FLOW OF A POWER LAW FLUID IN A ROTATING STRAIGHT PIPE – II. HEAT TRANSFER

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ABSTRACT

The heat transfer in the flow of a power law fluid in a straight pipe of circular cross-section rotating about a perpendicular axis is studied. The temperature field is evaluated up to the first order of the rotational Coriolis parameter ξ, when the wall of the pipe is maintained at a uniform temperature. It is found that the fluid is uniformly heated relative to the wall up to a critical value of the parameter α, which involves the Prandtl number and for higher values of α, relatively cooled and heated regions develop. The physical reasons for such a behaviour are analysed. The Nusselt number is not affected by rotation up to the first order of ξ.

1. INTRODUCTION

Many of the coolers use rotating devices and intrinsically the cooling of rotating devices themselves is of great importance in the modern industries. As mentioned by Morris[1], the cooling of turbine rotor blades can be done by the passage of a suitable coolant through internal passages. Since the non-Newtonian fluids are found to be better heat transport media the study of heat transfer phenomena due to flow of non-Newtonian fluids in rotating devices is of practical importance.

The influence of rotation on the velocity and temperature profiles for a Newtonian fluid flowing through a vertical tube rotating about a parallel axis was studied by Morris[1]. He obtained the non-symmetrical axial velocity and temperature profiles with a uniform temperature gradient using a series expansion in powers of the rotational Rayleigh number. His analysis is valid for low rates of heating only. Mori and Nakayama[2] have extended this problem for the case of a predominant secondary flow under the condition of constant wall temperature gradient. In their analysis the entire region is divided into a flow core region and a boundary layer along the wall. The same authors[3] have investigated the problem of heat transfer in the flow of a Newtonian fluid in a straight pipe rotating about a vertical axis. The analysis is similar to that of the previous investigation[2].

However, in all the above investigations the dissipative effects are neglected in the heat transfer problem. But this is not justified at higher Prandtl numbers as they play a dominant role in determining the temperature profiles as seen in § 2.

Here we consider the problem to study the effect of rotation on the heat transfer in a pressure driven flow of a power law fluid in a straight pipe of circular cross-section when the wall of the pipe is maintained at a constant temperature. We consider the pipe to be rotating about a perpendicular axis and all the dissipative effects are taken into consideration Broadly, the procedure adopted is as follows: the velocity field of the flow is obtained and then the temperature field due to convection and dissipation is evaluated. In a recent paper[4] the velocity field is obtained following the method developed by Barua[5] by expanding the flow field in terms of the rotational Coriolis parameter. Here we evaluate the temperature profiles up to the first order of the rotational parameter, when the wall temperature around the periphery of any cross-section is uniform. The isotherms and the variation of temperature for Newtonian, psuedoplastic and dilatant fluids are discussed in detail in § 4. We find that, as in the case of the heat transfer phenomena in the flow of a power law fluid in a curved pipe of circular cross-section, studied earlier[6], irrespective of the nature of the fluid, there exists a critical value of the non-dimensional parameter α, directly proportional to Prandtl number, below which the fluid is uniformly heated and beyond which relatively cooled and heated regions develop. The similarity between the flow in a curved pipe and in a rotating straight pipe can be explained by noting that the effects of centrifugal forces in either case are the same. But it is seen that the Coriolis force developed due to rotation is not as strong as that produced in the case of curved tubes. We have given a detailed discussion of the above phenomena in § 4.

The mean bulk temperature and the Nusselt number are also evaluated. It is found that they are not affected by the rotation up to the first order approximation.

2. Basic equations and formulation of the problem

Consider the flow of a power law fluid due to a constant axial pressure gradient, in a straight pipe rotating with a constnt angular speed Ω about a vertical axis. The constitutive equation for a power law fluid as given by Ostwald[7] and generalised by Tomita[8] is

$$T = -p\, I + \mu_p\, \Theta\, E \qquad [2.1]$$

where T is the stress tensor, p is the pressure, E is the rate of strain tensor, μ_p is a constant and

$$\Theta = \left[E_{11}^2 + E_{22}^2 + E_{33}^2 + 2\,(E_{12}^2 + E_{23}^2 + E_{31}^2) \right]^{(n-1)/1/2} \qquad [2.2]$$

n is the flow behaviour index.

The equations of motion referred to a Cartesian frame of reference X, Y, Z as given in[4] are

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=-\frac{\partial}{\partial X}(P/\rho)+\nu_p\left[\Theta\nabla^2 U+E_{xx}\frac{\partial\Theta}{\partial X}+E_{xy}\frac{\partial\Theta}{\partial Y}\right] \quad [2.3]$$

$$-2\,\Omega\,W+U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}=-\frac{\partial}{\partial Y}(P/\rho)+\nu_p\left[\Theta\nabla^2 V+E_{xy}\frac{\partial\Theta}{\partial X}+E_{yy}\frac{\partial\Theta}{\partial Y}\right] \quad [2.4]$$

$$2\,\Omega\,V+U\frac{\partial W}{\partial X}+V\frac{\partial W}{\partial Y}=-\frac{\partial}{\partial Z}(P/\rho)+\nu_p\left[\Theta\nabla^2 W+E_{zx}\frac{\partial\Theta}{\partial X}+E_{zy}\frac{\partial\Theta}{\partial Y}\right] \quad [2.5]$$

The equation of continuity is

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0 \quad [2.6]$$

Temperature is governed by the equation

$$U\frac{\partial T}{\partial X}+V\frac{\partial T}{\partial Y}=\frac{k}{\rho\,C_p}\left[\frac{\partial^2 T}{\partial X^2}+\frac{\partial^2 T}{\partial Y^2}\right]+\frac{1}{\rho\,C_p}\phi\,. \quad [2.7]$$

where C_p is the specific heat, k is the thermal conductivity, ϕ is the dissipation given by

$$\phi=T_{ij}\cdot E_{ij}\,. \quad [2.8]$$

Here $U(X, Y)$, $V(X, Y)$ and $W(X, Y)$ are the velocity components in the directions of X, Y, Z with X-axis as the axis of rotation and z-axis is along the axis of the pipe. $T(X, Y)$ is the temperature at any point (X, Y).

The boundary conditions are

$$\left.\begin{matrix} U=V=W=0 \\ T=T_w \end{matrix}\right] \text{ on } F(X, Y)=0\,, \quad [2.9]$$

where $F(x, Y)=0$ is the equation of the cross-section of the pipe and T_w is the constant wall temperature. We introduce the stream function Ψ such that

$$U=-\frac{\partial\Psi}{\partial Y}, \quad V=\frac{\partial\Psi}{\partial X}. \quad [2.10]$$

To study the temperature distribution of the flow in a circular pipe of radius a, we transform to cylindrical polar coordinates defined by

$$X=R\sin\theta, \quad Y=R\cos\theta, \quad Z=Z. \quad [2.11]$$

We introduce the non-dimensional variables $r, \psi, w, \Theta_1, \tilde{T}$ by substituting

$$R = ar$$

$$\Psi = \bar{U} a \Psi$$

$$W = \overline{W} w$$

$$\Theta = (\overline{W}^{n-1}/a^{n-1})\, \Theta_1$$

$$T = \tilde{T}\, T_c + T_w \quad . \qquad [2.12]$$

where

$$\overline{U} = \frac{2\,\Omega\, a^{n+1}}{\nu_p} \left(\frac{C\, a^{n+1}}{\nu_p} \right)^{(2-n)/n},$$

$$\overline{W} = \left(\frac{C\, a^{n+1}}{\nu_p} \right)^{1/n}.$$

$T_c = T_{11} \sim T_{12}$, difference in the inlet and outlet temperatures. $C = (1/\rho)(\partial P/\partial Z)$ is the given constant axial pressure gradient and $\nu_p = \mu_p/\rho$.

In the previous paper[4], the equations [2.3] to [2.6] in non-dimensional forms are solved for the velocity profile upto the second order of the rotation parameter

$$\xi = \frac{\overline{U}}{\overline{W} K}, \quad \left(K = \frac{Ca}{\overline{W}^2} \right), \qquad [2.13]$$

As in the case of Newtonian fluids here also the main flow is associated by a secondary flow induced due to rotation. The velocity components upto 1st order of ξ are

$$u = \xi \sin\theta \cos\theta \;[\,(1-s)\, B^1 r^{s-1} - 2\,(n+1)/n\; C^1 r^{2/n+2}], \qquad [2.14]$$

$$v = \xi\, [\cos^2\theta \;(A^1 + B^1 r^{s-1} + C^1 r^{2/n+2}) + \sin^2\theta \,\{A^1 + s\, B^1 r^{s-1}$$
$$+ (2/n+3)\; C^1 r^{2/n+2}\}], \qquad [2.15]$$

$$w_0 = w_0 + \xi\, w_1\,, \qquad [2.16]$$

where

$$w_0 = \frac{1}{2^{1/n}} \;\frac{n}{n+1}\; (1 - r^{1+1/n})\,, \qquad [2.17]$$

$$w_1 = \frac{n\, r^{1/n}}{2^{(2-n)/n}} \left[\frac{A^1}{2\,(n+1)^2}\,(1 - r^{1+1/n}) + \frac{B^1\,(1 - r^{s+1/n})}{(ns+1)\,(ns+2+n)} + \frac{C^1}{12\,(n+1)^2}\,(1 - r^{3/n+3}) \right] \cos\theta, \qquad [2.17]'$$

$$A^1 = C^1 \left[\frac{3n - ns + 2}{n\,(s-1)} \right]$$

$$B^1 = \frac{2\,C^1\,(n+1)}{n\,(1-s)}$$

$$C^1 = \frac{1}{2^{(n+2)/n}}\ \frac{n^4}{(n+1)\,(3n+1)\,(n^2+4n+1)}$$

and

$$s = \frac{n+1}{2n} + \frac{\sqrt{\{(\sqrt{17}\cdot n - 1)^2 + 2n\,(\sqrt{17} - 1)\}}}{2n}.$$

The non-dimensional form of the energy equation is

$$\frac{\partial^2 \tilde{T}}{\partial r^2} + \frac{1}{r}\,\frac{\partial \tilde{T}}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 \tilde{T}}{\partial \theta^2}$$

$$= \alpha\xi \left[\cos\theta\,(A^1 + B^1\, r^{s-1} + C^1\, r^{2/n+2})\,\frac{\partial \tilde{T}}{\partial r} - \sin\theta \{A^1 + s\,B^1\, r^{s-1} + (2/n+3)\, r^{2/n+2}\}\,\frac{1}{r}\,\frac{\partial \tilde{T}}{\partial \theta} \right] - \beta \left(\frac{r}{2}\right)^{(n+1)/n} \left[1 - \xi\,(n+1) \left(\frac{2}{r}\right)^{1/n} \frac{\partial w_1}{\partial r} \right] + 0\,(\xi^2), \qquad [2.18]$$

where

$$\alpha = (\bar{U}/\bar{W})\,\mathrm{Pr}, \qquad \beta = K.\ \mathrm{Pr.}\ \mathrm{Ec}, \qquad [2.19]$$

and

$\mathrm{Pr.} = \dfrac{\rho\, C_p\, a^2}{k} \left(\dfrac{Ca}{\nu_p}\right)^{1/n}$, the generalised Prandtl number,

$\mathrm{Ec.} = 2^{(n+1)/2}\, \dfrac{\overline{W}^2}{C_p T_c}$, the generalised Eckert number.

The second part on the right hand side of the equation [2 18] is the contribution due to the dissipative effects. The boundary condition is

$$\tilde{T}=0 \text{ on } r=1. \tag{2 20}$$

3 SOLUTION OF THE ENERGY EQUATION

Taking $\tilde{T}$ up to the first order in ξ as

$$\tilde{T}= T_0 (r, \theta)+\xi\, T_1 (r, \theta)\,, \tag{3.1}$$

substituting for the velocity components from [2.14] to [2.17] and separating various order terms, [2.18] reduces to

$$\frac{\partial^2 T_0}{\partial r^2}+\frac{1}{r}\frac{\partial T_0}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T_0}{\partial \theta^2}=-\frac{\beta}{2^{(n+1)/n}}\, r^{(n+1)/n}\,, \tag{3 2}$$

and

$$\frac{\partial^2 T_1}{\partial r^2}+\frac{1}{r}\frac{\partial T_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T_1}{\partial \theta^2}=\alpha\,(A^1+B^1 r^{s-1}+C^1 r^{2/n+2})\,\frac{\partial T_0}{\partial r}\cos\theta$$

$$+\,\beta\,\frac{(n+1)}{2}\,r\,\frac{\partial w_1}{\partial r}\,, \tag{3 3}$$

with the bounary conditions

$$T_0=0,\;\; T_1=0 \text{ on } r=1. \tag{3 4}$$

The solutions of [3.2] and [3.3] with [3.4] are

$$T_0=\frac{1}{2^{(n+1)/n}}\,\beta\,\frac{n^2}{(3n+1)^2}\,(1-r^{(3n+1)/n}) \tag{3.5}$$

and

$$T_1 (r, \theta)=-n^2 r\cos\theta\Bigg[\frac{E_1}{4\,(n+1)}\,(1-r^{2/n})+\frac{E_2\,(1-r^{1+3/n})}{3\,(n+1)\,(3+n)}$$

$$+\frac{E_3\,(1-r^{s+3/n})}{(ns+3+2n)\,(ns+3)}+\frac{E_4\,(1-r^{3+5/n})}{5\,(1+n)\,(5+3n)}$$

$$+\frac{E_5\,(1-r^{3+1/n})}{(3n+1)\,(5n+1)}+\frac{E_6\,(1-r^{s+1/n+2})}{(ns+2n+1)\,(ns+4n+1)}+\frac{F_7\,(1-r^{3+3/n})}{(3+7n)\,(3+5n)}\Bigg] \tag{3 6}$$

where

$$E_1 = \frac{\beta}{2^{2/n}} \cdot \frac{1}{12\,(n+1)} \cdot \left[6A^1 + \frac{12\,(n+1)^2\,B^1}{(ns+1)\,(ns+2+n)} + C^1 \right],$$

$$E_2 = -\frac{\beta}{2^{2/n}} \cdot A^1 \cdot \frac{(2+n)}{(n+1)},$$

$$E_3 = -\frac{\beta}{2^{2/n}} \cdot B^1 \cdot \frac{(n+1)\,(ns+2)}{(ns+1)\,(ns+2+n)}$$

$$E_4 = -\frac{\beta}{2^{2/n}} \cdot C^1 \cdot \frac{(4+3n)}{12\,(n+1)}$$

$$E_5 = -\frac{1}{2^{(n+1)/n}} \cdot \beta\alpha \cdot \frac{n}{3n+1} A^1$$

$$E_6 = -\frac{1}{2^{(n+1)/n}} \cdot \beta\alpha \cdot \frac{n}{3n+1} \cdot B^1$$

$$E_7 = -\frac{1}{2^{(n+1)/n}} \cdot \beta\alpha \cdot \frac{n}{3n+1} \cdot C^1.$$

4. Discussion of the Result

The solution of the energy equation up to the first order of the rotation parameter ξ consists of two parts T_0 and T_1, T_0 being the temperature profile when the rotation is neglected and T_1 is due to the rotation and with all the dissipation effects taken into consideration. We discuss these solutions for Newtonian and Non-Newtonian fluids separately.

(a) *Newtonian Fluids* :

The isotherms for a Newtonian fluid ($n=1.0$) are plotted in figure 1, taking $\alpha = 500$. It is found that the cross-section of the pipe is divided into cooled and heated regions by the $\tilde{T}=0$ isotherm and in each domain the non-zero isotherms form closed curves. The off-side domain towards which the Coriolis force acts is the heated rigion with the maximum relative rise in temperature being $M_1 = 0.057501$, while the other domain, the on-side end represents a cooled region, the maximum drop in temperature being $M_2 = -0.029700$. This asymmetry is due to the Coriolis force induced due to rotation which pushes the elements from the on-side end to the off-side end. This type of separation has been found in the case of a bent pipe also[6], there the induced force being due to the curvature of the pipe.

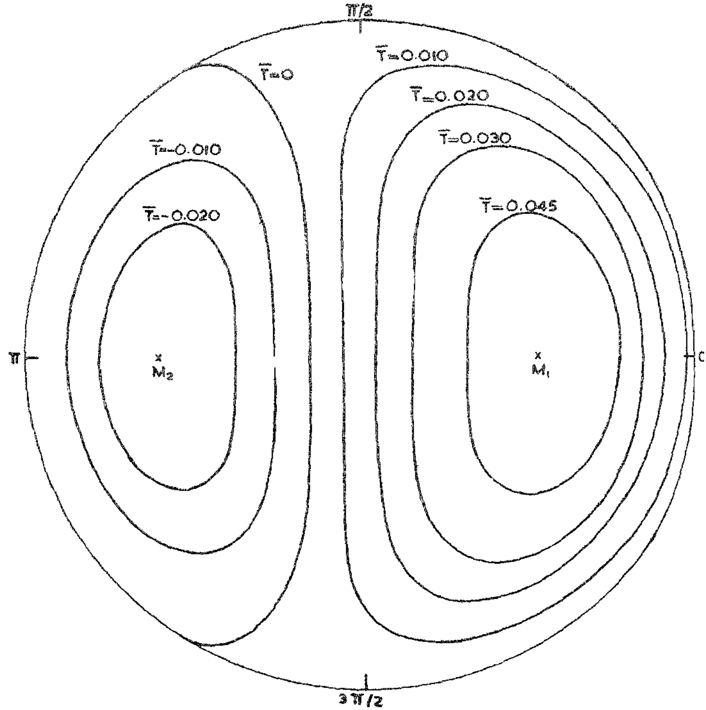

FIG. 1

Isotherms for Newtonian Fluids in the Cross Section ($n=1.0$)

Figure 2. gives the variation of temperature for a Newtonian fluid for different values of α, thus for different values of the Prandtl number. It is noticed that beyond a critical value of α, the cross-section of the pipe is divided into cooled and heated regions with $\bar{T}=0$ as the separating isotherm. But for lower values of α, the fluid is heated uniformly throughout the cross-section. It is evident from the energy equation that the increase of α can be thought of as an apparent increase in the Reynolds number. It is seen from [4] that as the rotational speeds increase that is, as the Reynolds number increases an almost shear free central region and a boundary layer type flow in the off-side end of the cross-section towards which the Coriolis force acts, develop. For low rotational speeds the swirling secondary flow

is weak and it is symmetrical about the plane of rotation in any cross-section. Thus as rotational speeds increase beyond a critical value the off-side region is highly sheared and there is an almost shear free mid region and the on-side end, thus producing cooled region on the on-side and heated region on the off-side. This seperation is due to the behaviour of the flow field and consequently needs a critical Prandtl number. For lower values of α, the shear free region is very less and hence no separation takes place. The figure 1 confirms this pattern and from numerical computations we find that the critical value of α is about 155.

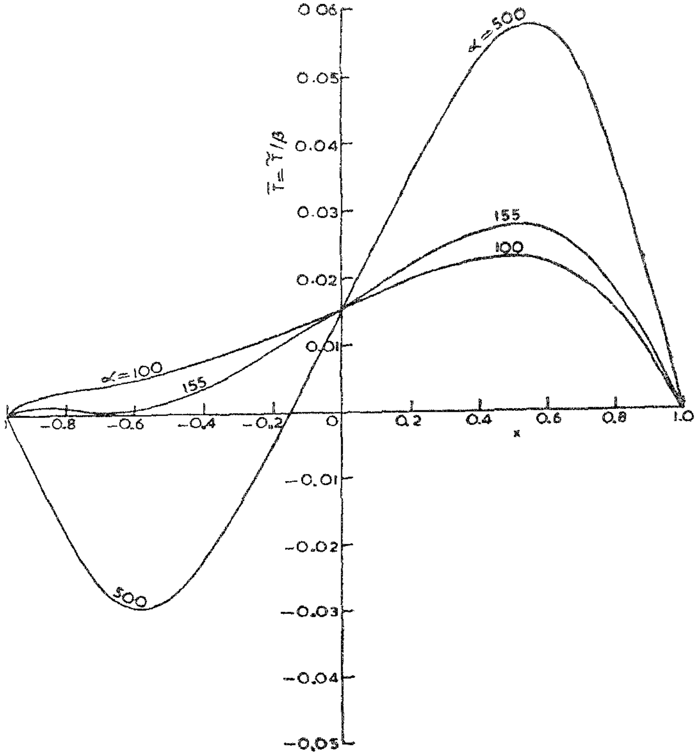

FIG. 2

Variation of temperature for Newtonian Fluid for different values of α when $\theta=0, \pi$.

(*b*) *Non-Newtonian fluids :*

A comparison of isotherms for dilatant ($n=1.2$) and psuedoplastic fluids ($n=0.8$) is given in figure 3. The general pattern for non-Newtonian fluids is similar to that of Newtontan fluids.

As it has been pointed out earlier[6], the psuedoplastic fluids ($n<1$) sustain less strain than the dilatant ($n>1$) or Newtonian fluids ($n=1.0$). Thus for the former fluids both heat generation and convective effects are less and for the later they are more. So a weaker separation of heated and cooled regions is expected for the psuedoplastic fluids which is evident from the figures 3 and 4. As in the case of Newtonian fluids we can find the critical Prandtl numbers for non-Newtonian fluids also. The following table gives the relative maximum and minimum temperatures and the critical values of α for different values of n.

n	Maximum relative heating $\alpha=500$	Minimum relative Cooling $\alpha=500$	Critical values of α
0 8	0 627447 (r=0.5)	−0.006352 (r=0 6)	310
1.0	0 057501 (r=0.5)	−0.029700 (r=0.6)	155
1.2	0.100017 (r=0.5)	−0 067318 (r=0 6)	95

The figure 4 gives the variation of temperature for Newtonian fluids for a fixed value of $\alpha=500$. From these results we can conclude that dilatant fluids are better and more efficient media that may be used as coolants. As compared with our previous results[6], we find that the relative heating and cooling is very much less in the present case,

Also, it is interesting to note that irrespective of the fluid either Newtonian or non-Newtonian, the heated and cooled regions develop in the same domain of the cross-section.

5. Nusselt number

The non-dimensional mean bulk temperature from [3.5] to [3 7] is given by

$$\bar{T}_v = 2^{1/n-1} \frac{\beta n^2 (4n+1)}{(5n+1)(3n+1)^2} + \frac{T_w}{T_c} \qquad [5.1]$$

The mean heat flux per unit length along the cross-section of the pipe in dimensional form is

$$q = -\frac{k}{2\pi a} \int_0^{2\pi} \left(\frac{\partial T}{\partial R}\right)_{R=a} a\, d\theta \qquad [5.2]$$

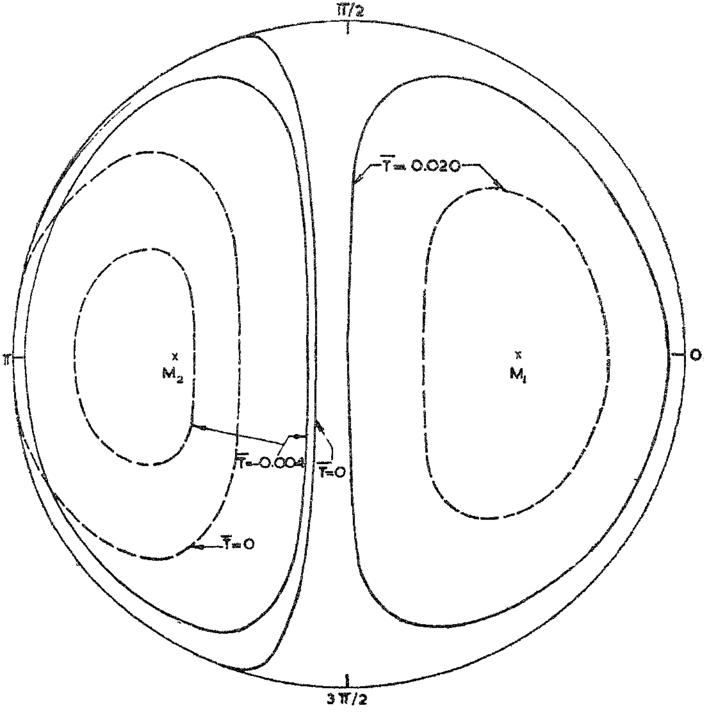

FIG. 3

Comparison of Isotherms for dilatant (n=1.2 –) and psuedoplastic (n=0.8 – —) Fluids.

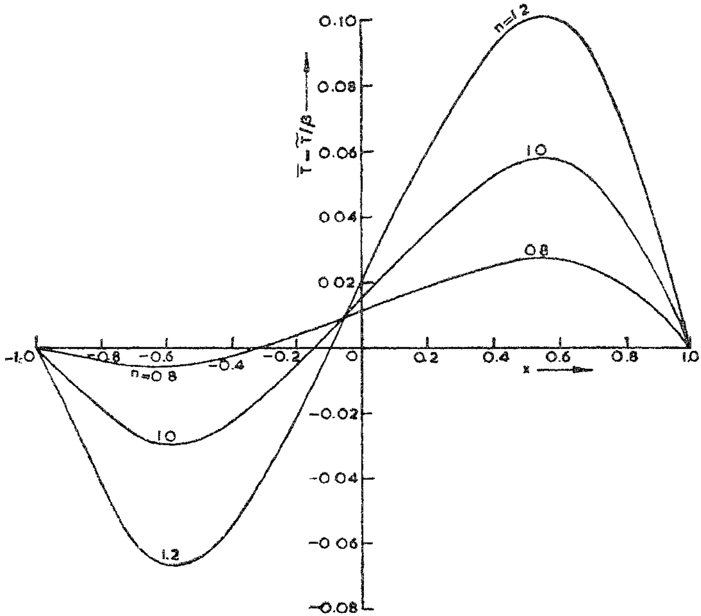

FIG. 4

Variation of temperature for different values of n when $\theta=0,\pi$ and $\alpha=500$

The Nusselt number defined by

$$Nu = 2\, aq/k\, \bar{T}_v \qquad [5.3]$$

is found to be

$$Nu = 1/[G\,(n) + H\,(n)] \qquad [5.4]$$

where

$$\left.\begin{aligned} G\,(n) &= 2^{2/n-1}\frac{n\,(4n+1)}{(5n+1)\,(3n+1)} \\ H\,(n) &= \frac{T_w}{T_c}\cdot\frac{2^{1/n}\,(3n+1)}{\beta\, n} \end{aligned}\right\} \qquad [5.5]$$

It is evident that up to first order of ξ, the Nusselt number is not affected by the rotation as in the case of bent pipes.

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