A NOTE ON A RIGID FOUNDATION ON A CROSS ANISOTROPIC HALF-SPACE

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ABSTRACT

The stress-displacement problem of a rigid circular foundation on a cross anisotropic half-space is presented. The distribution of the contact pressure and the foundation settlement are obtained. The contact pressure distribution is seen to be independent of the elastic constants.

1. INTRODUCTION

This has been established by several investigators that natural soils, in particular, overconsolidated clays, can be better represented by a transversely isotropic (cross anisotropic) elastic half-space rather than an isotropic one [1]. Michell [2] obtained solution for the case of a point load acting at the surface of a cross anisotropic medium whereas particular cases of cross anisotropy were considered by Wolf [3], Westergaard [4] and Barden [1]. Herein, the stress-displacement problem of a homogeneous cross anisotropic and elastic haf-space under the action of a rigid foundation is presented.

2. MATHEMATICAL FORMULATION

Let a rigid circular foundation of radius R_0 rest on the surface of a cross anisotropic half-space, the origin of the cylindrical coordinates r and z being at the centre of the contact surface. Further, it is assumed that the total load on the foundation is P and the equivalent uniformly distributed load at the contact surface is p.

Starting from the expressions of stresses and displacements expressed in terms of stress-function [5], using Hankel transforms [6] and following

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the same procedure as outlined in Snedon [6], the expressions for stresses and displacements become

$$\sigma_z = \int_0^\infty \eta^4 \{(s_1 (c - ds_1^2) A \cdot \exp(-s_1 \eta z) + s_2 (c - ds_2^2) \cdot B.$$
$$\times \exp(-s_2 \eta z)\} J_0 (\eta r)\, d\eta, \quad (1)$$

$$\sigma_r = - \int_0^\infty \eta^4 \{s_1 (1 - as_1^2) A \cdot \exp(s_1 \eta z + s_2 (1 - as_2^2) B.$$

$$\times \exp(-s_2 \eta z\} J_0 (\eta r)\, d\eta + \frac{1-b}{r} \int_0^\infty \eta^3 \{s_1 A \exp(-s_1 \eta z)$$

$$+ s_2 B \exp(-s_2 \eta z)\} J_1 (\eta r)\, d\eta, \quad (2)$$

$$\sigma_r + \sigma_\theta = \int_0^\infty \eta^4 \{ s_1 (2as_1^2 - b - 1) A \cdot \exp(-s_1 \eta z)$$
$$+ s_2 (2as_2^2 - b - 1) B \exp(-s_2 \eta z)\} J_0 (\eta r)\, d\eta, \quad (3)$$

$$\tau_{rz} = \int_0^\infty \eta^4 \{(1 - as_1^2) A \cdot \exp(-s_1 \eta z) + (1 - as_2^2) B \cdot \exp$$
$$\times (-s_2 \eta z)\} J_1 (\eta r)\, d\eta, \quad (4)$$

$$w = \frac{1}{nE} \int_0^\infty \eta^3 \{(m_1 + m_2 s_1^2) A \cdot \exp(-s_1 \eta z) + (m_1 + m_2 s_2^2) B$$
$$\times \exp(-s_2 \eta z)\} J_0 (\eta r)\, d\eta, \quad (5)$$

$$u = - \frac{(1 + \mu_{rr})(1 - b)}{nE} \int_0^\infty \eta^3 [s_1 A \cdot \exp(-s_1 \eta z) + s_2 B$$
$$\times \exp(-s_2 \eta z)] J_1 (\eta r)\, d\eta, \quad (6)$$

where $\sigma_z, \sigma_r, \sigma_\theta$ are normal stresses along vertical, radial and tangential directions; τ_{rz} is the shear stress along a vertical plane; w and u are vertical and radial displacements; A, B are constants to be evaluated from boundary conditions; $J_n(x)$ is the Bessel function of the first kind of the order n,

$$a = - \frac{\mu_{rz} (1 + \mu_{rr})}{n - \mu_{rz}^2}; \quad b = \frac{\mu_{rz}^2 + n\mu_{rr}^2 - nE\mu_{rz}/G_{rz}}{n - \mu_{rz}^2};$$

$$c = \frac{-\mu_{rz} (1 + \mu_{rr}) + nE/G_{rz}}{n - \mu_{rz}^2}; \quad d = \frac{1 - \mu_{rr}^2}{n - \mu_{rz}^2}; \quad (7)$$

$$s_1, s_2 = \sqrt{\frac{(a+c) \pm \sqrt{(a+c)^2 - 4d}}{2d}}\,; \tag{8}$$

$$m_1 = -cn - (1+b)\,\mu_{rz}\,;\; m_2 = dn + 2a\mu_{rz}\,; \tag{9}$$

and E, n, G_{rz}, μ_{rr} and μ_{rz} are elastic constants defined as

E = Young's modulus along any vertical direction;

nE = Young's modulus along any horizontal direction ;

μ_{rr} = Poisson's ratio for strain in any horizontal direction due to a horizontal direct stress;

μ_{rz} = Poisson's ratio for strain in any vertical direction due to a horizontal direct stress and

G_{rz} = Shear modulus in a vertical plane.

Boundary conditions:

The case of a rigid foundation leads to the following mixed boundary conditions:

$$(\tau_{rz})_{z=0} = 0,\ r > 0;\quad (w)_{z=0} = W,\ 0 < r < R_0 \text{ and}$$
$$(\sigma_z)_{z=0} = 0,\ r > R_0, \tag{10}$$

where W is the foundation settlement.

3. SOLUTION

The above mixed boundary conditions lead to the following dual integral equations

$$\int_0^\infty x^{-1} A(x)\, J_0(xR)\, dx = WR_0^4\, nEC_1,\ 0 < R < 1;$$
$$\int_0^\infty A(x)\, J_0(xR)\, dx = 0,\ R > 1 \tag{11}$$

where

$$x = \eta R_0,\quad R = r/R_0,\quad x^4 A = A(x) \tag{12}$$

and

$$C_1 = \frac{1 - as_2^2}{(m_1 + m_2 s_1^2)(1 - as_2^2) - (m_1 + m_2 s_2^2)(1 - as_1^2)}\,. \tag{13}$$

The solution of the above dual relations (eq. (13)) is [7]

$$A(x) = \frac{2\sqrt{2x}}{\pi} \left\{ \sqrt{-x}\, J_{-1/2}(x) \int_0^1 R_0^4 WnEC_1 \frac{y\,dy}{\sqrt{1-y^2}} + \int_0^1 \frac{y}{\sqrt{1-y^2}} \left[\int_0^1 R_0^4\, WnEC_1 (xt)^{3/2} J_{1/2}(xt)\, dt \right] dy \right\} \tag{14}$$

Using the condition of static equilibrium and after due simplifications, eq. (14) gives

$$A(x) = -P \frac{R_0^4 (1 - as_2^2)\sqrt{d}}{2(s_1 - s_2)(ac - d)} \sin x. \tag{15}$$

Contract pressure distribution:

The distribution of contact stress is obtained from eq. (1). As a first step, z is set t_0 zero, then B is expressed in terms of $A(x)$ *via* the first boundary condition and finally $A(x)$ is substituted from eq. (15). This gives

$$(\sigma_z)_{z=0} = -\frac{p}{2\sqrt{1-R^2}}. \tag{16}$$

Equation (16) shows that the distribution of contact stress under a rigid foundation on an anisotropic half-space is independent of the elastic constants and is same as that for an isotropic half-space [8].

Surface settlement:

The expression for surface settlement obtained same in the way as the contact stress is

$$W = \frac{\pi}{4} \frac{PR_0}{E} \sqrt{d}(s_1 + s_2)\left[1 - \frac{a(1-b)}{n(ac-d)} \mu_{rz}\right]. \tag{17}$$

The expression for W at the centre of a uniformly loaded area can be shown to be

$$W = \frac{PR_0}{E} \sqrt{d}(s_1 + s_2)\left[1 - \frac{a(1-b)}{n(ac-d)} u_{rz}\right]. \tag{18}$$

Comparing eqs. (17) and (18) it may be seen that as in the case of an isotropic half-space, the surface settlement of a rigid foundation is $\pi/4$ times the surface settlement at the centre of a perfectly flexible foundation [8]. For an isotropic material we have

$$n = 1, \mu_{rr} = \mu_{rz} = \mu, \; G_{rz} = \frac{E}{2(1+\mu)}, \tag{19}$$

which gives

$$W = \frac{2pR_0(1-\mu^2)}{E} \frac{\pi}{4},$$

a well-known result [8]

4. CONCLUDING REMARKS

The distribution of contact stress beneath a rigid circular foundation is independent of the elastic constants and is same as that corresponding to an isotropic medium. As for isotropic medium the settlement of a rigid foundation on a cross anisotropic medium is $\pi/4$ times the settlement at the centre of a perfectly flexible foundation.

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