A DIFFUSION PROCESS APPROACH TO A RANDOM EIGENVALUE PROBLEM

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ABSTRACT

A second order linear differential equation with random coefficients is studied here with reference to the distribution of eigenvalues. The random coefficient is assumed to be a function of the standard Brownian motion. Employing Ito Calculus a partial differential equation for the joint density function of the eigenvalues is derived.

Keywords: Random Vibrations, eigenvalues, Brownian motion, Ito Calculus, diffusion equations.

1. INTRODUCTION

Eigenvalue problems are of great interest in mathematics, physics and engineering. Though much attention has been given in the literature to deterministic problems, research on random eigenvalue problems is of recent origin. A typical example in this connection is the determination of the probability distribution of the eigenvalues of a string with a random mass distribution. This problem has been previously attempted by Boyce and his associates [1, 2, 3]. Two methods, styled ' honest ' and ' dishonest ' have been used to arrive at some interesting results for the eigenvalues of a random string. Under the ' honest ' methods the classical perturbation and the Fredholm integral equation approach have been discussed. The integral equation method leads to bounds on the moments of the eigenvalues. This has also been presented in a recent book by Bharucha-Reid [4]. In the ' dishonest ' approach some convenient, but questionable, assumptions regarding the statistical dependence of the solution with the random process appearing in the governing differential equation are introduced to obtain some approximate results.

In the present work we study the eigenvalue problem associated with a linear second order differential equation with a random coefficient. Such an equation arises in the study of free vibration of a string with random mass distribution. Our method differs in a substantial way from those employed in earlier works on the subject [1, 2, 3, 4]. We first convert the problem to that of studying the zeros of the solution of an initial value problem with a random coefficient. Next, when this coefficient is a function of the standard Brownian motion we reduce the problem to the determination of the solution of a first order Ito stochastic differential equation.

In the next section the concept of a random eigenvalue is defined *via* two approaches and their equivalence is established. In section 3 the initial value approach is discussed. In section 4 we derive a partial differential equation leading to the probability distribution of the eigenvalues when the random coefficient arises *via* Brownian motion. The paper ends with section 5, where some remarks and further problems arising out of this work are presented.

2. THE CONCEPT OF A RANDOM EIGENVALUE

Let us consider the equation

$$\frac{d^2 y(x, \omega)}{dx^2} + \lambda^2(\omega)\,[1 + \epsilon f(x, \omega)]\, y(x, \omega) = 0 \tag{1}$$

with boundary conditions

$$y(0, \omega) = 0 = y(L, \omega). \tag{2}$$

Here, $\{f(x, \omega);\ 0 \leqslant x \leqslant L\}$ is a stochastic process defined on some probability space (Ω, B, P) and $\epsilon > 0$ is a fixed number. We are interested in finding a stochastic process $\{y(x, \omega);\ 0 \leqslant x \leqslant L\}$ and a random variable $\lambda(\omega)$ on (Ω, B, P) such that equations (1) and (2) hold for almost all ω. Herein, we discuss the determination of only $\lambda(\omega)$.

Approach 1

Consider the system

$$\frac{d^2 y(x, \omega)}{dx^2} + \lambda^2\,[1 + \epsilon f(x, \omega)]\, y(x\ \omega) = 0 \tag{3}$$

with

$$y(0, \omega) = 0, \quad y'(0, \omega) = 1 \tag{4}$$

where λ is a parameter. It is well known from the existence theory of differential equations that for any reasonable f, say continuous in x for almost all ω, there exist a unique solution to equations (3) and (4). Let us denote this by $y(x, \omega, \lambda)$. Consider now the equation

$$y(L, \omega, \lambda) = 0 \tag{5}$$

in λ^2. All solutions of this equation will be called the *eigenvalues* of equations (1) and (2). It is a standard result in Sturm-Liouville theory [5] that when $[1 + \epsilon f(x, \omega)]$ is nonnegative for all x and ω the solution of equation (5) is a countable set of nonnegative real numbers and can be arranged in increasing order $\{\lambda_n^2(\omega);\ n = 1, 2 \cdots\}$.

Approach 2

Let $y(x, \omega, \lambda)$ be as in approach 1. For every fixed λ and ω consider the zeros of the function $y(x, \lambda^2, \omega)$. It is again a consequence of the Sturm-Liouville theory that these are countable and nonnegative and can be arranged in increasing order, say $\{Z_n(\lambda^2, \omega);\ n = 1, 2 \cdots\}$. Also, it is known that for every n, $Z_n(\lambda^2, \omega)$ is non-increasing in λ^2. Let $\lambda_n^{2*}(\omega)$ be the unique solution of the equation

$$Z_n(\lambda_n^{2*}(\omega), \omega) = L \tag{6}$$

Then, $\{\lambda_n^{2*}(\omega);\ n = 1, 2 \cdots\}$ will be called the eigenvalues of equations (1) and (2).

It turns out that both the above definitions lead to the same set of eigenvalues. That is, we have the following.

Lemma 1

Assume that $[1 + \epsilon f(x, \omega)]$ is nonnegative and continuous in x for almost all ω. Then for every n: $\lambda_n^2(\omega) = \lambda_n^{2*}(\omega)$ almost surely.

Proof.—Consider first the case $n = 1$. The function $y[x, \omega, \lambda_1^2(\omega)]$ is the solution corresponding to the first eigenvalue and hence has no zeros in the open interval $(0, L)$ [5]. Thus $Z_1[\lambda_1^2(\omega), \omega] = L$. On the other hand $\lambda_1^{2*}(\omega)$ is the unique solution of the equation

$$Z_1[\lambda_1^{2*}(\omega), \omega] = L$$

Thus

$$\lambda_1^{2*}(\omega) = \lambda_1^2(\omega).$$

Now, observe for a general n we need to use the fact that the function $y[x, \omega, \lambda_n^2(\omega)]$ has exactly $(n-1)$ zeros in the open interval $(0, L)$.

3. INITIAL VALUE APPROACH

It will be noted that whereas approach 1 is natural and intuitive it is approach 2 that will prove useful. This is primarily because of the identity

$$P\{\omega:\ \lambda_n^2(\omega) \leqslant t\} = P\{\omega:\ \lambda_n^{2*}(\omega) \leqslant t\} = P\{\omega:\ Z_n(t, \omega) \leqslant L\}.$$

The last function in this equation is in terms of the n-th zero of the stochastic process $\{y(x, \omega, t);\ x \geq 0\}$.

Now, let us make the so called Prufer substitution [5]

$$\begin{aligned} y(x, \omega, \lambda) &= R \sin \phi \\ y'(x, \omega, \lambda) &= R \cos \phi \end{aligned} \tag{7}$$

Here R and ϕ are functions of x, ω and λ. In terms of R and ϕ equations (3) and (4) become

$$\frac{d\phi}{dx} = \cos^2 \phi + \lambda^2 \sin^2 \phi\,[1 + \epsilon f(x, \omega)] \tag{8}$$

$$\frac{dR}{dx} = R \cos \phi \sin \phi\,[1 - \lambda^2\{1 + \epsilon f(x, \omega)\}]$$

and

$$\phi(0) = 0; \quad R(0) = 1. \tag{9}$$

From the nondecreasing nature of ϕ as a function of x it is obvious that the n-th zero of $y(x, \omega, \lambda)$ is the same as the root of the equation $\phi(x) = n\pi$. That is $Z_n(\lambda, \omega)$ is the random variable satisfying the equation

$$\phi[Z_n(\lambda, \omega), \omega, \lambda] = n\pi. \tag{10}$$

Thus, the study of $\{Z_n(\lambda, \omega);\ n = 1, 2 \cdots\}$ reduces to the study of the process $\phi(x, \omega, \lambda)$. Further, since $\phi(x)$ is nondecreasing it follows that

$$\begin{aligned} P\{Z_n(\lambda, \omega) \leqslant t\} &= P\{\phi(Z_n(\lambda, \omega), \omega, \lambda)\} \leqslant \phi(t, \omega, \lambda) \\ &= P\{n\pi \leqslant \phi(t, \omega, \lambda)\}. \end{aligned} \tag{11}$$

Hence, *in order to find the marginal distribution of* $\lambda_n^2(\omega)$ *it suffices to find the marginal distribution of* $\phi(t, \omega, \lambda)$ *for every fixed* t *and* λ.

4. A Diffusion Equation

Equation (8) may be recast as

$$\phi(x, \omega, \lambda) = \int_0^x [\cos^2 \phi(u, \omega, \lambda) + \lambda^2 \sin^2 \phi(u, \omega, \lambda) \times \{1 + \epsilon f(u, \omega)\}]\, du. \tag{12}$$

Now, if it is assumed that $f(x, \omega) = a(w(x, \omega))$ where $\{w(x, \omega);\ 0 \leqslant x\}$ is the standard Brownian motion, then we may think of $\phi(x, \omega, \lambda)$ as the unique nonanticipating solution, in the Ito sense [6] of the integral equation (12). Then it follows the vector process $\{\phi(x, \omega, \lambda),\ w(x, \omega),\ x \geqslant 0\}$ is Markov. It may be noted here that in order to validate the results of the Sturm-Liouville theory involed in the previous sections, $a(w(\omega, x))$ should be such that $\{1 + \epsilon a(w(x, \omega))\}$ is nonnegative for all $x \geqslant 0$ with probability one.

Let $p(x, \phi, w)$ denote the joint density of $\{\phi(x, \omega, \lambda),\ w(x, \omega)\}$ given that $\phi(0, \omega, \lambda) = 0 = w(0, \omega)$. If $H(\phi, \omega)$ is any twice continuously differentiable function of ϕ and w, then we see by Ito's formula [6] that $y(x, \omega) = H(\phi(x, \omega, \lambda), w(x, \omega))$ is a stochastic integral and its differential is given by

$$dy(x, \omega) = \{H_{10}(\phi, w) f_1(\phi, \omega) + \tfrac{1}{2} H_{02}(\phi, w) + H_{01}(\phi, w)\}\, du \tag{13}$$

where

$$H_{10} = \frac{\partial H}{\partial \phi}, \quad H_{01} = \frac{\partial H}{\partial w}, \quad H_{02} = \frac{\partial^2 H}{\partial w^2}$$

$$f_1(\phi, \omega) = \cos^2 \phi + \lambda^2 (\sin^2 \phi)\, [1 + \epsilon a(\omega(x, w))]. \tag{14}$$

Now, by taking expectations we get

$$EH(\phi(x, \omega, \lambda),\ w(x, \omega)) = E \int_0^x [H_{10}(\phi(u), w(u)) f_1(\phi(u), w(u)) + \tfrac{1}{2} H_{02}(\phi(u), w(u))]\, du.$$

This leads to a conclusion that the so called infinitesimal generator of the above vector process is the differential operator

$$\mathrm{A} = f_1(\phi, w) \frac{\partial}{\partial \phi} + \frac{1}{2} \frac{\partial^2}{\partial w^2}. \tag{15}$$

on the class of twice continuously differentiable functions. In particular, the density function $p(x, \phi, w, \lambda)$ satisfies the partial differential equation, generally called the backward equation;

$$\frac{\partial p}{\partial x} = f_1(\phi, w) \frac{\partial p}{\partial \phi} + \frac{1}{2} \frac{\partial^2 p}{\partial w^2} \tag{16}$$

with the initial condition

$$p(0, \phi, w) = \delta_\phi \delta w.$$

This equation, hopefully, could be solved by standard numerical procedures. *This, we feel, is a significant reduction in the solution of the random eigen. value problem posed at the beginning in as much as that the exact marginal distribution of the random eigenvalues are determined completely from the solution of the above partial differential equation.*

It may be further noted here that the argument in equation (11) could be generalized to yield the joint distribution of the eigenvalues in terms of the joint distribution of

$$\{\phi(t_1, \omega, \lambda), \quad \phi(t_2, \omega, \lambda), \ldots, \phi(t_n, \omega, \lambda)\}.$$

This density can be found in terms of the transition probability of the vector Markov process $\{\phi(t_1 w, \lambda), w(t, \omega)\}$. In turn, the transition probability density $p(t, \phi_1, w_1, \phi_2, w_2)$, which is the conditional density of $\{\phi(t, \omega), w(t, \omega)\}$ given $\phi(0, \omega) = \phi_1$ and $w(0, \omega) = w_1$, satisfies equation (16) with the initial condition

$$P(0, \phi_1, w_1, \phi_2, w_2) = \delta_{(\phi_2 - \phi_1)} \delta_{(\omega_2 - \omega_1)}.$$

5. CONCLUDING REMARKS

It has been demonstrated that many of the results of the deterministic Sturm-Liouville theory could be used beneficially in the stochastic case. In case the coefficient comes from a Brownian motion, the transition probability can be exploited to great advantage in studying the statistical properties of the eigenvalues. For example, we note that

$$E\lambda_n = \int_0^{n\pi} \int_{-\infty}^{\infty} \int_0^{\infty} p(L, \phi, w, t)\, dt\, dw d\phi.$$

In the deterministic case it is known that λ_n grows linearly with n. The above formula may help in showing the same for $E\lambda_n$.

Numerical methods may have to be employed in solving equation (16), since closed form solutions are extremely complicated, if at all available, for such partial differential equations. It should be of interest to study the solution,' at least for some special cases like $a = w^2$.

REFERENCES

[1] Boyce, W. E. .. Random vibrations of elastic strings and bars, *Proc. IV U.S. Nat. Congr. Appl. Mech., Berkeley*, 1962, pp. 77–85.

[2] Goodwin, B. E. and Boyce, W. E. Vibration of random elastic strings: method of integral equations, *Quart. Appl. Math.*, 1964, **22**, 261–266.

[3] Boyce, W. E. .. *Random Eigenvalue Problems, Probabilistic Methods in Applied Mathematics*, Vol. 1 (ed. A. T. Bharucha-Reid), Academic Press, New York, 1968.

[4] Bharucha-Reid, A. T. .. *Random Integral Equations*, Academic Press, New York, 1972.

[5] Birkoff, G. and Rota, G. C. *Ordinary Differential Equations*, Ch. 10, Ginn and Company, Boston, 1962.

[6] McKean, Jr., H. P, .. *Stochastic Integrals*, Academic Press, New Yoik, 1969.

Calendar of events: Conferences/Symposia at the Indian Institute of Science Campus

Sl. No.	Name of the School	Period	Sponsoring Department of the Institute
1.	Summer Institute in Molecular Structures	1–24 May 1975	Molecular Biophysics
2.	Indian National Science Academy	16–17 May 1975	Physics
3.	Intensive Course on High Voltage Technique	25 May to 8 June 1975	High Voltage Engineering
4.	Ferrous Foundry Technology	11–28 June 1975	Mechanical Engineering
5.	Molecular Biology of Bacteria and Bacterial Viruses	16 June to 12 July 1975	Microbiology and Pharmaco Laboratory
6.	Microwave and Millimeter Wave Communication	30 June to 12 July 1975	Electrical Communication Engineering
7.	All India Symposium on Biomedical Engineering	22–25 July 1975	Electrical Engineering
.	Geo-physical Fluid Dynamic Workshop	7–20 July 1975	Centre for Theoretical Studi
.	Organization of UNESCO Conference	1–15 September 1975	Mechanical Engineering
.	Intensive Course on Fluid Engineering	20 October to 2 November 1975	School of Automation
.	Lecture Course on Cavitation	Nov. to Dec. 1975	Chemical Engineering
.	Crystal Chemistry for College Teachers	December 1975	Inorganic and Physical Chemistry

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