A SECOND ORDER PROCESS FOR SOLVING POLYNOMIAL EQUATIONS

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ABSTRACT

Developed here is an accurate second order process for obtaining real and complex zeros of a real polinomial. This procedure is mainly concerned with McAuley method (or an equally good alternative method suggested) coupled with Newton's. A complex polynomial though not treated here directly, finds its place through a conversion algorithm (Conversion from complex to real polynomial) described here. A polynomial of repeated zeros, though remains notorious to both McAuley type and Newton methods [as $\{d\, p_N(x_i)/dx_i\} \to 0$], is very well bevaved with the aforesaid procedure. Algorithms for the evaluation of a real polynomial and its derivatives, as they form an integral part of the procedure, are also discussed.

1. THE PROCEDURE

Lin-Bairstaw[1], McAuley[2] and other methods of quadratic factors are not amenable to a polynomial having multiple zeros. The present procedure is a stable iterative one that provides us, unlike McAuley, Lin-Bairstaw and other methods[3], all the pairs of the roots having same accuracy. We first obtain two approximate zeros of an N-th degree polynomial $p_N(x)$ using McAuley method or the alternative method suggested below. Then to sharpen these two zeros we use a technique based on L' Hospital's rule coupled with Newton's method taking care of higher derivatives of $p_N(x)$ that may tend to zero for repeated roots.

Computational steps of McAuley and the suggested methods: We assume that the degree of the polynomial is even, for if it is not so it can be multiplied by $(x+1)$ introducing thereby an extraneous root of -1. A slightly altered and convenient form of common computational steps are given below. Let

$$p_N(x) = x^N + A_N x^{N-1} + \ldots + A_2 x + A_1 = 0$$

Writing $p_N(x)$ as

$$p_N(x) = (x^2 + mx + n)\, a(x) + R_1 x^{N-1} + R_2 x^{N-2}$$

where $a(x)$ is given by

$$a(x) = x^{N-2} + a_{N-2} x^{N-3} + \ldots + a_2 x + a_1.$$

We obtain a's and R_1, R_2 of the remainder term $R_1 x^{N-1} + R_2 x^{N-2}$ from the trial values of m and n as

(I) $a_i = \dfrac{A_i - ma_{i-1} - a_{i-2}}{n}$, $\quad a_0, a_{-1} = 0$; $i = 1, 2, \ldots, N-2$

(Ia) $a_{N-1} = 1$

(II) $R_1(m, n) = A_N - (m + a_{N-2})$

(III) $R_2(m, n) = A_{N-1} - (n + ma_{N-2} + a_{N-3})$

To obtain the changes in trial values of m and n in order to make the remainder term zero, McAuley method uses two term Taylor's series of two variables but the suggested method uses the aforesaid algorithms I, I*a*, II, III producing almost identical results.

McAuley computional steps to obtain $\dfrac{\partial R_1}{\partial m}$, $\dfrac{\partial R_2}{\partial m}$, $\dfrac{\partial R_1}{\partial n}$, $\dfrac{\partial R_2}{\partial n}$:

Let b's are given by

(IV) $b_i = \dfrac{a_i - mb_{i-1} - b_{i-2}}{n}$, $\quad b_0, b_{-1} = 0$; $i = 1, 2, \ldots, N-2$

then

(V) $\alpha_1 \equiv \dfrac{\partial R_1}{\partial m} = -(1 - b_{N-3})$

(VI) $\alpha_2 \equiv \dfrac{\partial R_2}{\partial m} = -(a_{N-2} - mb_{N-3} - b_{N-4})$

(VII) $\beta_1 \equiv \dfrac{\partial R_1}{\partial n} = b_{N-2}$

(VIII) $\beta_2 \equiv \dfrac{\partial R_2}{\partial n} = -(1 - mb_{N-2} - b_{N-3})$

Suggested computational steps to obtain $\dfrac{\partial R_1}{\partial m}$, $\dfrac{\partial R_2}{\partial m}$, $\dfrac{\partial R_1}{\partial n}$, $\dfrac{\partial R_2}{\partial n}$:

$$\text{(SV)} \quad \alpha_1 \equiv \frac{\partial R_1}{\partial m} = \frac{R_1(m+h, n) - R_1(m, n)}{h}$$

$$\text{(SVI)} \quad \alpha_2 \equiv \frac{\partial R_2}{\partial m} = \frac{R_2(m+h, n) - R_2(m, n)}{h}$$

$$\text{(SVII)} \quad \beta_1 \equiv \frac{\partial R_1}{\partial n} = \frac{R_1(m, n+k) - R_1(m, n)}{k}$$

$$\text{(SVIII)} \quad \beta_2 \equiv \frac{\partial R_2}{\partial n} + \frac{R_2(m, n+k) - R_2(m, n)}{k}$$

where h, k are two small quantities (say .1 or .05). The *rhs* quantities are obtained using relations I, Ia, II, III.

New common trial values are then given by

$$\text{(IX)} \quad \Delta m_K = \frac{\beta_1 R_2 - \beta_2 R_1}{\alpha_1 \beta_2 - \alpha_2 \beta_1}, \quad m_{K+1} = m_K + \Delta m_K$$

$$\text{(X)} \quad \Delta n_K = \frac{\alpha_2 R_1 - \alpha_1 R_2}{\alpha_1 \beta_2 - \alpha_2 \beta_1}, \quad n_{K+1} = n_K + \Delta n_K$$

This iterative process gives us the approximate quadratic factor of the polynomial $p_N(x)$, which in turn produces two approximate roots that are dealt with the following process.

Newton-Raphson's method coupled with L' Hospital's rule ; The Newton Raphson's iterative scheme is written as

$$x_{i+1} = x_i - \frac{p_N(x_i)}{p_N^{(1)}(x_i)}, \quad i = 0, 1, 2, \ldots \qquad \text{[A]}$$

where x_0 is an initial complex approximation. When $p_N(x)$ has s repeated roots

$$p_N^{(q)}(x_i) \to 0 \text{ as } x_i \to \text{the repeated root}$$

$$q = 1, 2, \ldots, s-1,$$

but $p_N^{(s)}(x_i)$ is non-zero. The iterative scheme [A] never allows us to obtain a repeated root as $p_N^{(1)}(x_i)$ always tends to zero near the root. The consequence is the oscillation of x_i' near the root as $p_N(x_i)$ tends to zero more rapidly than $p_N^{(1)}(x_i)$. Evidently Newton-Raphson's method has to be modified for repeated roots.

The finite degree polynomial with finite coefficients is a continuous and bounded function of x. If there are repeated roots, $p_N(x_i)$ and $p_N^{(1)}(x_i) \to 0$ as $x_i \to$ a repeated root resulting (0/0) form for

$$p_N(x_i)/p_N^{(1)}(x_i)$$

Numerically, when $|p_N^{(1)}(x_i)| <$ a small positive number, say, 10^{-3}, we expect repetition of the roots to which we are approaching. Consequently, we can replace, using L' Hospital's rule,

$$\lim_{x_i \to \text{ a repeated root}} \frac{p_N(x_i)}{p_N^{(1)}(x_i)} \text{ by } \lim_{x_i \to \text{ repeated root}} \frac{p_N^{(1)}(x_i)}{p_N^{(2)}(x_i)}$$

In general, for s (unknown) repeated roots we have, the general form

$$x_{i+1} = x_i - \frac{p_N^{(q)}(x_i)}{p_N^{(q+1)}(x_i)}, \quad q = 0, 1, 2, \ldots, s-1 \qquad [B]$$

$$i = 0, 1, 2, \ldots$$

x_0 is a good initial complex approximation.

It is, however, not necessary to know how many roots are repeated, since $p_N^{(q+1)}(x_i)$ will provide us the information through its values. The iterative scheme [B] necessitates the evaluation of $p_N(x)$ and its derivatives. These are obtained through Newton-Horner's recurrence scheme as given below :

Newton-Horner's recurrence scheme[3] *for evaluating $p_N(x)$ and its derivatives :*

Let

$$f(x) = d_0 x^N + d_1 x^{N-1} + \ldots + d_N$$

then

$$f(x) : \begin{cases} p_0 = d_0 \\ p_{i+1} = x p_i + a_{i+1} \\ \quad i = 0, 1, 2, \ldots, N-1 \end{cases} \qquad [C]$$

The value of p_N for a given x is the value of f. For k-th derivative, the scheme has the following general form :

$$f^{(k)}(x) : \begin{cases} p_0 = \prod_{j=0}^{k-1} (n-j)\, a_0 \\ p_{i+1} = x p_i + \prod_{j=0}^{k-1} [n-(i+1)-j] a_{i+1} \\ \quad i = 0, 1, 2, \ldots, N-k-1 \\ N > k \geq 1 \end{cases} \qquad [D]$$

Following the schemes [B], [C] and [D] for each of the two approximate roots, we obtain the two sharpened zeros of the quadratic factor and redetermine the accurate m and n. Taking out this quadratic factor x^2+mx+n from $p_N(x)$, we treat the resultant polynomial $p_{N-2}(x)$ using McAuley or the alternative process. The zeros of the second approximate quadratic factor thus obtained are treated through the schemes [B], [C] and [D] where [C] and [D] always uses the original polynomial $p_N(x)$ instead of $p_{N-2}(x)$. This ensures the same accuracy of all the N zeros of the polynomial.

2. Conversion of a Complex Polynomial to a Real Polynomial

$$p_N(x)=x^N+(a_{N-1}+j\,b_{N-1})\,x^{N-1}+(a_{N-2}+j\,b_{N-2})\,x^{N-2}+\;.\;.\;.\;+(a_1+j\,b_1)\,x$$
$$+(a_0+j\,b_0)=0$$

We have to obtain $p_N(x)\,p_N^*(x)$ which is a real polynomial of degree $2N$. Coefficient of x^m :

$$(a)\quad 2\left[\sum_{p=0}^{(m/2)-1}(a_{m-p}\,a_p+b_{m-p}\,b_p)\right]+\frac{a_m^2}{2}+\frac{b_m^2}{2}$$

for $0\leq m<N$ and m is even

$$(b)\quad 2\left[\sum_{p=0}^{(m-1)/2}a_{m-p}\,a_p+b_{m-p}\,b_p)\right]$$

for $0\leq m<N$ and m is odd

$$(c)\quad 2\,a_{m-N}+2\left[\sum_{p=m-N+1}^{(m/2)-1}(a_{m-p}\,a_p+b_{m-p}\,b_p)\right]+\frac{a_m^2}{2}+\frac{b_m^2}{2}$$

for $2N>m\geq N$ and m is even

$$(d)\quad 2\,a_{m-N}+2\left[\sum_{p=m-N+1}^{(m-1)/2}(a_{m-p}\,a_p+b_{m-p}\,b_p)\right]$$

for $2N>m\geq N$ and m is odd

(e) 1 for $m=2N$

3. Numerical Example

Besides increasing rounding errors associated with successive quadratic factors, McAuley and the alternative method usually work out well for a polynomial of all distinct zeros. The following simple example of a polynomial of multiple zeros, which is notorious to McAuley and similar other methods, illustrates its good behaviour with the present procedure Calculations are carried out in 25 dit floating point arithmetic (using CDC 3600 computer).

$$p_4(x) = x^4 + 12x^3 + 54x^2 + 108x + 81 = 0$$

McAuley method gives

$$-3.0005588912 \pm j\ 00041835544$$

and

$$-2.9992488620 \pm j\ .00090097350$$

as roots even with such a high precision calculation. The present procedure produces accurate roots *i e.*,

$$-3,\ -3,\ -3,\ -3$$

correct up to all 25 digits.

4. REMARKS

The successive approximation in [B] is of the form

$$x_{i+1} = \phi(x_i),\quad i = 0, 1, 2, \ldots \qquad [E]$$

where

$$\phi(x_i) = x_i - \frac{p_N^{(q)}(x_i)}{p_N^{(q+1)}(x_i)},\quad q = 0, 1, 2, \ldots, s-1$$

The sufficient condition for convergence of the scheme [E] is

$$\left|\frac{\partial \phi}{\partial x}\right|_{x=xi} < 1 \qquad [F]$$

This condition should be satisfied near and at the root for (B) to be stable. But for the repeated roots the condition (F) is not satisfied at the root when the scheme is Newton's. Irrespective of the multiplicity of the roots the condition (F) is satisfied for the scheme (B).

If there are N repeated roots a of $p_N(x) = 0$, we may think of the following scheme :

$$x_{i+1} = x_i - \frac{(x_i - a)^N}{N(x_i - a)^{N-1}} = x_i - \frac{x_i - a}{N} \qquad [G]$$

$$i = 0, 1, 2, \ldots$$

Since a is not known, we cannot use (G) as it is, even though the denominator is a non-zero quantity N. We can, however, write (G) in the following form :

$$x_{i+1} = x_i - \frac{[f(x_i)]^{1/N}}{N},\quad i = 0, 1, 2, \ldots \qquad [H]$$

If we can obtain $[f(x_i)]^{1/N}$ accurately, [H] could be a good scheme for repeated roots. But unfortunately $[f(x_i)]^{1/N}$ cannot be obtained at the root using exponential and logarithmic functions. Even successive square-rooting process[4] involves considerable rounding errors. Furthermore we have no knowledge of the number of the repeated roots in practice. The scheme [B] is thus an attractive scheme both for any number of repeated as well as for non-repeated roots.

Even though the present second order procedure provides us all pairs of roots one after the other, unlike Henrici's linear process[5, 6] for the simultaneous approximation of all zeros of a polynomial, accuracy of all the pairs remains same. The precision of the computer, however, sets the limitation to our desired accuracy.

The initial approximation for m and n in the trial quadratic factor x^2+mx+n should be such that

$$\left|\frac{x\Delta_1 m+\Delta_1 n}{x^2+mx+n}\right|<1$$

for convergence[2]. If we take $m=1.5\,A_N$ and $n=A_{N-1}$ $(A_{N-1}\neq 0)$, we usually achieve convergence.

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