; U_t(x,0) = 0 \qquad [16]$$

Rewriting eq. [16] in the form

$$(\alpha^2/4) U_{xx}\, \pi\; \alpha U_{xt} + U_{tt} - \partial/\partial x \{\tan^{-1}(U_x)\}$$

the nonlinear term $\tan^{-1}(U_x)$ is replaced by an equivalent linear term $p^2 U_x$ where,

$$p^2 = \frac{5}{H^5} \int_0^H U_x^3 [\tan^{-1}(U_x)]\, d\, U_x$$

$$= \frac{5}{H^5} \left\{ \frac{1}{4} (H^4 - 1) \tan^{-1}(H) - \frac{H^3}{12} + \frac{H}{4} \right\}$$

$$= \frac{5}{H^5} \left\{ \frac{1}{4} (H^4 - 1) \left(H - \frac{H^3}{3} + \frac{H^5}{5} - \frac{H^7}{7} \cdots \right) - \frac{H^3}{12} + \frac{H}{4} \right\}$$

$$= \frac{5\;1}{1.5} - \frac{H^2}{3.7} + \frac{H^4}{5.9} - \frac{H^6}{7.11} \cdots$$

$$= 5 \sum_{n=0}^{\infty} \left\{ \frac{H^{2n} (-1)^n}{(2n+1)(2n+5)} \right\} \qquad [17]$$

Equation [16] will then be approximated by

$$\left[\frac{\alpha^2}{4} - 5 \sum_{n=0}^{\infty} \frac{H^{2n} (-1)^n}{(2n+1)(2n+5)} \right] U_{xx} + \alpha U_{xt} + U_{tt} = 0 \qquad [18]$$

Assuming an approximate solution of the form as in eq. [4] and further assuming $X(x)$ to be of the form specified in eq. [5], the differential equation for $T(t)$, after setting the coefficient of $(\sin \pi x)$ to zere will be,

$$\ddot{T} + \left[5 \sum_{n=0}^{\infty} \frac{H^{2n} (-1)^n}{(2n+1)(2n+5)} - \frac{\alpha^2}{4}\right] T = 0 \qquad [19]$$

with $H = TU^*$, the approximate frequency will be

$$\omega = \pi \left[5 \sum_{n=0}^{\infty} \frac{H^{2n} (-1)^n}{(2n+1)(2n+5)} - \frac{\alpha^2}{4}\right]^{1/2} \qquad [20]$$

The frequency, by the harmonic balance method will be,

$$\omega = \pi \left[\left(1 + \frac{3}{16} \pi^2 U^{*2}\right)^{-1} - \frac{\alpha^2}{4}\right]^{1/2} \qquad [21]$$

Case 2

$$\frac{\alpha^2}{4} V^2 U_{xx} + \alpha V U_{xt} + U_{tt} = \frac{U_{xx}}{1 + U_x^2} \;; \qquad V = \frac{1}{1 + U_x^2} \qquad [22]$$

The boundary and initial conditions are the same as in Case 1. Eq. [22] can be rewritten in the form

$$\left(\frac{\alpha^2}{4} - 1\right)\frac{\partial}{\partial x}\{\tan^{-1} U_x\} + \alpha \frac{\partial}{\partial t}\{\log [U_x + \sqrt{(1 + U_x^2)}]\} + U_{tt} = 0 \qquad [23]$$

Replacing $\tan^{-1}(U_x)$ by $p^2 U_x$ and noting that the second term does not contribute to the $T(t)$ equation, the differential equation for $T(t)$ will be (by proceeding along the same lines as in Case 1).

$$\ddot{T} + \pi^2 \left(\frac{\alpha^2}{4} - 1\right)\left[5 \sum_{n=0}^{\infty} \frac{H^{2n} (-1)^n}{(2n+1)(2n+5)}\right] T = 0 \qquad [24]$$

Therefore, the frequency is, (with $H = \pi U^*$)

$$\omega = \pi \left[5\left(\frac{\alpha^2}{4} - 1 \right) \sum_{n=0}^{\infty} \frac{\pi^{2n} U^{*2n} (-1)^n}{(2n+1)(2n+5)}\right]^{1/2} \qquad [25]$$

$$\omega = \pi \left[\frac{[1 - (\alpha^2/4)]}{[1 + (3/16)\pi^2 U^{*2}]}\right]^{1/2} \qquad [26]$$

4. DISCUSSION AND CONCLUSION

From the results presented in the paper it is seen that the solution to the problem of the transverse vibrations of moving a non-linenr travelling string Equation [6] by the method of direct linearisation proposed by Panavko is in error by 4.79 % compared to the exact solution in terms of elliptic functions. Equation [8] on the other hand the solution obtained by the method of harmonic balance is in error by 25 %. This shows that the method of direct linearisation is more accurate than the method of harmonic balance. Having established the order of accuracy of the two approximate procedures, the method of direct linearisation is extended to the problem of the moving thread line and the approximate frequency obtained, whih is for all practical purposes, well within the realm of engineering accuracy.

References

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