TRIANGULAR DECOMPOSITIONS AND THEIR APPLICATIONS IN SOLVING EQUATIONS

By : SYAMAL KUMAR SEN

(Central Instruments and Services Laboratory, Indian Institute of Science, Bangalore-12, India)

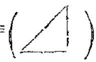
[Received January 6, 1971]

ABSTRACT

Discussed here are all possible triangular decompositions and their role to solve linear equations. Also presented are the computational formulae for five of these triangular decompositions to provide illustrations regarding their computational use

1. TRIANGULAR DECOMPOSITIONS

We define a triangular matrix as a square matrix having zero elements above (or below) the left diagonal (or right diagonal). The following four sets of triangular matrices are considered for the decomposition of a square matrix and the solution of linear equations. They are :

$\overset{1}{L_l}, \overset{2}{P_l} \equiv$ , $\overset{3}{L_r}, \overset{4}{P_r} \equiv$ 

$\overset{5}{R_l}, \overset{6}{U_l} \equiv$ , $\overset{7}{L_r}, \overset{8}{P_r} \equiv$ 

L_l, P_l are called lower triangular matrices of left diagonal L_r, P_r are called lower triangular matrices of right diagonal R_l, U_l are upper triangular matrices of left diagonal. R_r, U_r are upper triangular matrices of right diagonal. 1, 2, 3, 4, 5, 6, 7, 8 are numerical symbols to denote $L_l\,(=l_{ij})$, $P_l\,(=p_{ij})$, $L_r\,(=l'_{ij})$, $P_r\,(=p'_{ij})$, $R_l\,(=r_{ij})$, $U_l\,(=u_{ij})$, $R_r\,(=r'_{ij})$, $U_r\,(=u'_{ij})$, respectively. The two triangular matrices L_l and P_l have the identical form but different elements. Similar is the case with L_r. P_r ; R_l, U_l ; R_r, U_r.

The following chart (chart 1) gives us all possible triangular decompositions [*i e.*, decomposition of a square matrix into two triangular matrices (product form)] including invalid ones.

CHART 1

x 12	x 23	. 34	x 45	x 56	x 67	. 78
x 21	. 32	. 43	. 54	x 65	. 76	. 87
x 13	x 24	x 35	x 46	x 57	x 68	
. 31	. 42	. 53	. 64	. 75	. 86	
x 14	. 25	x 36	x 47	x 58		
. 41	. 52	. 63	x 74	. 85		
. 15	. 26	x 37	x 48			
. 51	. 62	x 73	x 84			
. 16	. 27	x 38				
. 61	x 72	x 83				
. 17	. 28					
x 71	x 82					
. 18						
x 81						

'x' indicates invalid triangular decompositions as these decompositions produce triangular forms in the product and consequently not valid to represent (or replace) a square matrix. There are totally 28 ($=6+4+6+6+4+2$) invalid triangular decompositions out of 56 ($=8^2-8$). '.' indicates the valid triangular decompositions. There are totally 28 ($=56-28$) valid triangular decompositions that can be used to replace a square matrix. All these 28 valid decompositions are not different as can be seen from the following chart (chart 2).

CHART 2

15 = 25 ≡ 16 ≡ 26	51 ≡ 52 ≡ 61 ≡ 62	17 ≡ 27 ≡ 18 ≡ 28
31 ≡ 41 ≡ 32 ≡ 42	75 ≡ 85 ≡ 76 ≡ 86	53 ≡ 54 ≡ 63 ≡ 64
34 ≡ 43	78 ≡ 87	

'≡' indicates 'is identical in nature'. Thus there are eight different triangular decompositions that can represent a square matrix uniquely.

Failures: When a matrix possesses one or more vanishing leading minors on left diagonal L_lR_l (*i.e.*, 15) algorithm collapses. The R_lL_l (*i.e.* 51) algorithm fails for one or more trailing minors that are zero on the left diagonal. L_lR_r (*i.e.*, 17) and L_rL_l (*i.e.*, 31) algorithms, on the other hand, blows up for some leading vanishing minors on the right diagonal. The R_rR_l

(*i.e.*, 75) decomposition meets the aforesaid ill-fate for some vanishing trailing minors on the right diagonal. Finally we put forth the fact that all the eight triangular decompositions collapse for a coefficient matrix A possessing vanishing leading and trailing minors on both left and right diagonals even if the coefficient matrix A may be non-singular. The following matrix, for instance,

$$\begin{bmatrix} 2 & 1 & -1 & -3 \\ 4 & 2 & 2 & 6 \\ 5 & 6 & -5 & -2 \\ 5 & 6 & -10 & -4 \end{bmatrix}$$

is not decomposable by any triangular decomposition though it is highly non-singular.

We are now confronted with two situations. Should we go on trying triangular decompositions one after the other or should we restrict ourselves to only one of the eight decompositions using the row (or column) interchanging technique ? In case we posess the idea of the leading and trailing minors, we can go ahead with the suitable decomposition. Otherwise, we follow the latter process.

2. COMPUTATIONAL ALGORITHMS

Derived here are the simple explicit computational recurrence relations for obtaining L_l, R_l, L_r and R_r and their inverses along with the solution vector $\vec{x}$ from the aforesaid five useful decompositions making use of matrix operation rules. The summation sign used below has the conventional meaning, *i e.*, the summation has to be taken as zero if its upper bound is less than the lower bound. The '=' sign in any recurrence relation has the identical meaning as that in Fortran. The '≡' sign, on the other hand, carries the meaning 'equivalent' all throughout.

We store the matrix A and the column vector $\vec{b}$ as below :

$$A=(a_{ij}), \quad i=1,\ 2,\ \ldots,\ n;\ j=1,\ 2,\ \ldots,\ n$$

and

$$\vec{b}=(b_1\ b_2,\ \ldots,\ b_n)'$$

where (′) indicates transpose. The sequence of subscripts i and j (to represent matrix elements) in

$$`i=1,\ 2,\ \ldots,\ n;\ j=1,\ 2,\ \ldots,\ n'$$

indicate that i is to take 1 (fixed) first and j is to be varied from 1 to n at an interval of 1. Then $i=2$ (fixed) and $j=1,\ 2,\ 3,\ \ldots,\ n$. Next $i=3$ (fixed),

$j=1, 2, \ldots, n$ and so on. At every change of subscripts, we determine different elements. A brief note for the mode of change of subscripts, however, is added in some cases for easy and quick access to the recurrence relations.

If none of the leading minors on the left diagonal are zeros, then

$$A = L_l R_l, \quad l_{ii}=1 \ \forall\ i \qquad [1]$$

$$\left.\begin{array}{ll} & j=1, 2, \ldots, n \\ R_l: & a_{ij} = a_{ij} - \sum\limits_{p=1}^{i-1} a_{ip}\, a_{pj} \\ & i=1, 2, \ldots, j \\ L_l: & a_{ij} = (a_{ij} - \sum\limits_{p=1}^{j-1} a_{ip}\, a_{pj})/a_{jj} \\ & i=j+1, j+2, \ldots, n \end{array}\right\} \qquad [1.1]$$

Letting $j=1$, we obtain a_{11} $(\equiv r_{11})$ from the first recurrence relation of [1.1] and then $a_{21}, a_{31}, \ldots, a_{n1}$ $(\equiv l_{21}, l_{31}, \ldots, l_{n1})$ from the second one. For $j=2$, a_{12}, a_{22} $(\equiv r_{12}, r_{22})$ are determined from the first relation and $a_{32}, a_{42}, \ldots, a_{n2}$ $(\equiv l_{32}, l_{42}, \ldots, l_{n2})$ from the second and so on. Lastly, for $j=n$ we find $a_{1n}, a_{2n}, \ldots, a_{nn}$ $(\equiv r_{1n}, r_{2n}, \ldots, r_{nn})$ from the first relation.

$$\text{Det } A = \prod_{i=1}^{n} a_{ii} \qquad [1.2]$$

In $L_l R_l \vec{x} = \vec{b}$ i.e., in $R_l \vec{x} = L_l^{-1} \vec{b} = \vec{c}$, we have the elements of L_l^{-1} matrix :

$$L_l^{-1}: \quad a_{ij} = -a_{ij} - \sum_{p=j+1}^{i-1} a_{ip}\, a_{pj} \qquad [1.3]$$

$$j=1, 2, \ldots, n-1;\ i=j+1, j+2, \ldots, n$$

The elements of the column vector $\vec{c}$ are :

$$\vec{c}: \quad b_i = b_i + \sum_{p=1}^{i-1} a_{ip}\, b_p \qquad [1.4]$$

$$i=n, n-1, \ldots, 2 \ (b_1 \text{ remains unchanged})$$

The solution vector $\vec{x}$ is :

$$\vec{x}: \quad b_i = (b_i - \sum_{p=i+1}^{n} a_{ip}\, b_p)/a_{ii} \qquad [1.5]$$

$$i=n, n-1, \ldots, 1$$

where $\vec{x}=(x_1,\ x_2,\ \ldots\ ,\ x_n)'\equiv(b_1,\ b_2,\ \ldots\ ,\ b_n)'$..

When none of the trailing minors on the left diagonal are zeros, then

$$A=R_l L_l,\ r_{ii}=1\quad\forall\ i \tag{2}$$

$$\left.\begin{array}{ll} & j=n,\ n-1,\ \ldots\ ,\ 1 \\ L_l: & a_{ij}=a_{ij}-\sum\limits_{p=i+1}^{n} a_{pj}\,a_{ip} \\ & i=n,\ n-1,\ \ldots\ ,\ j \\ R_l: & a_{ij}=(a_{ij}-\sum\limits_{p=j+1}^{n} a_{pj}\,a_{ip})/a_{jj} \\ & i=j-1,\ j-2,\ \ldots\ ,\ 1 \end{array}\right\} \tag{2 1}$$

The elements of L_l and R_l matrices have the identical subscripts with those of A.

$$\text{Det } A=\prod_{i=1}^{n} a_{ii} \tag{2 2}$$

In $R_l L_l \vec{x}=\vec{b}$ *i.e.*, in $L_l\vec{x}=R_l^{-1}\vec{b}=\vec{c}$, we have the elements of R_l^{-1} matrix:

$$R_l^{-1}:\quad a_{ij}=-a_{ij}-\sum_{p=i+1}^{j+1} a_{ip}\,a_{pj} \tag{2 3}$$

$$j=n,\ n-1,\ \ldots\ ,\ 2;\ i=j-1,\ j-2,\ \ldots\ ,\ 1$$

The elements of the column vector $\vec{c}$ are:

$$\vec{c}:\quad b_i=b_i+\sum_{p=i+1}^{n} a_{ip}\,b_p \tag{2.4}$$

$$i=1,\ 2,\ \ldots\ ,\ n-1$$

The solution vector $\vec{x}$ is:

$$x:\quad b_i=(b_i-\sum_{p=1}^{i-1} a_{ip}\,b_p)/a_{ii} \tag{2.5}$$

$$i=1,\ 2,\ \ldots\ ,\ n$$

where $\vec{x}=(x_1,\ x_2,\ \ldots\ ,\ x_n)'\equiv(b_1,\ b_2,\ \ldots\ ,\ b_n)'$

If none of the leading minors are vanishing on the right diagonal, then

$$A=L_l R_r,\ l_{ii}=1\quad\forall\ i \tag{3}$$

$$\left.\begin{aligned}
&i=1,\ 2,\ \ldots,\ n\\
L_l:\quad & a_{i,\,n-j+1}=(a_{i,\,n-j+1}-\sum_{p=1}^{j-1} a_{i,\,n-p+1}\,a_{p,\,n-j+1})/a_{j,\,n-j+1}\\
&j=1,\ 2,\ \ldots,\ i-1\\
R_r:\quad & a_{ij}=a_{ij}-\sum_{p=1}^{i-1} a_{i,\,n-p+1}\,a_{pj}\\
&j=1,\ 2,\ \ldots,\ (n-i+1)
\end{aligned}\right\}\qquad [3.1]$$

For $i=1$, we find $a_{11}, a_{12}, \ldots, a_{1n}\ (\equiv r'_{11}, r'_{12}, \ldots, r'_{1n})$ from the second recurrence relation of [3.1] For $i=2$, we obtain $a_{2n}\ (\equiv l_{21})$ from the first relation and then we find $a_{21}, a_{22}, \ldots; a_{2,\,n-1}\ (\equiv r'_{21}, r'_{22}, \ldots, r'_{2,\,n-1})$ from the second relation. For $i=3$, we determine $a_{3n}, a_{3,\,n-1}\ (\equiv l_{31}, l_{32})$ from the first relation and $a_{31}, a_{32}, \ldots a_{3,\,n-2}\ (\equiv r'_{31}, r'_{32}, \ldots, r'_{3,n-2})$ from the second relation, and so on. Finally, for $i=n$, we find $a_{nn}, a_{n,\,n-1}, \ldots, a_{n2}$ $(\equiv l_{n1}, l_{n2}, \ldots, l_{n,\,n-1})$ from the first relation and $a_{n1}\ (\equiv r'_{n1})$ from the second relation.

$$\text{Det } A=(-1)^{[n/2]}\prod_{i=1}^{n} a_{i,\,n-i+1} \qquad [3.2]$$

where $[n/2]$ is the integral part of $n/2$.

In $L_l R_r \vec{x}=\vec{b}$ *i.e*, in $R_r\vec{x}=L_l^{-1}\vec{b}=\vec{c}$, we have the elements of L_l^{-1} matrix :

$$L_l^{-1}:\quad a_{i,\,n-j+1}=-a_{i,\,n-j+1}-\sum_{p=j+1}^{i-1} a_{i,\,n-p+1}\,a_{p,\,n-j+1} \qquad [3.3]$$

$$j=1,\ 2,\ \ldots,\ n-1;\ i=j+1,\ j+2,\ \ldots,\ n$$

The elements of the column vector $\vec{c}$ are:

$$\vec{c}:\quad b_j=b_j+\sum_{p=1}^{j-1} a_{j,\,n-p+1}\,b_p \qquad [3.4]$$

$$j=n,\ n-1,\ \ldots,\ 2$$

b_1 remains unchanged

where $\vec{c}=(c_1, c_2, \ldots, c_n)'\equiv(b_1, b_2, \ldots, b_n)'$

The solution vector $\vec{x}$ is:

$$\vec{x}:\quad b_i=(b_i-\sum_{p=1}^{n-i} a_{ip}\,b_{n-p+1})/a_{i,\,n-i+1} \qquad [3.5]$$

$$i=n,\ n-1,\ \ldots,\ 1$$

where $\vec{x}=(x_1, x_2, \ldots, x_n)' \equiv (b_n, b_{n-1}, \ldots, b_1)'$

When none of the leading minors vanishes on the right diagonal,

$$A = L_r L_l, \quad l'_{i, n-i+1} = 1 \quad \forall\ i \tag{4}$$

$$j = n,\ n-1,\ \ldots,\ 1$$

$$L_l. \qquad a_{n-i+1, j} = a_{n-i+1, j} - \sum_{p=i+1}^{n} a_{n-p+1, j}\, a_{n-i+1, p}$$

$$i = n,\ n-1,\ \ldots,\ j \tag{4.1}$$

$$L_r: \qquad a_{ij} = (a_{ij} - \sum_{p=j+1}^{n} a_{n-p+1, j}\, a_{ip})/a_{n-j+1, j}$$

$$i = n-j+2,\ n-j+3,\ \ldots,\ n$$

For $j=n$, we find a_{1n} (l_{nn}) from the first recurrence relation and $a_{2n}, a_{3n}, \ldots, a_{nn}$ $(l'_{2n}, l'_{3n}, \ldots, l'_{nn})$ from the second relation. For $j=n-1$, we calculate $a_{1, n-1}, a_{2, n-1}$ $(\equiv l_{n, n-1}, l_{n-1, n-1})$ from the first relation and $a_{3, n-1}, a_{4, n-1}, \ldots, a_{n, n-1}$ $(\equiv l'_{3, n-1}, l'_{4, n-1} \ldots, l'_{n, n-1})$ from the second relation For $j=n-2$, we get $a_{1, n-2}, a_{2\ n-2}, a_{3, n-2}$ $(\equiv l_{n\ n-2}, l_{n-1\ n-2}, l_{n-2, n-2})$ from the first relation and $a_{4, n-2}, a_{5, n-2}, \ldots a_{n, n-2}$ $(\equiv l'_{4, n-2}, l'_{5, n-2}, \ldots, l'_{n, n-2})$ from the second relation and so on. Lastly, for $j=1$, we find $a_{11}, a_{21}, \ldots, a_{n1}$ $(\equiv l_{n1}, l_{n-1, 1}, \ldots, l_{11})$ from the first relation.

$$\text{Det } A = (-)^{[n/2]} \prod_{i=1}^{n} a_{n-i+1, i} \tag{4.2}$$

In $L_r L_l \vec{x} = \vec{b}$, *i.e.*, in $L_l \vec{x} = L_r^{-1} \vec{b} = \vec{c}$, we have the elements of L_r^{-1} matrix:

$$L_r^{-1}: \quad a_{ij} = -a_{ij} - \sum_{p=n-i+2}^{j-1} a_{ip}\, a_{n-p+1, j} \tag{4.3}$$

$$j = n,\ n-1,\ \ldots,\ 2;\ i = n-j+2,\ n-j+3,\ \ldots,\ n$$

The elements of the column vector $\vec{c}$ are:

$$\vec{c}: \qquad b_j = b_j + \sum_{p=n-j+2}^{n} a_{jp}\, b_{n-p+1}$$

$$j = n,\ n-1,\ \ldots,\ 2$$

b_1 remains unchanged [4.4]

where $\vec{c}=(c_1,\ c_2,\ \ldots,\ c_n)'\equiv(b_n,\ b_{n-1},\ \ldots,\ b_1)'$

The solution vector $\vec{x}$ is:

$$\vec{x}:\qquad b_i=\Big(b_i-\sum_{p=1}^{n-i}a_{ip}\,b_{n-p+1}\Big)\Big/a_{i,\ n-i+1} \qquad [4.5]$$

$$i=n,\ n-1,\ \ldots,\ 1$$

where $\vec{x}=(x_1,\ x_2,\ \ldots,\ x_n)'\equiv(b_n,\ b_{n-1},\ \ldots,\ b_1)'$

When none of the trailing minors vanishes on the right diagonal,

$$A=R_r\,R_l\,,\ r_{ii}=1\quad\forall\ i \qquad [5]$$

$$i=n,\ n-1,\ \ldots,\ 1$$

$$\left.\begin{aligned} R_r:\quad & a_{ij}=a_{ij}-\sum_{p=1}^{j-1}a_{ip}\,a_{n-p+1,\ j}\\ & j=1,\ 2,\ \ldots,\ n-i+1\\ R_l:\quad & a_{ij}=\Big(a_{ij}-\sum_{p=1}^{n-i}a_{ip}\,a_{n-p+1,\ j}\Big)\Big/a_{i,\ n-i+1}\\ & j=n-i+2,\ n-i+3,\ \ldots,\ n \end{aligned}\right\} \qquad [5.1]$$

For $i=n$, we find $a_{n1}\ (\equiv r'_{n1})$ from the first recurrence relation and a_{n2}, $a_{n3},\ \ldots\ a_{nn}\ (\equiv r_{12},\ r_{13},\ \ldots,\ \ldots,\ r_{1n})$ from the second relation. For $i=n-1$, $a_{n-1,\ 1},\ a_{n-1,\ 2}\ (\equiv r'_{n-1,\ 1},\ r'_{n-1,\ 2})$ are obtained from the first relation and $a_{n-1,\ 3},\ a_{n-1,\ 4},\ \ldots,\ a_{n-1,\ n}\ (\equiv r_{23},\ r_{24},\ \ldots,\ r_{2n})$ from the second relation. For $i=n-2$, we calculate $a_{n-2,\ 1},\ a_{n-2,\ 2}\ a_{n-2,\ 3}\ (\equiv r'_{n-2,\ 1},\ r'_{n-2,\ 2},\ r'_{n-2,\ 3})$ from the first relation and $a_{n-2,\ 4},\ a_{n-2,\ 5},\ \ldots,\ a_{n-2,\ n}\ (\equiv r_{34},\ r_{35},\ \ldots,\ r_{3n})$ from the second relation, and so on. Lastly, for $i=1$, we find $a_{11},\ a_{12},\ \ldots,$ $a_{1n}\ (\equiv r'_{11},\ r'_{12},\ \ldots,\ r'_{1n})$ from the first relation.

$$\text{Det}\ A=(-1)^{[n/2]}\ \prod_{i=1}^{n}a_{i,\ n-i+1} \qquad [5.2]$$

In $R_r\,R_l\,\vec{x}=\vec{b}$, *i e.*, in $R_l\,\vec{x}=R_r^{-1}\vec{b}=\vec{c}$, we have the elements of R_r^{-1} matrix:

$$\left.\begin{aligned} R_r^{-1}:\quad & a_{ij}=1/a_{ij}\ \text{ if }\ i+1>n-j+1\\ & \text{otherwise,}\\ & a_{ij}=\Bigg(-\sum_{p=i+1}^{n-j+1}a_{pj}\,a_{i,\ n-p+1}\Bigg)\Bigg/a_{i,\ n-i+1}\\ & j=1,\ 2,\ \ldots,\ n\ ;\ i=n-j+1,\ n-j,\ \ldots,\ 1 \end{aligned}\right\} \qquad [5\ 3]$$

The elements of the column vector $\vec{c}$ are :

$$\vec{c}: \qquad b_j = \sum_{p=j}^{n} a_{j,\, n-p+1}\, b_p \qquad [5\ 4]$$

$$j=1,\ 2,\ \ldots,\ n$$

where $\vec{c}=(c_1,\ c_2,\ \ldots,\ c_n)' \equiv (b_n,\ b_{n-1},\ \ldots,\ b_1)'$

The solution vector $\vec{x}$ is :

$$\vec{x}: \qquad b_i = b_i - \sum_{p=n-i+2}^{n} a_{ip}\, b_{n-p+1} \qquad [5\ 5]$$

$$i=1,\ 2,\ \ldots,\ n$$

where $\vec{x}=(x_1,\ x_2,\ \ldots,\ x_n)' \equiv (b_n,\ b_{n-1},\ \ldots,\ b_1)'$

3. REMARKS

To encounter the unforeseen cases where one or more leading (or trailing) minors vanish on the left (or the right) diagonal, we reserve a shortage of $(n+1)$ locations for the current row (or column) of the augmented matrix (A,b). As soon as an element on the left (or right) diagonal becomes zero, we swing to the interchange of rows (or columns) from the current row (or column) onwards. It is interesting to note that any one of the eight decompositions (that cannot be restarted because of the gradual destruction of the original matrix in the computer memory) can proceed to obtain the solution vector x for any nonsingular system.

Gaussian algorithms[1] (Gauss, Doolittle, Crout, Cholesky and Banachiewicz) are, however, basically the same as our $L_l R_l$ (*i.e.*, 15) algorithm and consequently have to be performed with row (or column) interchanging technique for vanishing leading minors on the left diagonal.

REFERENCES

1. L. Fox .. .. An introduction to Num. Linear Algebra, Clarendon Press, Oxford, 1964, 68, 99.