

AMPLITUDE-FREQUENCY CHARACTERISTIC OF NON-LINEAR VIBRATIONS OF A THIN ANISOTROPIC RIGHT-ANGLED TRIANGULAR PLATE RESTING ON ELASTIC FOUNDATIONS

B. M. KARMAKAR

(Department of Physics, Jalpaiguri Government Engineering College, Jalpaiguri, West Bengal, India)

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ABSTRACT

Following the approximate method given by Berger, the author has made an attempt to study mainly the amplitude frequency characteristic of a thin anisotropic right-angled triangular plate of constant thickness freely vibrating transversely and non-linearly on an elastic foundation. Applying Galerkin procedure a non-linear second order differential equation for the unknown time function is obtained and it is solved in terms of Jacobian elliptic functions. Relative period of linear and non-linear oscillations also are graphically represented against relative amplitude to compare the general nature of variation with the result obtained by Iwinski and Ismail based on Von Kármán equation.

Key words: Amplitude frequency, non-linear vibrations, Elastic foundation.

1. INTRODUCTION

An approximate method for investigating the large deflection of initially flat isotropic plates has been proposed by Berger [1]. Essentially this method is based on the neglect of the second invariant of the middle surface strains in the expression corresponding to the total potential energy of the system. An application of this technique to the case of orthotropic plates has been offered by Iwinski and Nowinski [2], and further boundary value problems associated with circular and rectangular plates have been investigated by Nowinski [3]. Large deflections of circular and rectangular plates resting on elastic foundations have been investigated very elegantly by Sinha, S. N. [4] following the technique offered by Berger. Nash and Modeer [5] found the large amplitude free vibrations of rectangular and circular plates applying this approximation of Berger. Nowinski and Ismail [6] investigated the large amplitude free vibrations of an orthotropic triangular plate, without elastic foundation, based on von Karman equation, and graphically exhibited the result,

In this paper a dynamic field equation, governing large transverse deflections having rectangular co-ordinates, is established by adopting Berger's approximated method, and the solution is subsequently obtained by applying Galerkin procedure.

The governing equation arrived at in this paper does not explicitly contain any mass term except the average density of the material of the plate. Thus the equation may be considered independent of the mass of either the plate or the elastic foundation.

The present author's endeavour is mainly to study the amplitude frequency relation of non-linear free transverse oscillations of an orthotropic right-angled triangular plate placed on an elastic foundation. The ratio of the linear and non-linear periods is also plotted against relative amplitude to make a qualitative comparison with the results obtained by Nowinski and Ismail [6].

In a variety of situations, motions may be generated which lead to vibrations with large amplitudes. These are of importance in elements such as plates whose deformational response is significantly sensitive in the direction of smaller dimension.

Problems, Equations, Boundary Conditions and Graphical Representations of Results Obtained with Physical Explanations.

2. FREE LARGE AMPLITUDE TRANSVERSE VIBRATIONS

The right-angled elastic plate (Fig. 1) is of constant thickness h and has sides of length a and b . Let x, y be the rectangular co-ordinates, the origin being at the vertex of the right-angle and the axes are along the sides. The plate is made of rectilinearly orthotropic material with axes parallel to the axes of co-ordinates. The transverse deflections are assumed to be of the order of the magnitude of the plate thickness. The elastic character of any layer may not be isotropic but only symmetrical with respect to the normal.

By adding the potential energy of the foundation reaction to the energy expression containing the strain energy due to bending and stretching of the middle surface of the plate and neglecting the second invariant of the middle surface strain e_2 , we get,

$$V = \frac{1}{2} \int \int \left[\left\{ D_x \left(\frac{\partial^2 W}{\partial x^2} \right)^2 + 2D_1 \frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} + D_y \left(\frac{\partial^2 W}{\partial y^2} \right)^2 + 4D_{xy} \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2 + D_x \cdot \frac{12}{h^2} \cdot e_1^2 \right\} + K_1 W^2 \right] dx dy \quad (1)$$

where the first invariant of the middle surface strain is defined by

$$e_1 = \frac{\partial u}{\partial x} + K \cdot \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 + \frac{K}{2} \left(\frac{\partial W}{\partial y} \right)^2 \quad (2)$$

and where,

h = plate thickness,

D_x, D_y = flexural rigidity along the x and y axes, respectively,

$D_{xy} = \frac{Gh^3}{12}$, where G = Modulus of elasticity in shear,

$D_1 = E'' h^3/12$, E'' = Young's modulus in y -direction,

W = deflection normal to the middle surface,

u, v = displacements along x and y axes,

K_1 = modulus of the foundation,

$$k = \sqrt{\frac{D_y}{D_x}}$$

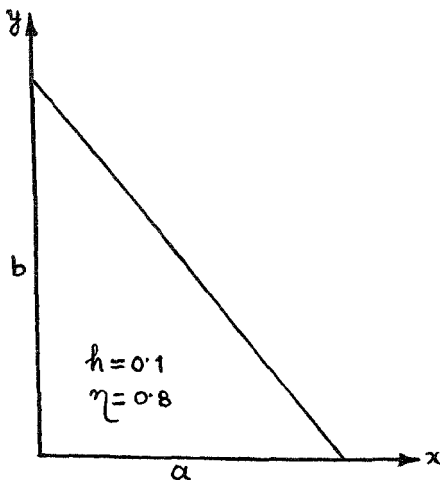


FIG. 1. Geometry of the Elastic Plate.

The kinetic energy of the plate is given by

$$T = \frac{\rho h}{2} \int \int (\dot{u}^2 + \dot{v}^2 + \dot{W}^2) dx dy \quad (3)$$

where ρ = average density of the plate, and the dots represent the derivatives with respect to time.

It is now possible to form the Lagrangian function

$$L = T - V \quad (4)$$

and according to Hamilton's principle

$$\delta \int_{t_1}^{t_2} L dt = 0 \quad (5)$$

and if we set

$$A = \int_{t_1}^{t_2} L dt \quad (6)$$

then

$$\delta A = 0. \quad (7)$$

Thus we have,

$$A = \int T dt - \int V dt. \quad (8)$$

Applying Euler's differential equations of variational problems and neglecting inertia effect in the plane of the plate, we obtain,

$$\begin{aligned} \frac{\partial^4 W}{\partial x^4} + \frac{2(D_1 + 2D_{xy})}{D_x} \cdot \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{D_y}{D_x} \frac{\partial^4 W}{\partial y^4} + \frac{k_1}{D_x} W \\ - \frac{12c}{h^2} \left(\frac{\partial^2 W}{\partial x^2} + k \frac{\partial^2 W}{\partial y^2} \right) f(t) + \frac{12}{h^2 c_s^2} \cdot \frac{\partial^2 W}{\partial t^2} = 0 \end{aligned} \quad (9)$$

where

$$c_s^{-2} = \frac{\rho h^3}{12D_x} \quad (10)$$

and

$$e_1 = c \cdot f(t) \quad (11)$$

c = normalised constant of integration.

For clamped edges, according to Nowinski and Ismail [6], the conditions imposed on the boundaries are

$$W = 0 \text{ and } W_{,n} = 0 \text{ for any time } t, \quad (12)$$

n = normal to the contour.

The boundary conditions given by equation (12) may be satisfied by the configurations of the forms, [A' being a constant],

$$u(x, y, t) = \frac{A'}{b} \left[x \left(1 - \frac{x}{a} - \frac{y}{b} \right) + \sin \frac{2\pi x}{a} \right] G(t) \quad (13)$$

$$v(x, y, t) = \frac{A'}{a} \left[y \left(1 - \frac{x}{a} - \frac{y}{b} \right) + \sin \frac{2\pi y}{b} \right] H(t) \quad (14)$$

$$W(x, y, t) = x^2 y^2 \left(1 - \frac{x}{a} - \frac{y}{b} \right)^2 F(t) \quad (15)$$

We have also,

$$\left. \begin{aligned} u = 0 \text{ at } x = 0 \\ v = 0 \text{ at } y = 0 \\ bu + av = 0 \text{ at } \frac{x}{a} + \frac{y}{b} = 1. \end{aligned} \right\} \quad (16)$$

Equations (13), (14) and (15), when put in equation (9), gives

$$F^2(t) = G(t) = H(t) = f(t) \quad (17)$$

Substituting for u , v and W , respectively from equations (13), (14) and (15) in equation (9) and remembering equation (17), we get, after integration over the whole area of the plate, the following relation for the constant c ,

$$c = \frac{a^2 b^3}{11.9.5} \cdot \frac{\sqrt{2} \sqrt{4}}{\sqrt{8}} \cdot (b^3 + a^2 k) \quad (18)$$

Now applying Galerkin procedure to equation (9) and putting the value of c obtained in equation (18), we get the equation for the time function as

$$\begin{aligned} \ddot{F} + \left[\frac{D_x}{\rho h} \left\{ 4\beta^2 (1 + \eta^2 m^2 + \eta^4 k^2) + \frac{k_1}{D_x} \right\} \right] F \\ + \left[\frac{13}{11.9.6.5.5} \cdot \eta \cdot \frac{D_x}{\rho} \left(\frac{b^2}{h} \right)^3 (1 + \eta^2 k)^2 \right] F^3 = 0, \quad (19) \end{aligned}$$

which may be written in a simpler form

$$\ddot{F} + \left(\alpha_1 \cdot \frac{D_x}{\rho h} + \frac{k_1}{D_x} \right) F + \left(\alpha_2 \cdot \frac{D_x}{\rho} \right) F^3 = 0 \quad (20)$$

where

$$\left. \begin{aligned} \eta &= \frac{a}{b} \\ \alpha_1 &= 4\beta^2 (1 + \eta^2 m^2 + \eta^4 k^2) \\ \alpha_2 &= \frac{13}{11 \cdot 9 \cdot 6 \cdot 5 \cdot 5} \cdot \eta \cdot \left(\frac{b^2}{h} \right)^3 (1 + \eta^2 k)^2 \\ m &= \sqrt{\frac{(D_1 + 2D_{xy})}{D_x}} \\ k &= \sqrt{\frac{D_y}{D_x}} \end{aligned} \right\} \quad (21)$$

and

$$\beta^2 = \frac{13 \cdot 11 \cdot 7}{a^4}$$

Let us now introduce the representation

$$F(t) = A_1 v(t) \quad (22)$$

where $v(t)$ is a new time function. We thus normalise the initial conditions as follows

$$v(0) = 1 \quad \text{and} \quad \dot{v}(0) = 0. \quad (23)$$

Equation (20) may now be carried into the form

$$\ddot{v} + \left[\alpha_1 \cdot \frac{D_x}{\rho h} + \frac{k_1}{D_x} \right] v + \left[\alpha_2 \cdot \frac{D_x h^2}{\rho} \xi^2 \right] v^3 = 0 \quad (24)$$

where $\xi = \frac{A_1}{h}$ is a representative of the non-dimensional amplitude of the fundamental mode of vibration.

Equation (24) is of the form

$$\ddot{v} + \gamma v + \delta v^3 = 0 \quad (25)$$

where

$$\gamma = \left[\alpha_1 \cdot \frac{D_x}{\rho h} + \frac{k_1}{D_x} \right] \quad (26)$$

and

$$\delta = \left[\alpha_2 \cdot \left(\frac{D_x h^2}{\rho} \right) \xi^2 \right] \quad (27)$$

Equation (25) represents a symmetric case. In such a case, only odd powers may occur in the law of force. The inertial force \ddot{v} is developed by virtue of acceleration or deceleration, according as the plate moves towards or away from the position of equilibrium. The middle term represents a linear elastic restorative force. The last term, which is proportional to the cube of the displacement, introduces non-linearity in this restorative force.

A solution to the non-linear differential equation (25) may be represented in terms of the cosine-type Jacobian elliptic function:—

$$v(t) = cn(\omega^* t, k^*) \quad (28)$$

where

$$\omega^{*2} = y + \delta \quad (29)$$

and

$$k^{*2} = \frac{\delta}{2(y + \delta)} \quad (30)$$

where again,

ω^* = fundamental frequency of non-linear free vibration,

\sqrt{y} = fundamental frequency of linear free vibration,

K^* = modulus of the elliptic function.

Here ω^* and K^* are positive constants and cn is Jacobi's elliptic function.

Substituting for y and δ from equations (26) and (27) respectively, we get, from equation (29),

$$A_1^2 = \frac{\omega^{*2} \rho}{\alpha_2 \cdot D_x} - \frac{\alpha_1}{\alpha_2} \cdot \frac{1}{h} - \frac{x_1 \rho}{\alpha_2 \cdot D_x^2} \quad (31)$$

This amplitude-frequency relation is displayed in Fig. 2, along with that of an isotropic plate vibrating under similar conditions like the orthotropic plate under investigation. The values taken for numerical calculations are those depicted in Table I given by Nowinski and Ismail [6].

For isotropy, equation (31) takes the form

$$A_1^2 = \frac{\omega^{*2} \rho}{a_2' D} - \frac{a_1'}{a_2'} \cdot \frac{1}{h} - \frac{k_1 \rho}{a_2' D^2} \quad (32)$$

where

$$\left. \begin{aligned} a_1' &= 4\beta^2 \left[1 + \eta^2 \cdot \frac{Eh^3}{12D(1-\nu^2)} + \eta^4 \right] \\ a_2' &= \frac{13}{11 \cdot 9 \cdot 6 \cdot 5 \cdot 5} \cdot \eta \cdot \left(\frac{b^2}{h} \right)^3 (1 + \eta^2)^2 \end{aligned} \right\} \quad (33)$$

and ν = Poisson's ratio.

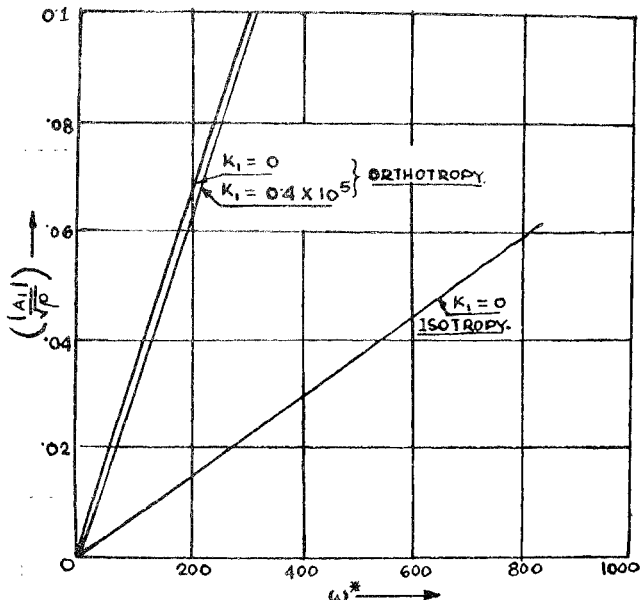


FIG. 2. Amplitude frequency response curve.

The period T^* of $cn(\omega^*, k^*)$ is given by

$$T^* = \frac{4K}{\omega^*} = \frac{4K}{\sqrt{\gamma + \delta}} = \frac{4K}{\left[\alpha_1 \frac{Dx}{\rho h} + \frac{k_1}{Dx} + \alpha_2 \cdot \frac{Dx}{\rho} h^2 \cdot \xi^2 \right]^{\frac{1}{2}}} \quad (34)$$

where K is the complete elliptic integral of the first kind.

Equation (34) corroborates the familiar phenomenon of a decrease of the non-linear time-period with increasing amplitude.

The linear time period is given by

$$T = \frac{2\pi}{\left[\alpha_1 \cdot \frac{Dx}{\rho h} + \frac{K_1}{Dx} \right]^{\frac{1}{2}}} \quad (35)$$

TABLE I

E_1, E_2, G_{12}, ν_1 and ν_2 are Young's modulus, shear modulus and Poisson's ratios respectively

E_1	E_2	G_{12}	ν_1	ν_2	k^2	m^2	$p^2 = \frac{E_2}{G_{12}} - \nu_2$
1×10^5	0.05×10^5	0.05×10^5	0.2	0.01	0.05	0.108	1
1×10^5	1×10^5	G	0.3	0.3	1	1	2

so that the ratio becomes

$$\frac{T^*}{T} = \frac{2K/\pi}{\left[1 + \frac{\alpha_2 Dx^2}{(\alpha_1 Dx^2 + \rho h k_1) h^3 \xi^2} \right]^{\frac{1}{2}}} \quad (36)$$

which in absence of a foundation ($X_1 = 0$), reduces to

$$\frac{T^*}{T} = \frac{2K/\pi}{\left[1 + \frac{\alpha_2}{\alpha_1} h^3 \xi^2 \right]^{\frac{1}{2}}} \quad (37)$$

The equation, corresponding to our equation (37), as obtained by Nowinski and Ismail, is given below

$$\frac{T^*}{T} = \frac{2K/\pi}{\left[1 + \frac{\alpha_2 E_2}{\alpha_1 D_1} h^3 \xi^2 \right]^{\frac{1}{2}}} \quad (38)$$

where D_1 corresponds to our D_2 , and γ_2 has a value different from ours.

The ratios obtained from equations (37) and (38) have been calculated from the values of the following Table I given by Nowinski and Ismail [6] and have been found to tally with their results within the limits of numerical calculations (Fig. 3). Thus Berger's approximate method, as applied to dynamic cases, are seen to be in good agreement with the results obtained from von Karman equations generalised to dynamic problems. For practical purposes, therefore, Berger's method may be used without any noticeable loss in accuracy. Further, the advantage of this method is the ease with which it can be employed.

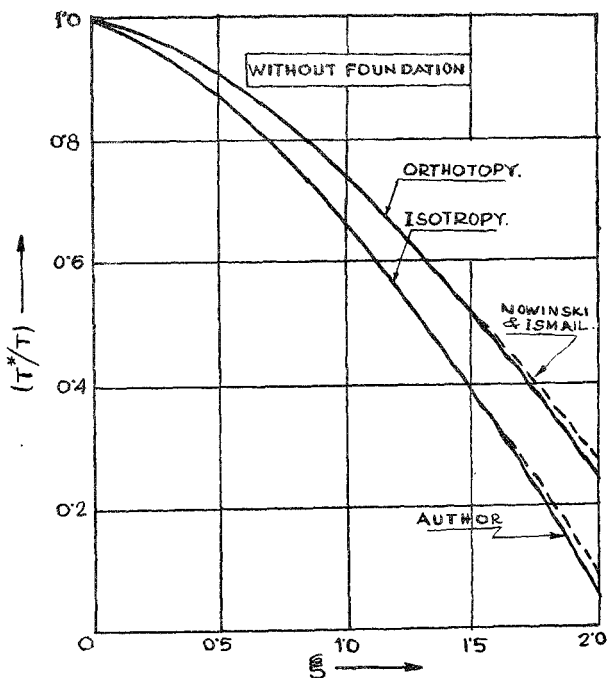


FIG. 3. Relative period vs relative amplitude.

We obtain again from equation (36), the ratio for isotropy as

$$\frac{T^*}{T} = \frac{2K/\pi}{\left[1 + \frac{\alpha_2 D^2}{(\alpha_1' D^2 + \rho h k_1)} h^3 \xi^2\right]^{\frac{1}{2}}} \quad (39)$$

It may be seen from equations (21, 33) that $\alpha_1 < \alpha_1'$, $\alpha_2 < \alpha_2'$ and $\alpha_1 < \alpha_2$, so that

$$\left(\frac{T^*}{T}\right)_{\text{iso}} < \left(\frac{T^*}{T}\right)_{\text{ortho}} \quad (40)$$

It may further be noted from equations (34) and (35) that only in the limiting case of harmonic motions the ratio (T^*/T) is unity, otherwise $(T^*/T) < 1$. Figure 3 displays the known general trend of this relative period in relation to the relative amplitude. In this Fig. 3, the numerical results obtained by the author have been compared with those of Nowinski and Ismail.

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