STEP FUNCTION RESPONSE OF NONLINEAR SPRING MASS SYSTEMS TAKING INTO ACCOUNT THE STATIC DEFLECTION

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ABSTRACT

In this paper the effect of static-deflection on the step function response of a non-linear spring mass system is investigated. The maximum displacement of the system is obtained by integrating the equation of motion straight away. An approximate method of determining the period of oscillation is considered and compared with the exact values, that can be obtained for some special cases of restoring force characteristics. The above analysis reveals that the static deflection has a prefound effect on both the peak displacement and the period of oscillation of the non-linear system.

List of Symbols :

m	...	mass
x	...	displacement
x^*	...	maximum displacement
V	...	velocity
$\bar{V}$	...	maximum velocity
$\bar{x}$	...	displacement corresponding to $\bar{V}$
t	...	time
t^*	...	period of oscillation
τ^*	...	non-dimensional period of oscillation
$\bar{\tau}^*$	...	approximate non-drmensional period
ϵ	...	percentage error in period of oscillation

Δ	...	static deflection
$\bar{\Delta}$	...	non-dimensional static deflection
F_0	...	Force
g	...	acceleration due to gravity
ξ	...	variable of integration
p	...	approximate frequency of oscillation
m_1	...	moment about centre of vibration
M	...	displacement of centre of vibration from the equilibrium position.
$\bar{E}^2$	...	mean square moment
y^*	...	maximum non-dimensional displacement
H_0	...	Non-dimensional force amplitude
$\alpha, \beta, \gamma, \nu$		non-linearity parameters

1. Introduction

The step function response of nonlinear spring mass systems exhibits certain features that are not encountered in linear systems, when the gravity effect of oscillating mass is considered. In linear spring mass systems, except for a shift in the equilibrium position, the gravity force of the oscillating mass has no effect on such salient characteristics of the motion as (*a*) the maximum displacement, (*b*) the period of oscillation. However, in nonlinear, systems, the static deflection of the system influences both these characteristics of the step response. In the present analysis the effect of static deflection on these features will be investigated. The maximum displacement of the system may be obtained by integrating the equation of motion straightway. To determine the period of oscillation a second integration of the equation of motion will be necessary. A closed form expression for the period cannot in general be obtained, because of the complicated nature of the integrand. In the analysis that follows an approximate method for determining the period of oscillation will be considered[1]. The accuracy of this approximation will be determined by comparison with the exact values that can be obtained for some special types of restoring force characteristics. The extension of the approximate procedure for more general types of restoring force characteristics will be indicated.

2. EQUATION OF MOTION AND SOLUTION

The Schematic diagram of the system is shown in Fig. (1-a). Let the restoring force characteristic of the system be represented by $f(x)$, a continuous, single valued, nonlinear odd function of the displacement x. If the

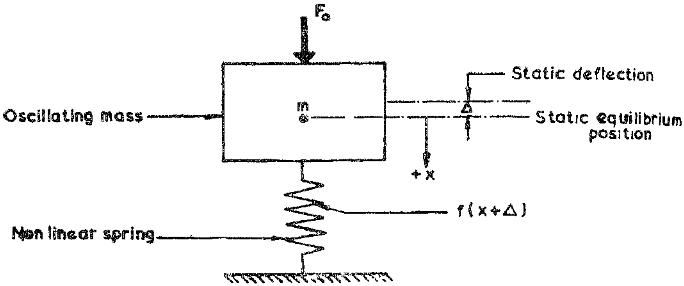

FIG. 1a

Schematic Diagram of the Oscillating System

characteristic is assumed to be of the hardening type then $f'(x)$ increases with increasing x. The equation of motion taking into account the static deflection will be,

$$m(d^2x/dt^2)+f(x+\Delta)=F_0+mg, \qquad F_0>0, \qquad [1]$$

where the static deflection Δ satisfies the static equilibrium condition,

$$f(\Delta)=mg \qquad [2]$$

If the system is initially assumed to be at rest, the initial conditions will be,

$$x=0,\ \dot{x}=0 \text{ at } t=0 \qquad [3]$$

Letting $\dot{x}=V$ and using equation [2], equation [1] may be rewritten in the differential form,

$$(dV/dx)=[F_0+f(\Delta)-f(x+\Delta)]/mV \qquad [4]$$

It can be seen from equation [1] that corresponding to the initial conditions in [3] the aceleration of the system will be positive as soon as the step forcing function is applied to it. Hence the velocity will be positive in the neighbourhood of $t=0$ and the displacement increases initially. The velocity reaches a maximum value of $\bar{V}$ when the aceleration is zero. The corresponding displacement $\bar{x}$, will be the solution of:

$$f(\bar{x}+\Delta)=F_0+f(\Delta). \qquad [5]$$

The first integral of the system with initial conditions as specified in equation [3] is,

$$m(\dot{x}^2/2)+\int_0^x f(\xi+\Delta)\,d\xi=[F_0+f(\Delta)]\,x \qquad [6\text{-a}]$$

The maximum displacement x^* of the system is obtained by letting $x=x^*$, $\dot{x}=0$ in equation [6-a]. This yields,

$$\int_0^{x^*} f(\xi+\Delta)\,d\xi=[F_0+f(\Delta)]\,x^* \qquad [6\text{-b}]$$

Further, equation [6-a] shows that the phase trajectory is symmetrical about the x-axis. Hence the oscillations are periodic and the system oscillates between the positions $x=0$ and $x=x^*$.

Separating the variables in equation [6-a] and integrating,

$$\int_0^t dt=t=(m)^{1/2}\int_0^x \left\langle 2\left[\{F_0+f(\Delta)\}x-\int_0^x f(\xi+\Delta)\,d\xi\right]\right\rangle^{-1/2} dx$$

ξ being a dummy variable of integration. Observing that the time taken by the system in undergoing the displacement from $x=0$ to $x=x^*$ is $\frac{1}{2}\,t^*$, and using equation [6-b], the period of oscillation t^* will be,

$$t^*=\sqrt{(2mx^*)}\int_0^{x^*}\left[x\int_0^{x^*} f(\xi+\Delta)\,d\xi-x^*\int_0^x f(\xi+\Delta)\,d\xi\right]^{-1/2} dx \qquad [7\text{-a}]$$

3. Approximate Solution by Direct Linearisation Method: [2]

An approximate expression for the period of oscillation of the system in equation [1] can be obtained by Panovko's Direct linearisation method. This procedure consists in replacing the nonlinear restoring force characteristic $\phi(x)=(1/m)\,\{f(x+\Delta)-F_0-f(\Delta)\}$ by an approximate linear characteristic $p^2(x-M)$, where M represents the displacement of the centre of vibration from the equilibrium position $x=0$ and p, the approximate frequency of oscillation. In the present case as the system oscillates between the positions $x=0$ and $x=x^*$, M will have the value of $(x^*/2)$. The constant p^2 of the approximating linear characteristic is so chosen as to make the mean square of the moment of the difference between $\phi(x)$ and $p^2(x-M)$, about the centre of vibration a minimum over the range $0 \leq x \leq x^*$.

Referring to figure (1-b) the difference $r(x)$ between the exact and approximating characteristics is,

$$r(x) = \phi(x) - p^2(x-M)$$
$$= [(1/m)\{f(x+\Delta) - F_0 - f(\Delta)\} - p^2(x-M)]$$

and the moment about the centre of vibration is,

$$m_1(x) = [(1/m)\{f(x+\Delta) - F_0 - f(\Delta)\} - p^2(x-M)] \cdot (x-M)$$

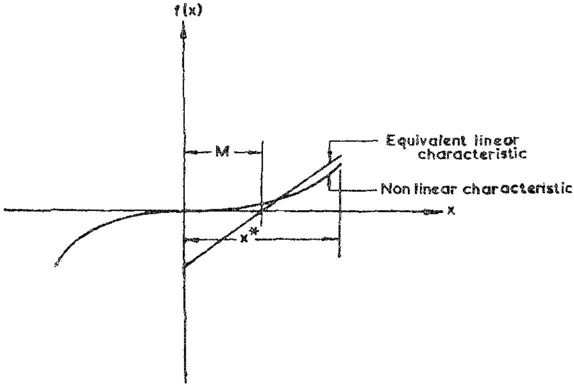

FIG. 1b

Diagram Illustrating the Principle of Direct Linearisation Method

The mean square moment $\bar{E}^2$ will be,

$$\bar{E}^2 = (1/x^*) \int_0^{x^*} [(1/m)\{f(x+\Delta) - F_0 - f(\Delta)\} - p^2(x-M)]^2 (x-M)^2 \, dx$$

For this mean square moment to be a minimnm, $(\partial E^2/\partial p^2)=0$. This yields,

$$p^2=\frac{5}{m\,[(x^*-M)^5+M^5]}\int_0^{x^*}[f(x+\Delta)-F_0-f(\Delta)]\,(x-M)^3\,dz$$

Substituting $M=x^*/2$,

$$p^2=\frac{80}{m\,x^{*5}}\int_0^{x^*}[f(x+\Delta)-F_0-f(\Delta)]\cdot\left(x-\frac{x^*}{2}\right)^3dx$$

Let $\quad x-x^*/2=u$

Then,

$$p^2=\frac{80}{mx^{*5}}\int_{-x^*/2}^{x^*/2}f\left(u+\Delta+\frac{x^*}{2}\right)u^3\,du \qquad [7\text{-b}]$$

Some particular cases of restoring force characteristics that permit exact evaluation of the integral in equation [7-a] will now be considered. The expressions for the approximate period will be developed alongsides, using equation [7-*b*]. The accuracy of the approximate period relative to the exact period will be determined in each of these cases, to judge the feasibility of satisfactorily applying the direct linearisation mothod for such systems whose restoring force characteristics do not permit an exact evaluation of the period of oscillation.

4. Particular Cases

(i) $f(x)=\alpha x+\beta\,(Sgn\,x)\,|x|^2$, *$\alpha$ and β being positive constants*

From equation [6-b], the maximum displacement will be given by,

$$(\alpha/2)\,x^{*2}+\alpha\,\Delta x^*+(\beta/3)\,x^{*3}+\beta\,\Delta\,x^{*2}+\beta\,\Delta^2x^*=F_0\,x^*+\alpha\,\Delta x^*+\beta\,\Delta^2\,x^*$$

Substituting,

$y^*=(\beta/\alpha)\,x^*$ = maximum nondimensional displacement,

$\bar{\Delta}=(\beta/\alpha)\,\Delta$ = nondimensional static deflection,

$H_0=(\beta/\alpha^2)\,F_0$ = nondimensional force amplitude, the above equation simplifies to,

$$3\,y^*\,(1+2\bar{\Delta})+2\,y^{*2}=6\,H_0$$

or $$y^*=\frac{3\,(1+2\bar{\Delta})}{4}\;\{[1+16\,H_0/\langle 3\,(1+2\,\Delta)^2\rangle^{-1}]^{1/2}-1\} \qquad [8]$$

The variation of y^* with H_0 for various values of the non-dimensional static deflection parameter $\bar{\Delta}$ is shown in figure (2).

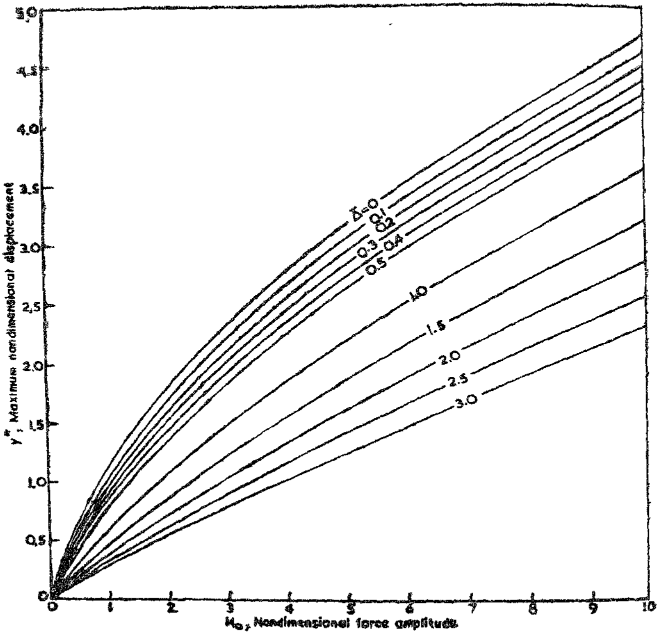

FIG. 2

Variation of y^* with H_0: $\{f(x)-f(x)=\alpha x+\beta\,(\mathrm{sgn}\,x)\,|x|^2,\ \alpha>0,\ \beta>0\}$

From equation [7-a] the period of oscillation will be

$$t^*=\sqrt{(2mx^*)}\int_0^{x^*}\left[\begin{array}{l} x\{(\alpha x^{*2}/2)+\alpha\,\Delta x^*+(\beta x^{*3}/3)+\beta\,\Delta x^{*2}+\beta\,\Delta^2\,x^*\} \\ -x^*\{(\alpha x^2/2)+\alpha\,\Delta x+(\beta x^3/3)+\beta x^2\,\Delta+\beta\,\Delta^2 x\}\end{array}\right]^{-1/2} dx$$

If $\tau^* = \sqrt{(\alpha/m)^{1/2}} \cdot t^*$ = nondimensional period of oscillation the above equation simplifies to,

$$\tau^* = (6)^{1/2} \int_0^{y^*} \frac{dy}{\sqrt{\{-y\,(y-y^*)\,[y+y^*+(3/2)\,(1+2\bar{\Delta})]\}}} \qquad [9]$$

Consider the definite integral [3, pp. 44]

$$I = \int_{e_2}^{e_1} \frac{dy}{\sqrt{\{-4\,(y-e_1)\,(y-e_2)\,(y-e_3)\}}}$$

$$= \frac{1}{\sqrt{(e_1-e_3)}}\; K\; \{[(e_1-e_2)/(e_1-e_3)]^{1/2})\},\quad e_1 > e_2 > e_3\,;\; e_1, e_2 \text{ and } e_3 \text{ being all real} \qquad [10]$$

Comparing the integrals in equations [9] and [10],

$e_1 = y^* > e_2 = 0 > e_3 = -\{y^* + (3/2)\ (1+2\bar{\Delta})\}$, since y^* and $\bar{\Delta}$ are greater than zero.

Therefore,

$$\tau^* = \frac{4\sqrt{3}}{\sqrt{\{4y^*+3\,(1+2\bar{\Delta})\}}} \cdot\ K\ \left[\left(\frac{2\,y^*}{4y^*+3\,(1+2\bar{\Delta})}\right)^{1/2}\right]$$

Substituting for y^* from equation [8]

$$\tau^* = 4\left[\frac{3}{\{16\,H_0 + 3\,(1+2\bar{\Delta})^2\}}\right]^{1/4} K\left[\frac{1}{2}\left\{1-\left(\frac{3\,(1+2\bar{\Delta})^2}{16\,H_0+3\,(1+2\bar{\Delta})^2}\right)^{1/2}\right]^{1/2} \qquad [11]$$

The variation of τ^* with H_0 for various values of the static deflection parameter $\bar{\Delta}$ is shown in figure [3].

The opproximate frequency of oscillation obtained by using equation [7-b] will be,

$$p = \left[\frac{80}{mx^{*5}} \int_{-x^*/2}^{x^*/2} [\alpha\ (u + \Delta + x^*/2) + \beta\ (u + \Delta + \{x^*/2\})^2]\, u^3\, du\right]^{1/2}$$

$$= \left(\frac{\alpha + 2\,\beta\,\Delta + \beta x^*}{m}\right)^{1/2}$$

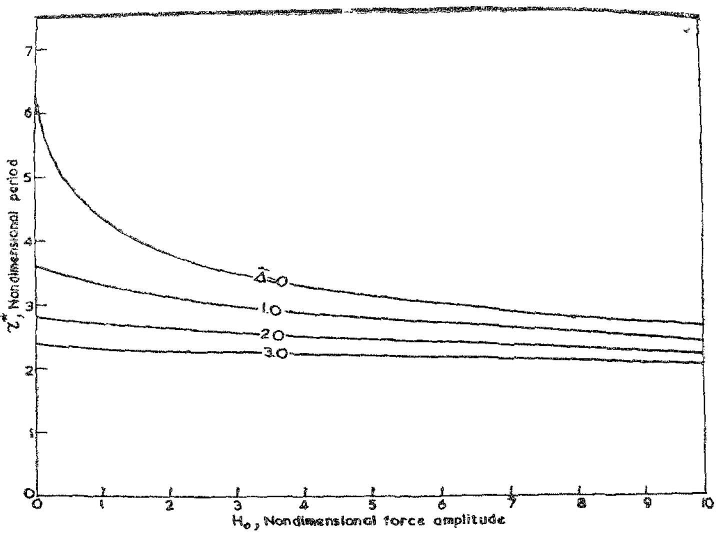

FIG. 3

Variation of τ^* with H_0: $[f(x)=\alpha x+\beta\,(\text{sgn}\,x)\,|x|^2]$

Hence the approximate non-dimensional period, $\bar{T}^*$, will be,

$$\bar{T}^* = \frac{2\pi}{\sqrt{(1+2\bar{\Delta}+y^*)}} = \frac{4\pi}{[1+2\bar{\Delta})+\{9\,(1+2\bar{\Delta})^2+48\,H_0\}^{1/2}]^{1/2}} \qquad [12]$$

If the percentage error in the period of oscillation is defined as,

$$\epsilon = 100\,[1-(\bar{T}^*/T^*)] \qquad [13]$$

then from equations [11] and [12],

$$\epsilon = 100\left[1-\frac{\pi\,[(16/3)\,H_2+(1+2\bar{\Delta})^2]^{1/4}}{\{(1+2\bar{\Delta})+[9\,(1+2\bar{\Delta})^2+48\,H_0]^{1/2}\}^{1/2}}\cdot\frac{1}{K(k)}\right] \qquad [14]$$

where.

$$k^2 = 2\,y^*/[4\,y^*+3\,(1+2\bar{\Delta})]$$

Figure [4] shows the variation of ϵ with H_0 for various values of $\bar{\Delta}$.

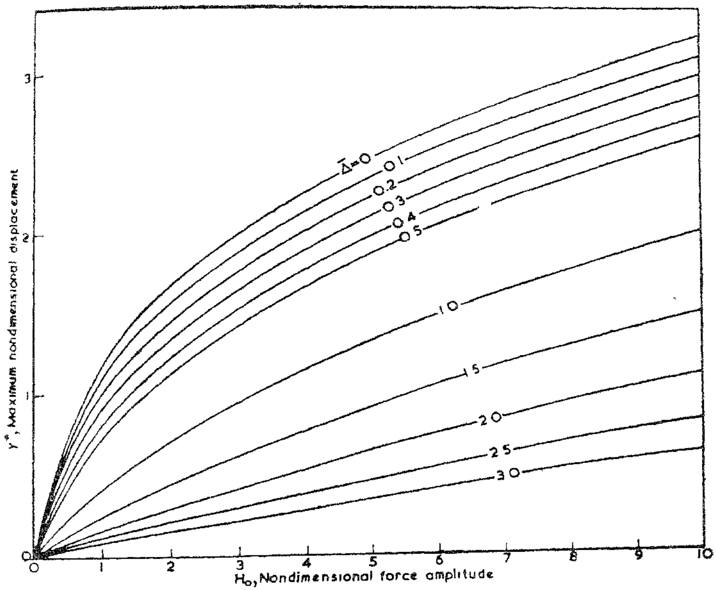

FIG. 4

Variation of ϵ with H_0. [$f(x) = \alpha x + \beta$ (sgn x) $|x|^2$]

(ii) $f(x) = \beta\,(\text{Sgn}\,x)\ |x|^2,\quad \beta > 0:$

From equation [6-b], the maximum displacement x^* is determined by,

$$\beta\,(x^{*3}/3) + \beta\,\Delta\,x^{*2} + \beta\,\Delta^2 x^* = F_0 x^* + \beta\,\Delta^2 x^*$$

With, $y^* = (\beta/F_0)^{1/2} \cdot x^* =$ nondimensional displacement

$\bar{\Delta} = (\beta/F_0)^{1/2} \cdot \Delta =$ nondimensional static deflection,

the above equation reduces to

$$y^{*2} + 3\,y^*\,\bar{\Delta} = 3 \qquad [15]$$

The variation of y^* with $\bar{\Delta}$ is shown in figure (5).

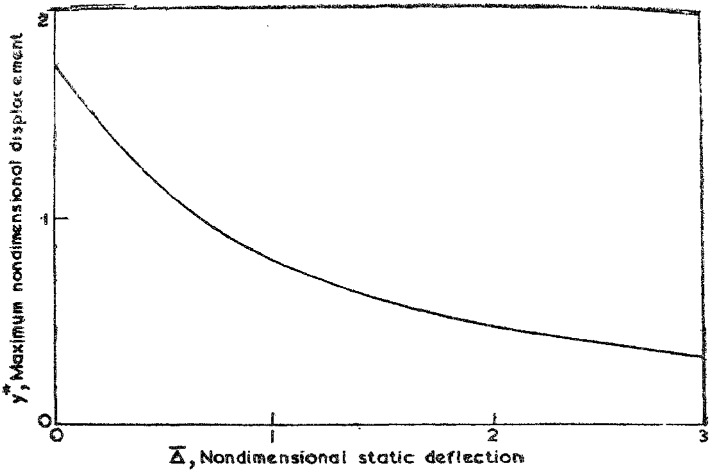

FIG. 5

Variation of y^* with $\bar{\Delta}$: [$f(x) = \beta \operatorname{sgn} x) \, |x|^2$]

The period of oscillation as given by equation [7-a] will be,

$$t^* = \sqrt{(2\, m x^*)} \int_0^{x^*} \left[x \left\{ \frac{\beta x^{*3}}{3} + \beta \Delta x^{*2} + \beta \Delta^2 x^* \right\} - x^* \left\{ \frac{\beta x^3}{3} + \beta \Delta x^2 + \beta \Delta^2 x \right\} \right]^{-1/2} dx$$

with $T^* = [(\beta F_0)^{3/4}/m] \cdot t^*$ = nondimensional period, the above integral may be rewritten as,

$$T^* = (6)^{1/2} \int_0^{y^*} [-y\,(y - y^*)\,(y + y^* + 3\bar{\Delta})]^{-1/2}\, dy \qquad [16]$$

Comparing the integrals in equations [16] and [9]

$e_1 = y^* > e_2 = 0 > e_3 = -(y^* + 3\bar{\Delta})$, since y^* and $\bar{\Delta}$ are both positive quantities.

Hence.

$$\tau^* = \frac{2\sqrt{6}}{\sqrt{(2y^* + 3\bar{\Delta})}} \cdot K\left[\left(\frac{y^*}{2y^* + 3\bar{\Delta}}\right)^{1/2}\right]$$

Substituting for y^* from equation [15]

$$\tau^* = 2\left\{\frac{12}{4 + 3\bar{\Delta}^2}\right\}^{1/4} \cdot K\left[\left\{\frac{1}{2}\left(1 - \frac{3\bar{\Delta}}{\sqrt{(9\bar{\Delta}^2 + 12)}}\right)\right\}^{1/2}\right] \qquad [17]$$

A plot of equation [17] is shown in figure (6)

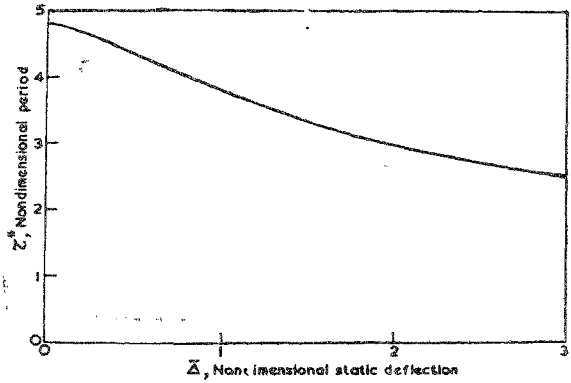

FIG. 6

Variation of τ^* with $\bar{\Delta}$: $[f(x) = \beta\,(\text{sgn}\,x)\,|x|^2]$

The approximate frequency of oscillation from equation [7-b] will be

$$p = \left[\frac{80}{mx^{*5}} \int_{-x^*/2}^{x^*/2} \beta\,[u + \Delta + (x^*/2)]^2 u^3\,du\right]^{1/2}$$

$$= [(2\beta\Delta + \beta x^*)/m]^{1/2}$$

Hence the approximate nondimensional period will be,

$$\tilde{\tau}^* = \frac{2\pi}{\sqrt{(2\bar{\Delta} + y^*)}} = \frac{2\,(2)^{1/2}\,\pi}{[\bar{\Delta} + (9\bar{\Delta}^2 + 12)^{1/2}]^{1/2}} \qquad [18]$$

Substituting equations [17] and [18] into [13] the percentage error and will be,

$$\epsilon = 100\left[1 - \frac{\pi}{3^{1/4}} \frac{(4+3\bar{\Delta}^2)^{1/4}}{[\bar{\Delta}+(9\bar{\Delta}^2+12)^{1/2}]^{1/2}} \frac{1}{K(k)}\right] \qquad [19]$$

where $k^2 = \frac{1}{2}\left[1 - \frac{3\bar{\Delta}}{(9\bar{\Delta}^2+12)^{1/2}}\right]$.

The variation of ϵ with $\bar{\Delta}$ is shown in figure (7)

(iii) $f(x) = \alpha x + \gamma x^3, \;\; \alpha > 0, \;\; \gamma > 0$

From equation [6-b] the expression for maximum displacement will be,

$$(\alpha x^{*2}/2) + \alpha \Delta x^2 + (\gamma x^{*4}/4) + \gamma x^{*3}\Delta + \tfrac{3}{2}\gamma x^{*2}\Delta^2 + \gamma\Delta^3 x^* = F_0 x^* + \alpha\Delta x^* + \gamma\Delta^3 x^*$$

Let $y^* = (\gamma/\alpha)^{1/2} \cdot x^*$ = maximum nondimensional displacement

$\bar{\Delta} = (\gamma/\alpha)^{1/2}\Delta$ = nondimensional static deflection,

$H_0 = (\gamma^{1/2}/\alpha^{3/2}) \cdot F_0$ = nondimensional force amplitude

Then the expression for nondimensional displacement reduces to,

$$2\,y^*\,(1+3\bar{\Delta}^2) + 4\bar{\Delta}\,y^{*2} + y^{*3} = 4\,H_0 \qquad [20]$$

A plot of y^* against H_0 for various values of the static deflection parameter $\bar{\Delta}$ is shown in figure [8].

From equation [7-a] the period of oscillation will be,

$$t^* = (2\,mx^*)^{1/2} \int_0^{x^*} \left\{ \begin{array}{l} x\,\{(\alpha/2)\,x^{*2} + \alpha\,\Delta x^* + (\gamma x^{*4}/4) \\ \quad + \gamma\Delta x^{*3} + \frac{3}{2}\gamma x^{*2}\Delta^2 + \gamma\Delta^3 x^*\} \\ -x^*\,\{(\alpha/2)\,x^2 + \alpha\,\Delta x + (\gamma\,x^4/4) \\ \quad + \gamma\Delta x^3 + \frac{3}{2}\gamma x^2\Delta^2 + \gamma\Delta^3 x\} \end{array} \right\}^{-1/2} dx$$

$$= (2\,m)^{1/2} \int_0^{x^*} \{-x\,(x-x^*)\,[(\alpha/2) + (\gamma/4)\,(x^2+xx^*+x^{*2}) + \gamma\,\Delta(x+x^*) + (3/2)\,\gamma\Delta^2]^{-1/2}\,dx$$

If $\tau^* = [\alpha/m]^{1/2} \cdot t^*$ is the nondimensional period, then

$$\tau^* = 2(2)^{1/2} \int_0^{y^*} \{-y\,(y-y^*)\,[\,y^2 + y\,(y^*+4\bar{\Delta}) + (2+y^{*2}+4\bar{\Delta}\,y^* + 6y^{*2}]^{-1/2}\,dy$$

$$= 2(2)^{1/2} \int_0^{y^*} \left\langle -y\,(y-y^*)\;\{[\,y+(y^*/2)+2\bar{\Delta}]^2 + (\tfrac{3}{4}y^{*2} + 2\bar{\Delta}\,y^* + 2\bar{\Delta}^2+2)\}\right\rangle^{1/2} dy \qquad [21]$$

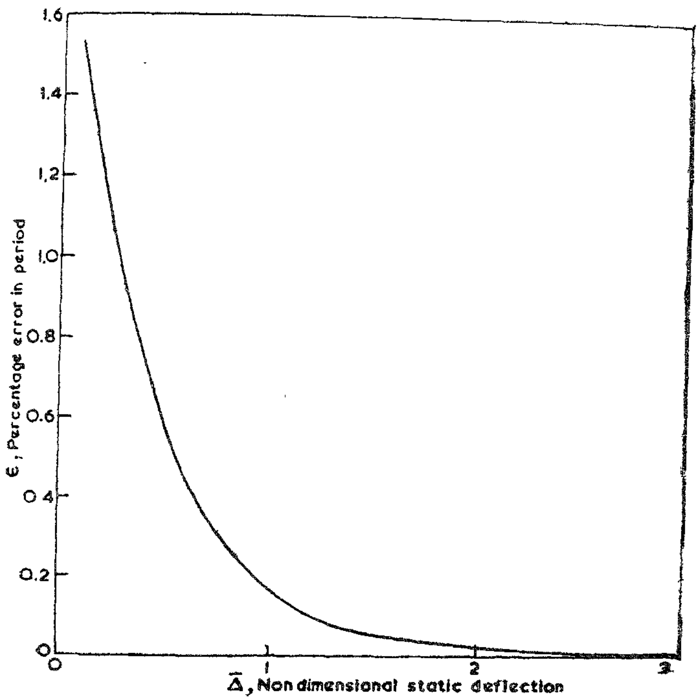

FIG. 7

Variation of ϵ with $\bar{\Delta}$: $[f(x)=\beta\,(\text{sgn}\,x)\,|x|^3]$

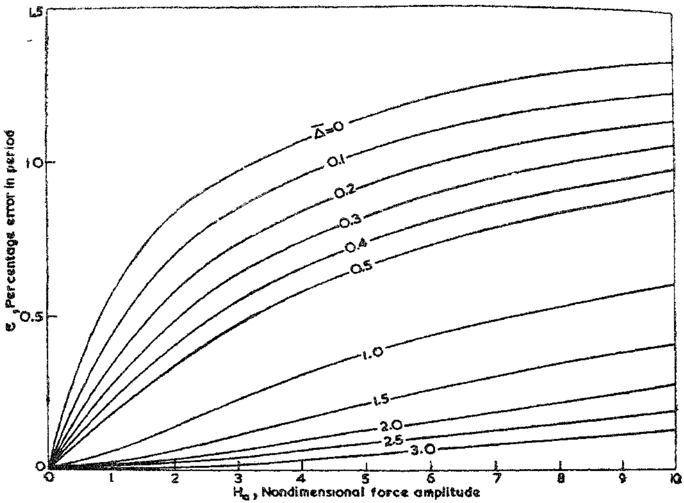

FIG. 8

Variation of y^e **with** H_0 **:** $[f(x)=\alpha x+\gamma x^3,\ \alpha>0,\ \gamma>0]$

Consider the definite integral [3, pp, 47]

$$I=\int_{\bar{\alpha}_2}^{\bar{\alpha}_1}\left\{-(y-\bar{\alpha}_1)(y-\bar{\alpha}_2)\left[(y-r)^2+S^2\right]\right\}^{1/2}dy$$

$$=\frac{2\,K(k)}{\left\{[S^2+(\bar{\alpha}_1-r)^2][S^2+(\bar{\alpha}_2-r)^2]\right\}^{1/4}},\quad \bar{\alpha}_1>\bar{\alpha}_2,\ S>0, \qquad [22]$$

where

$$k^2=\tfrac{1}{2}\left\{1-\frac{S^2+(\bar{\alpha}_1-r)(\bar{\alpha}_2-r)]}{\left\{[S^2+(\bar{\alpha}_1-r)^2][S^2+(\bar{\alpha}_2-r)^2]\right\}^{1/2}}\right\} \qquad [23]$$

Comparing the integrals in equations [21] and [22],

$\bar{\alpha}_1 = y^* > \bar{\alpha}_2 = 0$; since y^* is positive

$r = -[(y^*/2) + 2\bar{\Delta}]$

$S = [\frac{3}{4} y^{*2} + 2\bar{\Delta} y^* + 2\bar{\Delta}^2 + 2 > 0]^{1/2}$, since y^* and $\bar{\Delta}$ are both positive.

Therefore,

$$T^* = \frac{4(\sqrt{2})\, K(k)}{[3y^{*2} + 8\bar{\Delta} y^* + 6\bar{\Delta}^2 + 2)\,(y^{*2} + 4\bar{\Delta} y^* + 6\bar{\Delta}^2 + 2)]^{1/4}} \quad [24]$$

where

$$k^2 = \frac{1}{2}\left\{1 - \frac{(3y^{*2} + 12\bar{\Delta} y^* + 12\bar{\Delta}^2 + 4)}{2[3y^{*2} + 8\bar{\Delta} y^* + 6\bar{\Delta}^2 + 2)(y^{*2} + 4\bar{\Delta} y^* + 6\bar{\Delta}^2 + 2)]^{1/2}}\right\} \quad [25]$$

The variation of T^* with H_0 for various values of $\bar{\Delta}$ is shown in figure (9). The approximate frequency of oscillation as given by equation [7-b] is,

$$p = \left[\frac{80}{mx^{*5}} \int_{-x^*/2}^{x^*/2} \{\alpha\,(u + \Delta + (x^*/2)] + \gamma\,[u + \Delta + (x^*/2)^3]\, u^3\, du\right]$$

$$= (1/\sqrt{m})\,[\alpha + 3\gamma\Delta\,(\Delta + x^*) + (13/14)\,\gamma x^{*2}]^{1/2}$$

Hence the approximate nondimensional period will be,

$$\tilde{T}^* = (2\pi/p)$$

$$= [1 + 3\bar{\Delta}^2 + 3\bar{\Delta} y^* + (13/14)\, y^{*2}]^{-1/2} \cdot 2\pi \quad [26]$$

Substituting equations [24] and [26] into [13], the percentage error in the approximate period will be,

$$\epsilon = 100\left[1 - \frac{\pi}{2\sqrt{2}}\, \frac{[(3y^{*2} + 8\bar{\Delta} y^* + 6\bar{\Delta}^2 + 2)\,(y^{*2} + 4\bar{\Delta} y^* + 6\bar{\Delta}^2 + 2)]^{1/4}}{[1 + 3\bar{\Delta}^2 + 3\bar{\Delta} y^* + (13/14)\, y^*]^{1/2}} \cdot \frac{1}{K(k)}\right] [27]$$

k being given by equation [25].

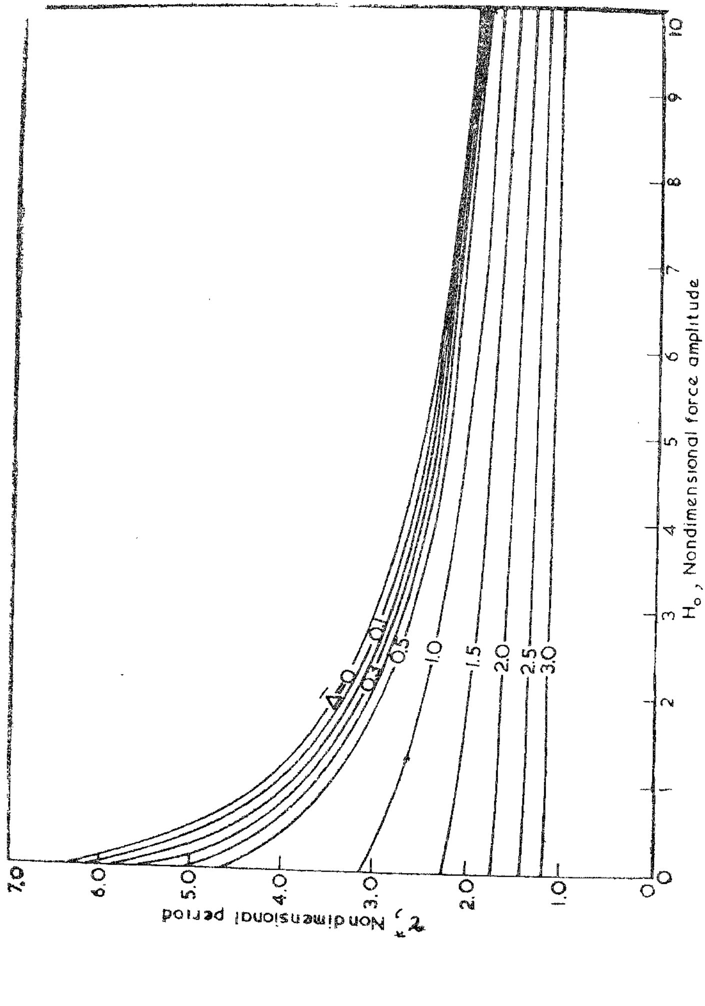

FIG. 9

Variation of τ^* with H_0: $[f(x)=\alpha x+\gamma x^3,\ \alpha>0,\ \gamma>0]$

The variation of the percentage error ϵ for various values of H_0 is shown in figure (10).

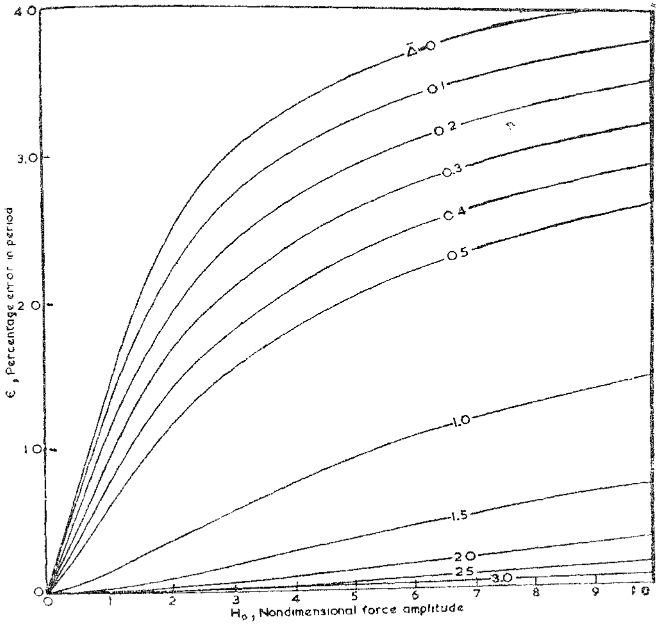

FIG. 10

Variation of ϵ with H_0: $[f(x)=\alpha x+\gamma x^3,\ \alpha>0,\ \gamma>0]$

(iv) $f(x)=\gamma x^3,\ \ \gamma>0$

The expression for the maximum displacement x^*, as obtained by substituting $f(x)=\gamma x^3$ into equation [6-b] is,

$$(\gamma/4)\,x^{*4}+\gamma x^{*3}\Delta+(3/2)\,\gamma x^{*2}\Delta^2+\gamma x^*\Delta^3=F_0x^*+\gamma\Delta^3x^*.$$

With $y^*=(y/F_0)^{1/3}x^*=$ maximum nondimensional displacement,

$\bar{\Delta}=(\gamma/F_0)^{1/3}\ \Delta=$ nondimensional static deflection,

the above equation reduces to

$$y^{*3}+4\bar{\Delta}y^{*2}+6\bar{\Delta}^2y^*=4 \qquad [28]$$

The variation of y^* with $\bar{\Delta}$ is shown in figure (11).

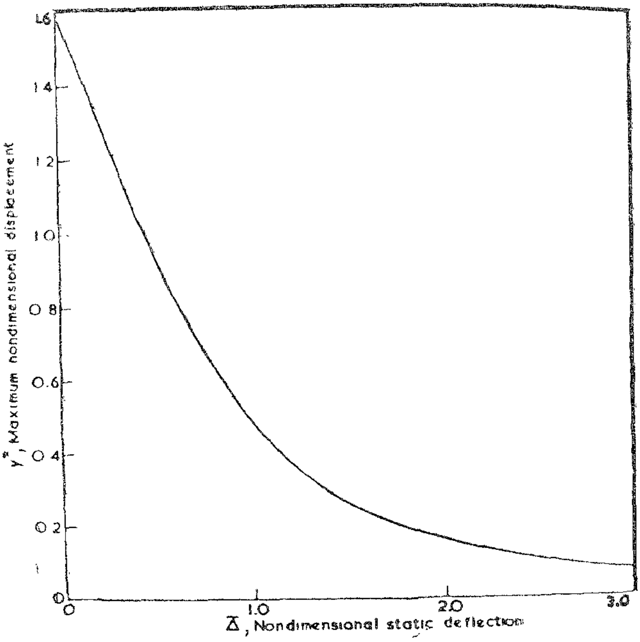

FIG. 11

Variation of y^* with $\bar{\Delta}$: $[f(x) = \gamma x^3, \gamma > 0]$

From equation [7-a] the exact period of oscillation is,

$$t^* = (2mx^*)^{1/2} \int_0^{x^*} \left[\begin{array}{l} x \left\{ (\gamma x^{*4}/4) + \gamma \Delta x^{*3} + \frac{3}{2} \gamma \Delta^2 x^{*2} + \gamma \Delta^3 x^* \right\} \\ - x^* \left\{ (\gamma x^4/4) + \gamma \Delta x^3 + \frac{3}{2} \gamma \Delta^2 x^2 + \gamma \Delta^3 x \right\} \end{array} \right]^{1/2} dx$$

Letting, $\tau^* = [(\gamma F_0^2)^{1/6}/\sqrt{m}] \cdot t^*$ = nondimensional period of oscillation, the above integral becomes,

$$\tau^* = 2(2)^{1/2} \int_0^{y^*} \left\langle -y(y-y^*) \left\{ [y + (y^*/2) + 2\bar{\Delta}]^2 + \frac{3}{4} y^{*2} + 2\bar{\Delta} y^* + 2\bar{\Delta}^2 \right\} \right\rangle^{-1/2} dy \quad [29]$$

Comparing the integrals in equations [29] and [22], it can be seen that

$\bar{\alpha}_1 = y^* > \bar{\alpha}_2 = 0$; since y^* is positive

$r = -[(y^*/2) + 2\bar{\Delta}]$,

$s = [\frac{3}{4} y^{*2} + 2\bar{\Delta} y^* + 2\bar{\Delta}^2]^{1/2} > 0$, since y^* and $\bar{\Delta}$ are positive

Hence the period of oscillation will be

$$T^* = \frac{4\sqrt{2} \cdot K(k)}{[(3y^{*2} + 8\bar{\Delta} y^* + 6\bar{\Delta}^2)(y^{*2} + 4\bar{\Delta} y^* + 6\bar{\Delta}^2)]^{1/4}} \qquad [30]$$

where

$$k^2 = \frac{1}{2}\left\{1 - \frac{3}{2} \frac{(y^{*2} + 4\bar{\Delta} y^* + 4\bar{\Delta}^2)}{[(3y^{*2} + 8\bar{\Delta} y^* + 6\bar{\Delta}^2)(y^{*2} + 4(y^*\bar{\Delta} + 6\bar{\Delta}^2)]^{1/2}}\right\} \qquad [31]$$

The variation of T^* with $\bar{\Delta}$ is shown in figure (12).

The approximate frequency of oscillation resulting from direct linearisation procedure will be,

$$p = \sqrt{(80/mx^{*5})} \left\{ \int_{-x^*/2}^{x^*/2} \gamma (u + \Delta + x^*/2)^3 \cdot u^3 \, du \right\}^{1/2}$$

$$= [1/\sqrt{m}] \, [3\gamma\Delta (\Delta + x^*) + (13/14) \gamma x^{*2}]^{1/2}$$

Hence the approximate period $\bar{T}^*$, will be,

$$\bar{T}^* = 2\pi/p$$

$$= [3\bar{\Delta} (\bar{\Delta} + y^*) + (13/14) y^{*2}]^{-1/2} \cdot 2\pi \qquad [32]$$

Substituting equations [30] and [32], into [13], the percentage error in approximate period will be,

$$\epsilon = 100 \left\{ 1 - \frac{\pi}{2\sqrt{(2)}} \frac{[3y^{*2} + 8\bar{\Delta} y^* + 6\bar{\Delta}^2)(y^{*2} + 4\bar{\Delta} y^* + 6\bar{\Delta}^2)]^{1/4}}{[3\bar{\Delta}(\bar{\Delta} + y^*) + (13/14) y^{*2}]^{1/2}} \cdot \frac{1}{K(k)} \right\} \qquad [33]$$

where k is given by equation [31].

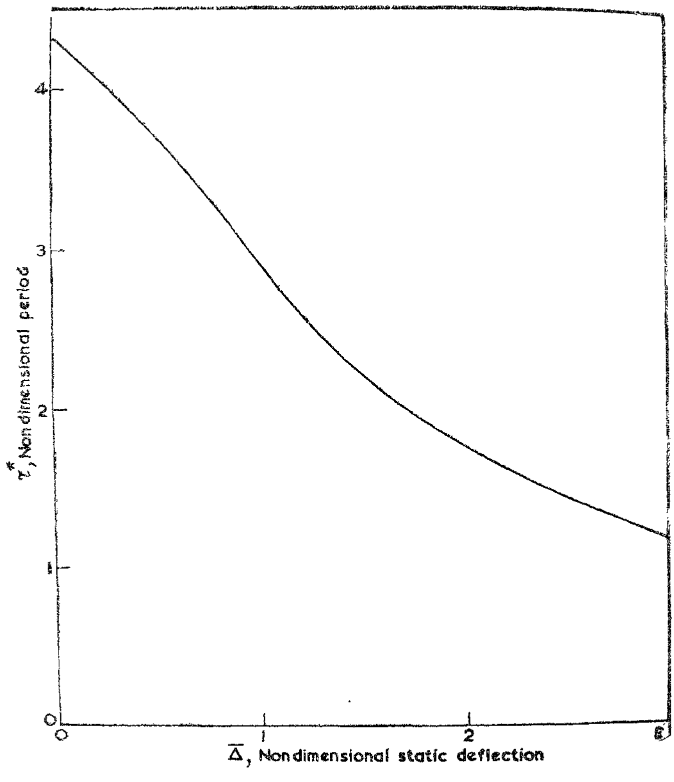

FIG. 12

Variation of τ^* with $\bar{\Delta}$: [$f(x)=\gamma x^3$, $\gamma>0$]

The variation of ϵ with $\bar{\Delta}$ is shown in figure (13)

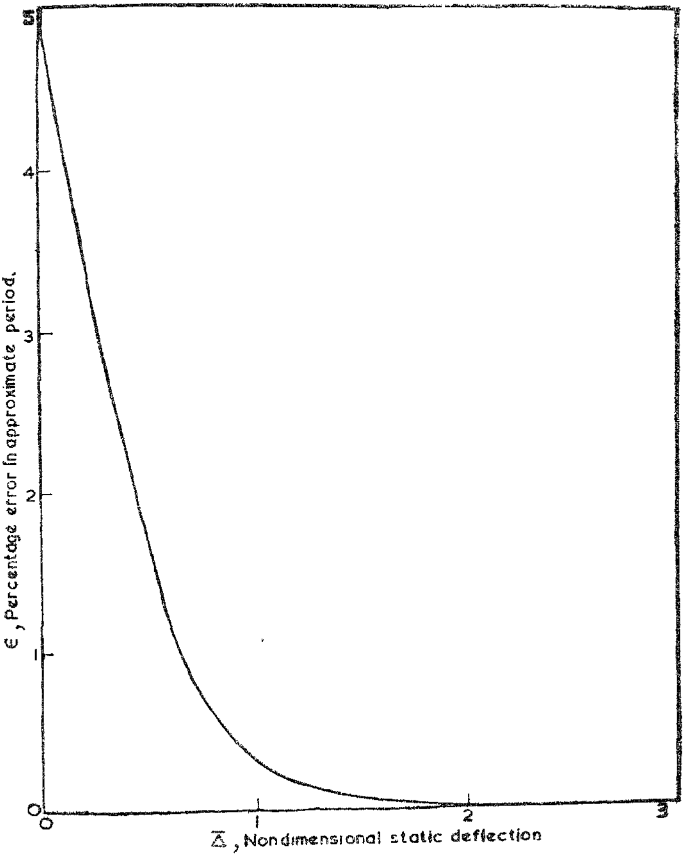

FIG. 13

Variation of ϵ with $\bar{\Delta}$: $[f(x)=\gamma x^3, \gamma>0]$

(v) $f(x) = \beta\,(\mathrm{Sgn}\,x)\,|x|^2 + \gamma x^3$; $\beta > 0$, $\gamma > 0$

From equation [6-b] the maximum displacement is given by

$$\beta[x^{*3}/3 + \Delta x^{*2} + \Delta^2 x^*] + \gamma[x^{*4}/4 + \Delta x^{*3} + \tfrac{3}{2}\Delta^2 x^{*2} + \Delta^3 x^*]$$
$$= F_0 x^* + \beta\Delta^2 x^* + \gamma\Delta^3 x^*$$

Upon introducing,

$y^* = (\gamma/\beta)\,x^*$ = nondimensional displacement,

$\bar{\Delta} = (\gamma/\beta)\cdot\Delta$ = nondimensional static deflection,

$H_0 = (\gamma^2/\beta^3)\,F_0$ = nondimensional force amplitude,

the above equation reduces to,

$$3y^{*3} + 4y^{*2}(1 + 3\bar{\Delta}) + 6\bar{\Delta}y^*(2 + 3\bar{\Delta}) = 12\,H_0 \qquad [34]$$

The variation of y^* with H_0 for various values of the static deflection parameter $\bar{\Delta}$ is shown in figure (14).

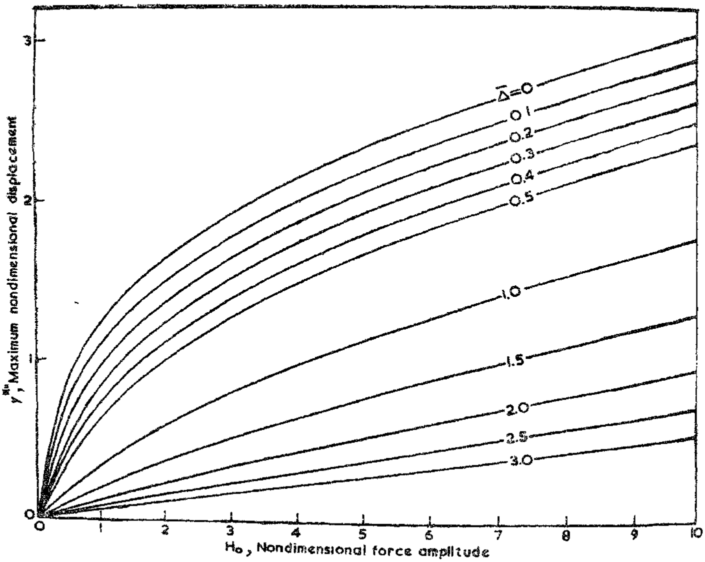

FIG. 14

Variation of y^* with H_0 : $[f(x) = \beta\,(\mathrm{sgn}\,x)\,|x|^2 + \gamma x^3,\ \beta > 0,\ \gamma > 0]$

From equation [7-a] the exact period of oscillation will be,

$$t^* = \sqrt{(2mx^*)} \int_0^{x^*} \left\{ \begin{array}{l} x\{\beta([x^{*3}/3] + \Delta x^{*2} + \Delta^2 x^*) + \gamma([x^{*4}/4] + \Delta x^{*3} \\ \quad + [3/2]\Delta^2 x^{*2} + \Delta^3 x^*)\} \\ -x^*\{\beta([x^3/3] + \Delta x^2 + \Delta^2 x) + \gamma([x^4/4] + \Delta x^3 \\ \quad + [3/2)\Delta^2 x^2 + \Delta^3 x)\} \end{array} \right\}^{-1/2} dx$$

With $\tau^* = (\beta/\sqrt{\gamma m})\, t^*$ = nondimensional period of oscillation,

this reduces to

$$\tau^* = 2(2)^{1/2} \int_0^{y^*} \left[\begin{array}{l} -y(y-y^*)[(y + \{y^*/2\} + 2\bar{\Delta} + \{2/3\})^2 + \{2/3\} y^*(1 + 3\bar{\Delta}) \\ \quad + \{2/3\}\bar{\Delta}(2 + 3\bar{\Delta}) - 4/9 + \{3/4\} y^{*2}] \end{array} \right]^{-1/2} dy \qquad [35]$$

Two cases should be considered in the evaluation of this integral.

(a) If $(3/4)\, y^{*2} + (2/3)\, y^*(1 + 3\bar{\Delta}) + (2/3)\bar{\Delta}(2 + 3\bar{\Delta}) > (4/9)$

Comparing the integrals in equations [35] and [22]

$\bar{\alpha}_1 = y^* > \bar{\alpha}_2 = 0$, since y^* is positive

$r = -(y^*/2 + 2\bar{\Delta} + 2/3)$

$S = \{(3/4)\, y^{*2} + (2/3)\, y^*(1 + 2\bar{\Delta}) + (2/3)\,\bar{\Delta}\,(2 + 3\bar{\Delta})\}^{1/2} > 0$

Hence,

$$\tau^* = \frac{4\sqrt{6} \cdot K(k)}{\{[9y^{*2} + 8y^*(1 + 3\bar{\Delta}) + 6\bar{\Delta}(3\bar{\Delta} + 2)][3y^{*2} + 4y^*(1 + 3\bar{\Delta}) + 6\bar{\Delta}(3\bar{\Delta} + 2)]\}^{1/4}} \qquad [36]$$

where,

$$k^2 = \frac{1}{2}\left\{1 - \frac{3}{2}\, \frac{[3y^{*2} + 4y^*(1 + 3\bar{\Delta}) + 4\bar{\Delta}(2 + 3\bar{\Delta})]}{\{[9y^{*2} + 8y^*(1 + 3\bar{\Delta}) + 6\bar{\Delta}(3\bar{\Delta} + 2)]\,[3y^{*2} + 4y^*(1 + 3\bar{\Delta}) + 6\bar{\Delta}(3\bar{\Delta} + 2)]\}^{1/2}}\right\} \qquad [37]$$

(b) If $(3/4)\, y^{*2} + (2/3)\, y^*(1 + 3\bar{\Delta}) + (2/3)\bar{\Delta}(2 + 3\bar{\Delta}) < 4/9$

Equation [35] may be rearranged as,

$$\tau^* = 2(2)^{1/2} \int_0^{y^*} \frac{dy}{\sqrt{\{-y(y - y^*)(y - \bar{\alpha}_3)(y - \bar{\alpha}_4)\}}} \qquad [38]$$

where

$$\overline{\alpha_3} = -(y^*/2 + 2\bar{\Delta} + 2/3) + \sqrt{\{-(3/4)\, y^{*2} - (2/3)\, y^* (1 + 3\bar{\Delta}) - (2/3)\,\bar{\Delta}(2 + 3\bar{\Delta}) + (4/9)\}}$$

$$\overline{\alpha_4} = -(y^*/2 + 2\bar{\Delta} + 2/3) - \sqrt{\{-(3/4)\, y^{*2} - \{2/3\} y^* (1 + 3\bar{\Delta}) - (2/3)\,\bar{\Delta}(2 + 3\bar{\Delta}) + (4/9)\}}$$

Consider the definite integral [3, pp 47],

$$I = \int_{\bar{\alpha}_2}^{\bar{\alpha}_1} \frac{dy}{\sqrt{\{-(y-\overline{\alpha_1})(y-\overline{\alpha_2})(y-\overline{\alpha_3})(y-\overline{\alpha_4})\}}}$$

$$= \frac{2}{\sqrt{\{(\overline{\alpha_1}-\overline{\alpha_3})(\overline{\alpha_2}-\alpha_4)\}}} K\left[\left\{\frac{(\overline{\alpha_1}-\overline{\alpha_2})(\overline{\alpha_3}-\overline{\alpha_4})}{(\overline{\alpha_1}-\overline{\alpha_3})(\overline{\alpha_2}-\alpha_4)}\right\}^{1/2}\right];$$

$$\overline{\alpha_1} > \overline{\alpha_2} > \overline{\alpha_3} > \overline{\alpha_4}\,. \qquad [39]$$

Comparing equations [38] and [39]

$\bar{\alpha}_1 = y^* > \bar{\alpha}_2 = 0, > \bar{\alpha}_3 > \bar{\alpha}_4$ as y^* and $\bar{\Delta}$ are bosth positive.

Hence the period of oscillation will be,

$$T^* = \frac{8\sqrt{3}\cdot K(k)}{\left[\begin{array}{l} 9y^{*2} + 12y^*(1+3\bar{\Delta}) + 12\bar{\Delta}(2+3\bar{\Delta}) + \\ \quad y^*\{16 - 27y^{*2} - 24y^*(1+3\bar{\Delta}) - 24\bar{\Delta}(2+3\bar{\Delta})\}^{1/2} \end{array}\right]^{1/2}}$$

where

$$k^2 = \frac{2y^*\sqrt{\{16 - 27y^{*2} - 24y^*(1+3\bar{\Delta}) - 24\bar{\Delta}(2+3\bar{\Delta})\}}}{\left[\begin{array}{l} 9y^{*2} + 12y^*(1+3\bar{\Delta}) + 12\bar{\Delta}(2+3\bar{\Delta}) + \\ \quad y^*\sqrt{\{16 - 27y^{*2} - 24y^*(1+3\bar{\Delta}) - 24\bar{\Delta}(2+3\bar{\Delta})\}} \end{array}\right]} \qquad [41]$$

The variation of τ^* with H_0 is shown plotted in figure [15]

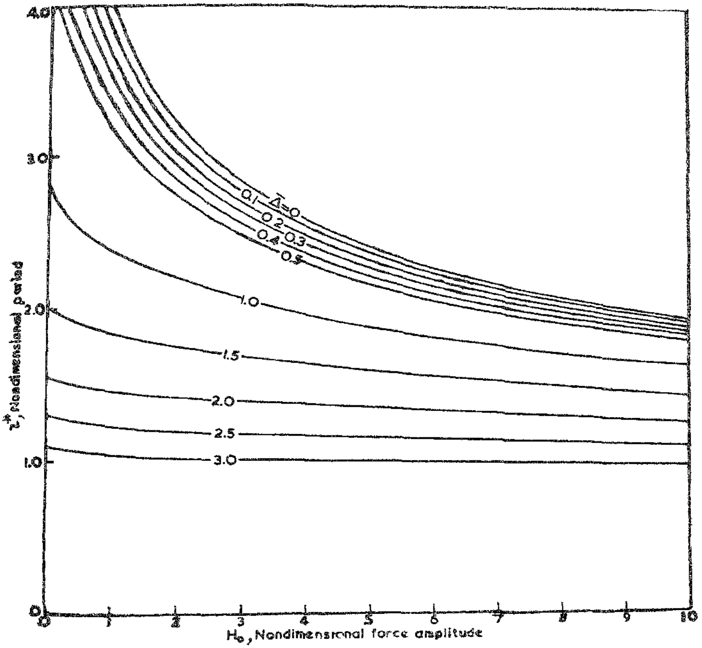

FIG. 15

Variation of τ^* with H_0: $[f(x) = \beta\,(\text{sgn}\,x)\,|x|^2 + \gamma x^3]$

The approximate period of oscillation obtained by using equation [7-b] is

$$p = \left\{\frac{80}{m\,x^{*5}} \int_{-x^*/2}^{x^*/2} [\beta\,(u + \Delta + \{x^*/2\})^2 + \gamma\,(u + \Delta + \{x^*/2\})^3]\,u^3\,du\right\}^{1/2}$$

$$= \frac{1}{\sqrt{m}} \left\{\Delta\,(2\beta + 3\gamma\Delta) + x^*(\beta + 3\gamma\Delta) + \frac{13}{14}\gamma x^{*2}\right\}^{1/2}$$

Hence the nondimensional period of oscillation will be,

$$\tilde{\tau}^* = \frac{2\pi}{p} = \frac{2\pi}{\sqrt{\{\bar{\Delta}(2+3\bar{\Delta})+y^*(1+3\bar{\Delta})+\frac{13}{14}y^{*2}\}}} \quad [42]$$

The error in period as determined by the direct linearisation method will be,

$$\epsilon = \begin{cases} 100\left[1-\dfrac{\pi}{2\sqrt{6}}\,\dfrac{\{[9y^{*2}+8y^*(1+3\bar{\Delta})+6\bar{\Delta}(2+3\bar{\Delta})][3y^{*2}+4y^*(1+3\bar{\Delta})+6\bar{\Delta}(2+3\bar{\Delta})]\}^{1/4}}{K(k)\cdot\sqrt{\{\bar{\Delta}(2+3\bar{\Delta})+y^*(1+3\bar{\Delta})+\frac{13}{14}y^{*2}\}}}\right], \\ \text{if } [\frac{3}{4}y^{*2}+\frac{2}{3}y^*(1+3\bar{\Delta})+\frac{2}{3}\bar{\Delta}(2+3\bar{\Delta})] > \frac{4}{9},\ k^2 \text{ being as in equation [37]} \\ 100\left[1-\dfrac{\pi}{4\sqrt{3}}\left\{\dfrac{9y^{*2}+12y^*(1+3\bar{\Delta})+12\bar{\Delta}(2+3\bar{\Delta})+y^*\sqrt{\{16-27y^{*2}-24y^*(1+3\bar{\Delta})-24\bar{\Delta}(2+3\bar{\Delta})\}}}{\bar{\Delta}(2+3\bar{\Delta})+y^*(1+3\bar{\Delta})+\frac{13}{14}y^{*2}}\right\}^{1/2}\right] \end{cases}$$

if $[\frac{3}{4}y^{*2}+\frac{2}{3}y^*(1+3\bar{\Delta})+\frac{2}{3}\bar{\Delta}(2+3\bar{\Delta})] < \frac{4}{9}$, k^2 being as in eqn [41] [43]

The variation of ϵ with H_0 for various values of $\bar{\Delta}$ is shown in figure [16]

(vi) $f(x) = \alpha x + \beta(Sgn\ x)\ |x|^2 + \gamma x^3,\ \ \alpha > 0,\ \beta > 0,\ \gamma > 0$

The maximum displacement x^* of the system as given by equation [6-b] is,

$$\alpha[(x^{*2}/2)+\Delta x^*]+\beta[x^{*3}/3+\Delta x^{*2}+\Delta^2 x^*]+\gamma[(x^{*4}/4)+\Delta x^{*3}+\tfrac{3}{2}\Delta^2 x^{*2}+\Delta^3 x^*] = (F_0+\alpha\Delta+\beta\Delta^2+\gamma\Delta^3)\,x^*$$

Let,

$y^* = (\beta/\alpha)\cdot x^*$ = maximum nondimensional displacement

$\bar{\Delta} = (\beta/\alpha)\,\Delta$ = nondimensional static deflection

$H_0 = (\beta/\alpha^2)F_0$ = nondimentional force amplitude

$\nu = (\gamma\alpha/\beta^2)$ = nonlinearity parameter.

Then the above equation reduces to

$$6y^*(1+2\bar{\Delta}+3\nu\bar{\Delta}^2)+4(1+3\nu\bar{\Delta})y^{*2}+3\nu y^{*3} = 12H_0 \quad [44]$$

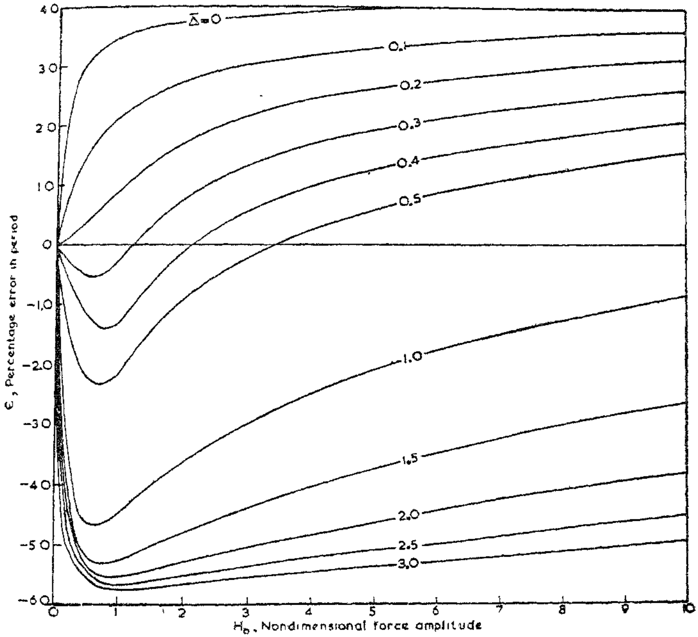

FIG. 16

Variation of ϵ with H_0: $[f(x)=\beta\,(\text{sgn}\,x)\,|\,x\,|^2+\gamma x^3,\ \beta\geq 0,\ \gamma>0]$

The variation of y^* with H_0 for various values of the parameters $\bar{\Delta}$ and $\nu=0.2,\ 5.0$, is shown in figures [17] and [18]. The exact period of oscillation as obtained from equation [7-a] will be,

$$t^*=\sqrt{(2\,mx^*)}\int_0^{x^*}\left[\left\{x\int_0^{x^*}[\alpha\,(\xi+\Delta)+\beta(\xi+\Delta)^2+\gamma(\xi+\Delta)^3]\,d\xi\right\}-\left\{x^*\int_0^{x}[\alpha(\xi+\Delta)+\beta(\xi+\Delta)^2+\gamma(\xi+\Delta)^3]\,d\xi\right\}\right]^{-1/2}dx$$

$$-2[2m/\alpha\nu]^{1/2}\int_0^{y^*}\left\{\begin{array}{l}-y(y-y^*)\cdot\{[y+(y^*/2)+(2/3\nu)+2\bar{\Delta}]^2+\frac{3}{4}y^{*2}\\ \quad+(2y^*/3\nu)(1+3\nu\bar{\Delta})+(2/\nu)(1+2\bar{\Delta}+3\nu\bar{\Delta}^2)\\ \quad-[(2/3\nu)+2\bar{\Delta}]^2\}\end{array}\right\}^{-1/2}dy$$

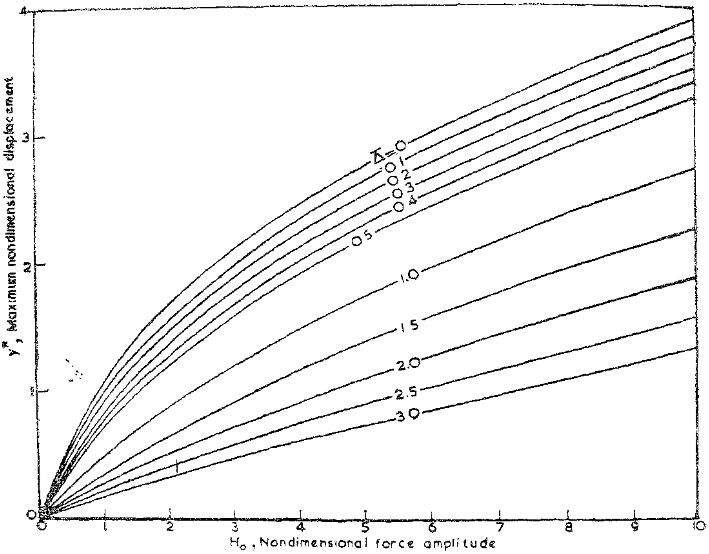

FIG. 17

Variation of y^* with H_0: $[f(x)=\alpha x+\beta\,(\mathrm{sgn}\,x)\,|x|^2+\gamma x^3,\ \nu=0{\cdot}2]$

If $\tau^*=\sqrt{(\alpha/m)}\ t^*$ is the nondimensional period of oscillation, then

$$\tau^*=2\sqrt{(2/\nu)}\int_0^{*}\left\{\begin{array}{l}-y(y-y^*)\{[y+\frac{1}{2}y^*+2\bar{\Delta}+2/3\nu]^2+\frac{3}{4}y^{*2}\\ \quad+(2y^*/3\nu)(1+3\nu\bar{\Delta})+(2/\nu)\ (1+2\ \bar{\Delta}+3\ \nu\bar{\Delta}^2)\\ \quad-(4/9\nu^2)\ (1+3\ \nu\ \bar{\Delta})^2\}\end{array}\right\}^{-1/2}dy \quad [45]$$

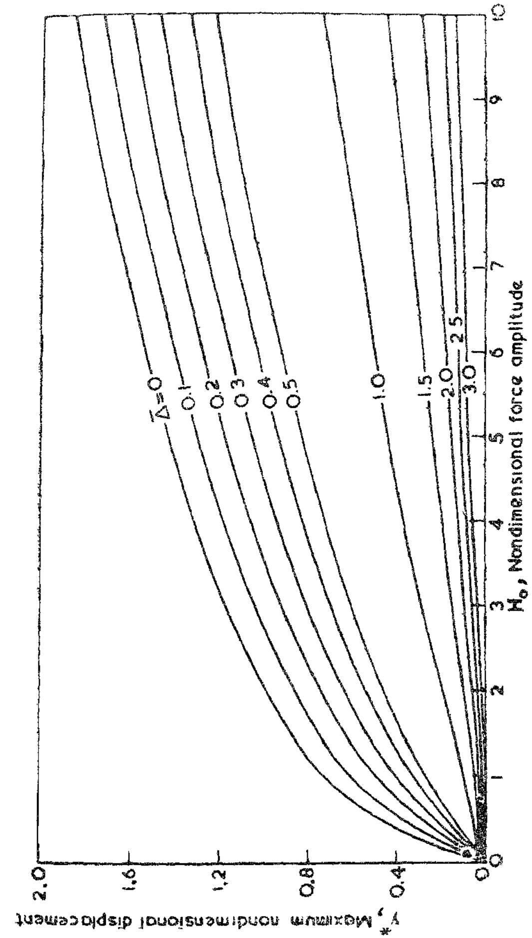

FIG. 18

Variation of y^* with H_0 : $[f(x)=\alpha x \cdot \beta\,(\text{sgn}\,x)\,|x|^2+\gamma x^3,\ \nu=5\cdot 0]$

Two cases arise in evaluating this integral.

(1) If $27\nu^2 y^{*2} + 24\nu(1+3\nu\bar{\Delta})y^* + 72\nu(1+2\bar{\Delta}+3\nu\bar{\Delta}^2) > 16(1+3\nu\bar{\Delta})^2$:

Comparing the integrals in equations [45] and [22]

$$\bar{\alpha}_1 = y^* > \bar{\alpha}_2 = 0; \quad r = -[y^*/2 + 2/3\nu + 2\bar{\Delta}]$$

$$S = (1/6\nu)\,[27\nu^2 y^{*2} + 24\nu(1+3\nu\bar{\Delta})y^* + 72\nu(1+2\bar{\Delta}+3\nu\bar{\Delta}^2) - 16(1+3\nu\bar{\Delta})^2]^{1/2} > 0$$

and

$$T^* = \frac{4\sqrt{6}\cdot K(k)}{\left\{\begin{array}{l}[9\nu y^{*2} + 8(1+3\nu\bar{\Delta})y^* + 6(1+2\bar{\Delta}+3\nu\bar{\Delta}^2)] \times \\ \quad [3\nu y^{*2} + 4(1+3\nu\bar{\Delta})y^* + 6(1+2\bar{\Delta}+3\nu\bar{\Delta}^2)]\end{array}\right\}^{1/4}}$$

where

$$k^2 = \tfrac{1}{2}\left\{1 - \tfrac{3}{2}\,\frac{3\nu y^{*2} + 4(1+3\nu\bar{\Delta})y^* + 4(1+2\bar{\Delta}+3\nu\bar{\Delta}^2)}{\left\{\begin{array}{l}[9\nu y^{*2} + 8(1+3\nu\bar{\Delta})y^* + 6(1+2\bar{\Delta}+3\nu\bar{\Delta}^2)]. \\ \times[3\nu y^{*2} + 4(1+3\nu\bar{\Delta})y^* + 6(1+2\bar{\Delta}+3\nu\bar{\Delta}^2)]\end{array}\right\}^{1/2}}\right\} \qquad [47]$$

(2) If $27\nu^2 y^{*2} + 24(1+3\nu\bar{\Delta})\nu y^* + 72\nu(1+2\bar{\Delta}+3\nu\bar{\Delta}^2) < 16(1+3\nu\bar{\Delta})^2$:

Equation [45] may be rearranged as,

$$T^* = 2\sqrt{(2/\nu)} \int_0^{y^*} [-y(y-y^*)(y-\bar{\alpha}_3)(y-\bar{\alpha}_4)]^{-1/2}\, dy \qquad [48]$$

where,

$$\bar{\alpha}_3 = -\left(\frac{y^*}{2} + \frac{2}{3\nu} + 2\bar{\Delta}\right) + \frac{1}{6\nu}\left\{\begin{array}{l}16(1+3\nu\bar{\Delta})^2 - 27\nu^2 y^{*2} - 24(1+3\nu\bar{\Delta})y^*\nu - \\ \qquad 72\nu(1+2\bar{\Delta}+3\nu\bar{\Delta}^2)\end{array}\right\}^{1/2}$$

$$\bar{\alpha}_4 = -\left(\frac{y^*}{2} + \frac{2}{3\nu} + 2\bar{\Delta}\right) - \frac{1}{6\nu}\left\{\begin{array}{l}16(1+3\nu\bar{\Delta})^2 - 27\nu^2 y^{*2} - 24(1+3\nu\bar{\Delta})y^*\nu \\ \qquad -72\nu(1+2\bar{\Delta}+3\nu\bar{\Delta}^2)\end{array}\right\}^{1/2}$$

Comparing the integrals in equations [48] and [39],

$\bar{\alpha}_1 = y^* > \bar{\alpha}_2 = 0 > \bar{\alpha}_3 > \bar{\alpha}_4$, since $\bar{\Delta}$, ν and y^* are all positive.

Hence,

$$\tau^* = \frac{8\sqrt{3}\cdot K(k)}{\left\{ \begin{array}{l} 9\nu y^{*2} + 12\,(1+3\nu\bar{\Delta})\,y^* + 12\,(1+2\bar{\Delta}+3\nu\bar{\Delta}^2) + \\ + y^*\,[16\,(1+3\nu\bar{\Delta})^2 - 27\nu^2 y^{*2} - 24\nu\,(1+3\nu\bar{\Delta})\,y^* \\ \qquad - 72\nu\,(1+2\bar{\Delta}+3\nu\bar{\Delta}^2)]^{1/2} \end{array} \right\}^{1/2}} \qquad [49]$$

where

$$k^2 = \frac{2y^*\,[16(1+3\nu\bar{\Delta})^2 - 27\nu^2 y^{*2} - 24\nu\,(1+3\nu\bar{\Delta})\,y^* - 72\nu(1+2\bar{\Delta}+3\nu\bar{\Delta}^2)]^{1/2}}{\left\{ \begin{array}{l} 9\nu y^{*2} + 12\,(1+3\nu\bar{\Delta})\,y^* + 12\,(1+2\bar{\Delta}+3\nu\bar{\Delta}^2) + \\ y^*\,[16\,(1+3\nu\bar{\Delta})^2 - 27\nu^2 y^{*2} - 24\nu\,(1+3\nu\bar{\Delta})\,y^* - 72\nu\,(1+2\bar{\Delta}+3\nu\bar{\Delta}^2)]^{1/2} \end{array} \right\}} \qquad [50]$$

The variation of τ^* with H_0 for various values of $\bar{\Delta}$ and $\nu = 0.2$, 5 is shown plotted in figures (19) and (20).

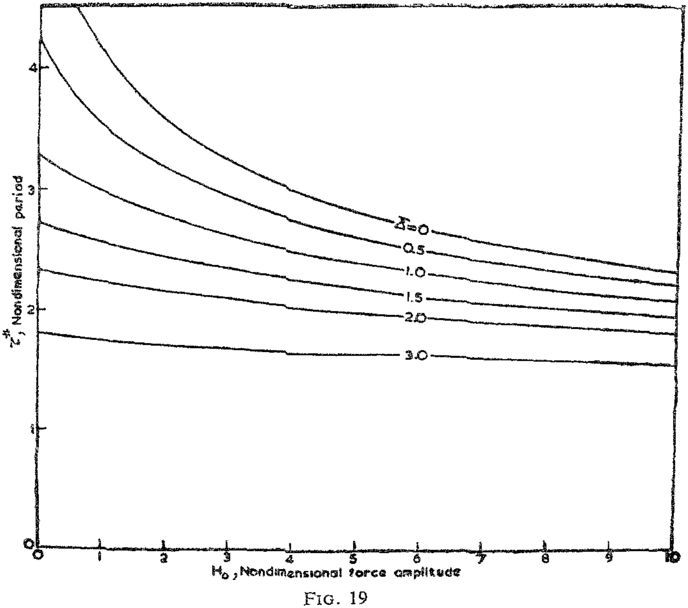

FIG. 19

Variation of τ^* with H_0: $[f(x) = \alpha x + \beta\,(\text{sgn}\,x)\,|x|^2 + \gamma x^3,\ \nu = 0{\cdot}2]$

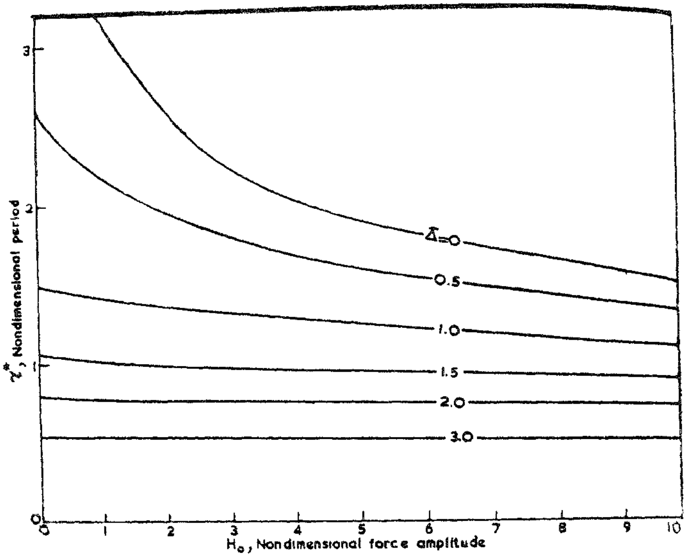

FIG. 20

Variation of τ^* with H_0: $[f(x) = \alpha x + \beta\,(\mathrm{sgn}\,x)\,|x|^2 + \gamma x^3,\ \nu = 5 \cdot 0]$

The approximate frequency of oscillation resulting from the application of the direct linearisation method eqn. [7-a] is,

$$p = \left[\frac{80}{mx^{*5}} \int_{-1/2x^*}^{1/2x^*} \left\{ \alpha\,[u + \Delta + (x^*/2)] + \beta\,[u + \Delta + (x^*/2)]^2 + \gamma\,[u + \Delta + (x^*/2)]^3 \right\} \cdot u^3 du \right]^{1/2}$$

$$= \sqrt{(\alpha/m)} \cdot \sqrt{[(1 + 2\bar{\Delta} + 3\nu\bar{\Delta}^2) + y^*(1 + 3\nu\bar{\Delta}) + \tfrac{13}{14}\nu y^{*2}]}$$

The nondimenfional approximate period $\bar{\tau}^*$ is,

$$\bar{\tau}^* = [(1 + 2\bar{\Delta} + 3\nu\bar{\Delta}^2) + y^*(1 + 3\nu\bar{\Delta}) + \tfrac{13}{14}\nu y^{*2}]^{-1/2} \cdot 2\pi \qquad [51]$$

From equations [13], [46], [49] and [50], the percentage error in the approximate period will be,

$$\epsilon = \begin{cases} 100\left\{1 - \dfrac{\pi}{2(6)^{1/2}\cdot K(k)}\dfrac{\{9\nu y^{*2}+8(1+3\nu\bar{\Delta})y^*+6(1+2\bar{\Delta}+3\nu\bar{\Delta}^2)]\}^{1/4}\times\{[3\nu y^{*2}+4(1+3\nu\bar{\Delta})y^*+6(1+2\bar{\Delta}+3\nu\bar{\Delta}^2)]\}^{1/4}}{[(1+2\bar{\Delta}+3\nu\bar{\Delta}^2)+y^*(1+3\nu\bar{\Delta})+\frac{13}{14}\nu y^{*2}]^{1/2}}\right\}, \\ \quad \text{if } 27\nu^2y^{*2}+24\nu(1+3\nu\bar{\Delta})y^*+72\nu(1+2\bar{\Delta}+3\nu\bar{\Delta}^2) > 16(1+3\nu\bar{\Delta})^2 \\ \quad k^2,\ \text{being as in equation [47]} \\ 100\left\{1 - \dfrac{\pi}{4\sqrt{3}\cdot K(k)}\left\{\dfrac{9\nu y^{*2}+12(1+3\nu\bar{\Delta})y^*+12(1+2\bar{\Delta}+3\nu\bar{\Delta}^2)+y^*[16(1+3\nu\bar{\Delta})^2-27\nu^2y^{*2}-24\nu(1+3\nu\bar{\Delta})y^*-72\nu(1+2\bar{\Delta}+3\nu\bar{\Delta}^2)]^{1/2}}{(1+2\bar{\Delta}+3\nu\bar{\Delta}^2)+(1+3\nu\bar{\Delta})y^*+\frac{13}{14}\nu y^{*2}}\right\}^{1/2}\right\} \\ \quad \text{if } 27\nu^2y^{*2}+24\nu(1+3\nu\bar{\Delta})y^*+72\nu(1+2\bar{\Delta}+3\nu\bar{\Delta}^2) < 16(1+3\nu\bar{\Delta})^2 \\ \quad k^2,\ \text{being given by equation [50]} \end{cases} \quad [52]$$

The variation of ϵ with H_0 for various values of the static deflection parameter $\bar{\Delta}$, for $\nu = 0.2,\ 5$ is shown in figures [21] and [22].

An examination of figures (2), (5), (8), (11), (14), (17) and (18) shows that greater the static deflection, smaller will be the maximum response. It can also be seen from figures (3), (6), (9), (12), (15), (19) and (20), that the period of oscillation decreases with increasing static deflection for a fixed value of the exciting force amplitude. Further a study of figures (4), (7), (10), (13), (16), (21) and (22) shows that the error in the approximate period of oscillation as obtained by direct linearisation procedure is quite small for all values of $\bar{\Delta}$ and H_0. Thus for cases that are not amenable to closed form integration, equation [7-b] provides a simple and sufficiently accurate formula for approximately evaluating the frequency of oscillation.

As an example the restoring force characteristic,

$f(x) = \alpha x + \lambda(\text{Sgn } x)\ |x|^n$, α, λ and n, all positive, will be considered.

From equation [6-b] the maximum displacement will be,

$$\int_0^{x^*} [\alpha(x+\Delta) + \lambda(x+\Delta)^n]\,dx = (F_0 + \alpha\Delta + \lambda\Delta^n)\,x^*$$

or
$$x^{*2} + \frac{2\lambda}{\alpha\,(n+1)}\left[\,(x^* + \Delta)^{n+1} - \Delta^{n+1}\,\right] = \frac{2}{\alpha}\,(F_0 + \lambda\,\Delta^n)\,x^*$$

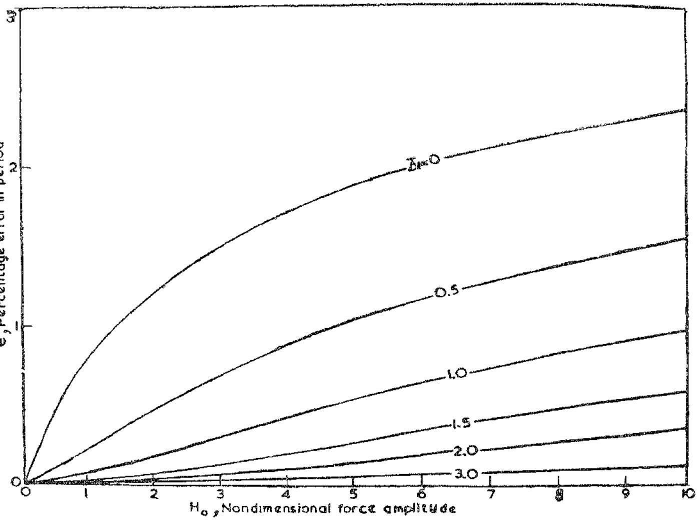

FIG. 21

Variation of ϵ with $H_0\,t$ $[\,f(x) = \alpha x + \beta\,(\mathrm{sgn}\,x)\,|x|^2 + \gamma x^3,\ \nu = 0{\cdot}2\,]$

Let $y^* = (2\lambda/\alpha)^{1/(n-1)}\cdot x$ = nondimensional maximum displacement

$\bar{\Delta} = (2\lambda/\alpha)^{1/(n-1)}\,\Delta$ = nondimensional static deflection

$H_0 = F_0 \lambda^{1/(n-1)}\,(2/\alpha)^{n/(n-1)}$ = nondimensional force amplitude

Then

$$y^{*2} + [1/(n+1)]\,[(\bar{\Delta} + y^*)^{n+1} - \bar{\Delta}^{n+1}] = (H_0 + \bar{\Delta}^n)\,y^* \qquad [53]$$

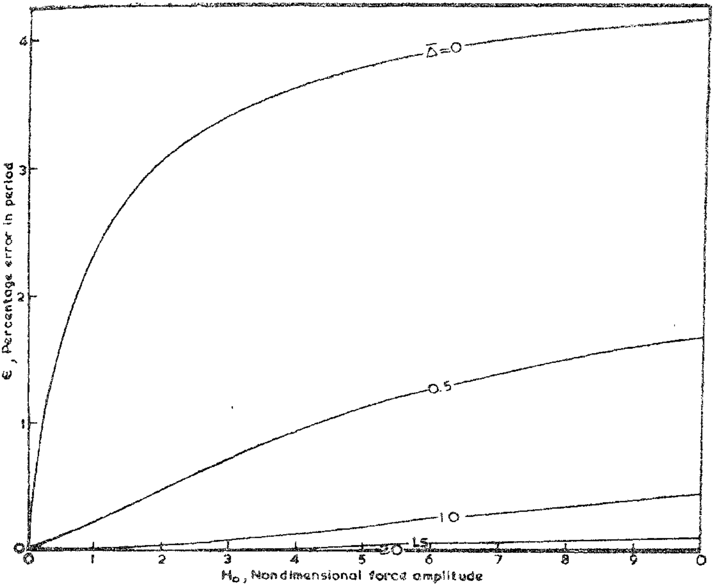

FIG. 22

Variation of ϵ with H_0: $[f(x)=\alpha x+\beta\,(\mathrm{sgn}\,x)\,|x|^2+\gamma x^3,\ \nu=5{\cdot}0]$

The approximate frequency as given by equation [7-b] is

$$p^2=\frac{80}{m\,x^{*5}}\int_{-1/2x^*}^{1/2x^*}\left[\alpha\,(u+\Delta+\tfrac{1}{2}x^*)+\lambda\,(u+\Delta+\tfrac{1}{2}x^*)^n\right]u^3\,du$$

$$=\frac{\alpha}{m}+\frac{80\lambda}{mx^{*5}}\int_{-1/2x^*}^{1/2x^*}(u+\Delta\tfrac{1}{2}x^*)^n\,u^3\,du$$

$$=\frac{\alpha}{m}+\frac{80\lambda}{mx^{*5}}\left\{\frac{(\Delta+x^*)^{n+4}-\Delta^{n+4}}{(n+4)}-3(\Delta+\tfrac{1}{2}x^*)\cdot\frac{(\Delta+x^*)^{n+3}-\Delta^{n+3}}{(n+3)}\right.$$

$$\left.+3(\Delta+\tfrac{1}{2}x^*)^2\cdot\frac{(\Delta+x^*)^{n+2}-\Delta^{n+2}}{(n+2)}-(\Delta+\tfrac{1}{2}x^*)^3\,\frac{(\Delta+x^*)^{n+1}-\Delta^{n+1}}{(n+1)}\right\}$$

Expressed in terms of nondimensional quantities

$$p^2 = \frac{\alpha}{m}\left[1 + \frac{40}{y^{*5}}\left\{\frac{(\bar{\Delta} + y^*)^{n+4} - \bar{\Delta}^{n+4}}{(n+4)} - 3(\bar{\Delta} + \tfrac{1}{2}y^*) \cdot \frac{(\bar{\Delta} + y^*)^{n+3} - \bar{\Delta}^{n+3}}{(n+3)}\right.\right.$$

$$+ 3(\bar{\Delta} + \tfrac{1}{2}y^*)^2 \cdot \frac{(\bar{\Delta} + y^*)^{n+2} - \bar{\Delta}^{n+2}}{(n+2)}$$

$$\left.\left. - (\bar{\Delta} + \tfrac{1}{2}y^*)^3 \cdot \frac{(\bar{\Delta} + y^*)^{n+1} - \bar{\Delta}^{n+1}}{(n+1)}\right\}\right] \qquad [54]$$

The nondimensional period, defined as $\tilde{T}^* = (2\pi/p)$, will then be given by equations [53] and [54] for any given set of parameters α, λ, n and H_0

4. CONCLUSIONS

The above analysis reveals that the static deflection has a profound effect on both the peak displacement and the period of oscillation of the nonlinear system. This behaviour is a characteristic feature of the nonlinear system that is not present in the linear counterpart. For linear systems the static deflection alters only the equilibrium position of the system and the gravity force of the oscillating mass is always balanced by the force set up in the spring due to static deflection, so that the equation of motion and hence the solution remains unchanged. However, for the nonlinear system, additional terms occur in the equation of motion, when the static deflection is taken into account. The resulting maximum displacement and period of oscillatiou therefore depend on the static deflection in addition to the system and excitation parameters.

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