J. Indian Inst. Sci., May–June 2006, **86**, 265–278

Solution of a singular integral equation and its application to water wave problems

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Received on June 7, 2004, Revised on November 9, 2005.

Abstract

In the present paper, the solution of a singular integral equation with logarithmic kernel in two disjoint intervals $(0, a)\cup(b, \infty)$, (a, b are finite) is obtained by using function theoretic method. The two cases are considered when the unknown function satisfying the integral equation is unbounded or bounded at both nonzero finite end points of the interval. In the latter case, two solvability conditions are to be satisfied in order that the solution of the integral equation exists. We have used these two solvability conditions to evaluate the amplitude of waves at infinity for the three well-known water wave problems. These are: (i) scattering of water waves by a vertical plate submerged in deep water, (ii) generation of waves due to a line source in front of a vertical plate submerged in deep water, and (iii) generation of waves due to rolling of a submerged vertical plate.

Keywords: Singular integral equation, logarithmic kernel, two disjoint intervals, function theoretic method, scattering problem, radiation problem.

1. Introduction

The singular integral equation

$$\frac{1}{p}\int_G g(u)\ln\left|\frac{u+x}{u-x}\right| du = h(x), \qquad \text{for } x \in G \tag{1}$$

is considered here for solution where G consists of two disjoint intervals $(0, a)$ and (b, ∞) and $g(x)$, $h(x)$ are differentiable functions. This integral equation was first studied by Hardy [1] long back for $G = [0, \infty)$. In recent years, several researchers have investigated this integral equation for explicit solution for a single or double intervals [2–4]. The method of Riemann–Hilbert problem in the theory of complex variable was mostly employed in the mathematical analysis. This type of integral equation arises in the problems of linearised theory of water waves involving scattering and radiation of water waves by a thin vertical barrier. The need for study of the integral equation (1) lies in the fact that its kernel involves logarithmic singularity which is weakly singular in nature. Usually in the literature (cf. [5], [6]), equation (1) is solved by reducing it to an integral equation whose kernel involves Cauchy-type singularity which is strong in nature. As it is always numerically advantageous to evaluate integrals with weak singularity, so researchers were motivated to

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study (1) for various intervals without converting it to integral equations with strong singularity. The solution of (1) depends on the behaviour of the unknown function $g(u)$ at the end points of G. This behaviour is dictated by the physics of the problem in which it arises. Here the solution of (1) is obtained for $G = (0, a) \cup (b, \infty)$ by reducing it to a Riemann–Hilbert problem when $g(u)$ has the following two types of behaviour

$$\text{(i)} \quad g(u) \sim O((|u-c|)^{-\frac{1}{2}}) \quad \text{as } u \to c, \tag{2}$$

$$\text{(ii)} \quad g(u) \sim O((|u-c|)^{\frac{1}{2}}) \quad \text{as } u \to c, \tag{3}$$

c being a or b. In case (3), $h(x)$ has to satisfy two solvability conditions in order that the solution of (1) exists. We have used these two solvability conditions while investigating some scattering and radiation problems arising in the linearised theory of water waves. These involve scattering of surface water waves by a vertical plate submerged in deep water, generation of waves due to presence of a line source in front of a submerged vertical plate and of waves due to rolling of a submerged vertical plate. These are somewhat classical problems in the linearised theory of water waves (cf. [7–9]). Evans [7] used complex variable theory together with function theoretic method to obtain the wave amplitude at infinity of the problem of scattering of surface waves due to rolling of a thin plate submerged in deep water. The method used in [7] is rather elaborate. Mandal [8] used a simpler method based on judicious application of Green's integral theorem to obtain the wave amplitude at infinity without solving the problem in closed form. In the present paper, the boundary value problems associated with these problems were first reduced to multiple integral equations which in turn were reduced to (1) with $G = (0, a) \cup (b, \infty)$, where the unknown function $g(u)$ satisfies (3). The two solvability conditions here give rise to two simple simultaneous algebraic equations in two unknowns, which are solved very easily to recover the reflection and transmission coefficients in the scattering problem and amplitude of radiated waves at infinity for the radiation problems.

In Section 2, we give an outline of method of solution of (1). In Section 3, we describe the genesis of the integral equation and finally in Section 4, we evaluate the wave amplitude of the waves at infinity of the water wave scattering and radiation problems mentioned above.

2. Solution of a logarithmic singular integral equation

We consider the singular integral equation

$$\frac{1}{p} \int_G g(u) \ln \left| \frac{u+x}{u-x} \right| \mathrm{d}u = h(x), \quad \text{for } x \in G, \tag{4}$$

where $g(u)$ may have the following behaviour:

I)

$$g(u) \sim \begin{cases} O((|u-a|)^{-\frac{1}{2}}) & \text{as } u \to a, \\ O((|u-b|)^{-\frac{1}{2}}) & \text{as } u \to b, \end{cases} \tag{5}$$

II)

$$g(u) \sim \begin{cases} O((|u-a|)^{\frac{1}{2}}) \text{ as } u \to a, \\ O((|u-b|)^{\frac{1}{2}}) \text{ as } u \to b. \end{cases} \tag{6}$$

We first solve (4) where $g(u)$ has the behaviour given by (5).

Let

$$F(z) = \frac{d}{dz}\left(\int_G g(u) \ln \frac{u+z}{u-z} du \right), \tag{7}$$

where $F(z)$ represents sectionally analytic function in the complex z-plane ($z = x + iy$) cut along $(-\infty, -b) \cup (-a, 0) \cup (0, a) \cup (b, \infty)$ and $F(z) \sim O(\frac{1}{z^2})$ as $|z| \to \infty$.

We denote $F^{\pm}(x) = \lim_{y \to 0\pm} F(z)$ so that

$$F^{\pm}(x) = \frac{d}{dx}\left(\int_G g(u) \ln \left| \frac{u+x}{u-z} \right| du \right) \pm ip\, g(x),\ x \in (0, a) \cup (b, \infty)$$

and

$$F^{\pm}(x) = \frac{d}{dx}\left(\int_G g(u) \ln \left| \frac{u+x}{u-z} \right| du \right) \mp ip\, g(-x),\ x \in (-\infty, -b) \cup (-a, 0).$$

These give rise to

$$F^{+}(x) + F^{-}(x) = s(x) \tag{8}$$

and

$$F^{+}(x) - F^{-}(x) = l(x) \tag{9}$$

where,

$$s(x) = \begin{cases} 2p\, h'(x), & \text{for } x \in (0, a) \cup (b, \infty), \\ 2p\, h'(-x) & \text{for } x \in (-\infty, -b) \cup (-a, 0), \end{cases} \tag{10}$$

$$l(x) = \begin{cases} 2p\, ig(x), & \text{for } x \in (0, a) \cup (b, \infty) \\ -2p\, ig(-x) & \text{for } x \in (-\infty, -b) \cup (-a, 0). \end{cases} \tag{11}$$

Here (8) is a Riemann–Hilbert problem whose solution can be obtained in the following manner.

Let

$$F(z) = F_0(z) F_1(z) \tag{12}$$

where $F_0(z)$ satisfies

$$F_0^+(x) + F_0^-(x) = 0, \quad \text{for} \quad x \in (G \cup G') \tag{13}$$

and $G = (0, a) \cup (b, \infty)$, $G' = (-\infty, -b) \cup (-a, 0)$.

Let

$$F_0(z) = \frac{1}{\sqrt{(z^2 - a^2)(z^2 - b^2)}} \tag{14}$$

whereas $z \to x \pm i0$

$$F_0^{\pm}(x) = \begin{cases} \mp \dfrac{1}{\sqrt{(a^2 - x^2)(b^2 - x^2)}}, & x \in (-a, 0) \cup (0, a), \\ \pm \dfrac{1}{\sqrt{(x^2 - a^2)(x^2 - b^2)}}, & x \in (-\infty, -b) \cup (b, \infty). \end{cases} \tag{15}$$

Putting (12) in (8), we get for $x \in G \cup G'$

$$F_1^+(x) - F_1^-(x) = \frac{s(x)}{F_0^+(x)},$$

which by Plemelj formula gives

$$F_1(z) = Q + \frac{1}{2p\,i} \int_{G \cup G'} \frac{s(t)}{F_0^+(t)(t - z)}\, dt,$$

where Q is an arbitrary constant.

Thus

$$F(z) = F_0(z) \left[Q + \frac{1}{2p\,i} \int_{G \cup G'} \frac{s(t)}{F_0^+(t)(t - z)}\, dt \right]. \tag{16}$$

Using Plemelj formula and (9) we obtain

$$2p\,ig(x) = -\frac{2}{R(x)} \left[Q + \frac{1}{2p\,i} \int_{G \cup G'} \frac{s(t)}{F_0^+(t)(t - x)}\, dt \right], \text{ for } 0 < x < a, \tag{17}$$

$$2p\,ig(-x) = \frac{2}{R(x)} \left[Q + \frac{1}{2p\,i} \int_{G \cup G'} \frac{s(t)}{F_0^+(t)(t - x)}\, dt \right], \text{ for } -a < x < 0, \tag{18}$$

and

$$2p\,ig(x) = \frac{2}{R(x)} \left[Q + \frac{1}{2p\,i} \int_{G \cup G'} \frac{s(t)}{F_0^+(t)(t - x)}\, dt \right], \text{ for } b < x < \infty, \tag{19}$$

$$2p\,ig(-x) = -\frac{2}{R(x)}\left[Q + \frac{1}{2p\,i}\int_{G\cup G'} \frac{s(t)}{F_0^+(t)(t-x)}\,dt\right], \text{ for } -\infty < x < -b, \tag{20}$$

where $R(x) = |(x^2 - a^2)(x^2 - b^2)|^{\frac{1}{2}}$.

Now,

$$\int_{G\cup G'} \frac{s(t)}{F_0^+(t)(t-x)}\,dt = 4p\,x\left[-\int_0^a \frac{h'(t)R(t)}{t^2 - x^2}\,dt + \int_b^\infty \frac{h'(t)R(t)}{t^2 - x^2}\,dt\right]. \tag{21}$$

We substitute (21) in (17), (18), (19) and (20). Next we compare (17) with (18) and (19) with (20) to get $Q = 0$. Thus we get the solution of (4) satisfying (5) as

$$g(x) = \begin{cases} -\dfrac{2x}{p\,R(x)}P(x) & \text{for} \quad 0 < x < a, & (22) \\[2ex] \dfrac{2x}{p\,R(x)}P(x) & \text{for} \quad b < x < \infty, & (23) \end{cases}$$

where

$$P(x) = -\int_0^a \frac{h'(t)R(t)}{t^2 - x^2}\,dt + \int_b^\infty \frac{h'(t)R(t)}{t^2 - x^2}\,dt.$$

We now obtain the solution of (4) with $g(u)$ satisfying (6). In this case, we rearrange $P(x)$ such that

$$P(x) = R^2(x)\left[-\int_0^a \frac{h'(t)}{R(t)(t^2 - x^2)}\,dt + \int_b^\infty \frac{h'(t)}{R(t)(t^2 - x^2)}\,dt\right] + |\,a^2 - x^2\,|\left[\int_0^a \frac{h'(t)}{R(t)}\,dt - \int_b^\infty \frac{h'(t)}{R(t)}\,dt\right]$$

$$+\left[\int_0^a h'(t)\left(\left|\frac{t^2 - b^2}{t^2 - a^2}\right|\right)^{\frac{1}{2}} dt + \int_0^\infty h'(t)\left(\left|\frac{t^2 - b^2}{t^2 - a^2}\right|\right)^{\frac{1}{2}} dt\right].$$

Substituting $P(x)$ in (22) and (23) we have

$$g(x) = -\frac{2R(x)}{p\,x}\left[-\int_0^a \frac{t^2 h'(t)}{R(t)(t^2 - x^2)}\,dt + \int_b^\infty \frac{t^2 h'(t)}{R(t)(t^2 - x^2)}\,dt + \int_0^a \frac{h'(t)}{R(t)}\,dt - \int_b^\infty \frac{h'(t)}{R(t)}\,dt\right]$$

$$-\left(\frac{a^2 - b^2}{b^2 - x^2}\right)^{\frac{1}{2}}\left(\frac{2x}{p}\right)\left[\int_0^a \frac{h'(t)}{R(t)}\,dt - \int_b^\infty \frac{h'(t)}{R(t)}\,dt\right]$$

$$+\frac{2x}{p\,R(x)}\left[\int_0^a h'(t)\sqrt{\frac{b^2 - t^2}{a^2 - t^2}}\,dt + \int_b^\infty h'(t)\sqrt{\frac{t^2 - b^2}{t^2 - a^2}}\,dt\right], \; 0 < x < a \tag{24}$$

and

$$g(x)=\frac{2R(x)}{p\,x}\left[-\int_0^a \frac{t^2 h'(t)}{R(t)(t^2-x^2)}\,dt+\int_b^\infty \frac{t^2 h'(t)}{R(t)(t^2-x^2)}\,dt+\int_0^a \frac{h'(t)}{R(t)}\,dt-\int_b^\infty \frac{h'(t)}{R(t)}\,dt\right]$$

$$+\left(\frac{x^2-a^2}{x^2-b^2}\right)^{\frac{1}{2}}\left(\frac{2x}{p}\right)\left[\int_0^a \frac{h'(t)}{R(t)}dt-\int_b^\infty \frac{h'(t)}{R(t)}dt\right]$$

$$-\frac{2x}{p\,R(x)}\left[\int_0^a h'(t)\sqrt{\frac{t^2-b^2}{t^2-a^2}}\,dt+\int_b^\infty h'(t)\sqrt{\frac{t^2-b^2}{t^2-a^2}}\,dt\right],\ \ b<x<\infty. \tag{25}$$

If $g(x)$ has to be bounded as x tends to a and b, i.e. $g(x)$ satisfies (6), we must have following conditions:

(i)
$$\int_0^a \frac{h'(t)}{R(t)}dt-\int_b^\infty \frac{h'(t)}{R(t)}\,dt=0, \tag{26}$$

(ii)
$$\int_0^a h'(t)\sqrt{\frac{b^2-t^2}{a^2-t^2}}\,dt+\int_b^\infty h'(t)\sqrt{\frac{t^2-b^2}{t^2-a^2}}\,dt=0. \tag{27}$$

Thus, solution of (4) in this case is

$$g(x)=\begin{cases}-\frac{2R(x)}{p(x)}x(x) & \text{for}\quad 0<x<a,\\ \frac{2R(x)}{p(x)}x(x) & \text{for}\quad b<x<\infty,\end{cases} \tag{28}$$

where,

$$x(x)=-\int_0^a \frac{t^2 h'(t)}{R(t)(t^2-x^2)}dt+\int_b^\infty \frac{t^2 h'(t)}{R(t)(t^2-x^2)}\,dt. \tag{29}$$

Thus, (28) together with (26) and (27) give the solution of (1) with $g(u)$ satisfying (6).

In the next section, we discuss about the genesis of the integral equation (1).

3. Genesis of the integral equation (1)

Under the assumption of the linearized theory, a general two-dimensional radiation problem involving a vertical plate submerged in deep water is described by the following boundary-value problem for Laplace equation (cf [5, 8, 9]):

$$\nabla^2 y = 0 \quad \text{in} \quad y \geq 0, \tag{30}$$

$$Ky + y_y = 0, \quad \text{on} \quad y=0, \tag{31}$$

where, $K = \frac{s^2}{g}$, g is acceleration due to gravity and s is the circular frequency,

$$y_x = f(y); \quad \text{for} \quad x = 0, \quad y \in B, \quad B = (a, b), \tag{32}$$

where, $f(y)$ is known. Here B denotes the vertical barrier occupying the interval $a < y < b$.

Also,

$$\nabla y \to 0 \quad \text{as} \quad y \to \infty, \tag{33}$$

$$r^{1/2}\nabla y \text{ is bounded as } r \to 0,\ r \text{ being the sharp edges of the barrier.} \tag{34}$$

$$y \sim \begin{cases} A_1 \exp(-Ky + iKx) & \text{as} \quad x \to \infty, \\ A_2 \exp(-Ky - iKx) & \text{as} \quad x \to -\infty, \end{cases} \tag{35}$$

where A_1 and A_2 are amplitudes of the radiated waves at infinity on either side of the wall. Here the function $Re(y(x, y)\exp(-ist))$ represents the velocity potential of the irrotational motion produced due to a small motion of a vertical plate $x = 0$, $y \in B$ submerged in deep water $y \geq 0$ where the usual cartesian coordinate system is chosen with y axis vertically downwards.

The problem of scattering of incoming waves represented by $\exp(-Ky + iKx)$ by a thin vertical plate submerged in deep water can be described by velocity potential $Re(f(x, y)\exp(-ist))$ where f satisfies (30), (31), (33), (34) and

$$f_x = 0, \quad \text{for} \quad x = 0, \quad y \in B \tag{36}$$

and

$$f \sim \begin{cases} R \exp(-Ky - iKx) + \exp(-Ky + iKx) & \text{as} \quad x \to -\infty, \\ T \exp(-Ky + iKx) & \text{as} \quad x \to \infty, \end{cases} \tag{37}$$

where, R, T are the reflection and transmission coefficients. This problem can be formulated as radiation problem by defining

$$y = f - \exp(-Ky + iKx),$$

where, y satisfies (30) to (35) with

$$f(y) = -iK \exp(-Ky) \tag{38}$$

and

$$A_1 = T - 1,\ A_2 = R. \tag{39}$$

Now we shall reduce the boundary value problem for y into singular integral equation (4).

By Havelock's expansion of velocity potential, a suitable representation of $y(x, y)$ satisfying (30), (31), (32) and (34) is

$$y(x, y) \sim \begin{cases} A_1 \exp(-Ky + iKx) + \int_0^\infty C(k)L(k, y)\exp(-kx)dk, & x > 0, \\ A_2 \exp(-Ky + iKx) + \int_0^\infty E(k)L(k, y)\exp(kx)dk, & x < 0, \end{cases} \tag{40}$$

where $C(k)$, $E(k)$, A_1, A_2 are unknown and

$$C(k) = -E(k),$$

$$L(k, y) = k\cos ky - K\sin ky.$$

Using (40) in (32) we have

$$\int_0^\infty kC(k)L(k,y)dk = iKA_1\exp(-Ky) - f(y),\ \ y\in B. \tag{41}$$

Also, $y(+0, y) = y(-0, y)$, $y \in G$, so that using (40) we get

$$\int_0^\infty C(k)L(k,y)dk = -A_1\exp(-Ky),\ \ y\in G. \tag{42}$$

The dual integral equations (41) and (42) can be put in the alternative form as

$$\int_0^\infty kC(k)\sin ky dk = D_1\exp(Ky) - \frac{iA_1}{2}\exp(-Ky)$$
$$-\exp(Ky)\int_a^y h(t)\exp(-Kt)dt,\ \ y\in(a,b) \tag{43}$$

and

$$\int_0^\infty C(k)\sin ky dk = \begin{cases} \frac{A_1}{2K}\exp(-Ky) + D_2\exp(Ky),\ \ 0<y<a, \\ \frac{A_1}{2K}\exp(-Ky) + D_3\exp(Ky),\ \ b<y<\infty, \end{cases} \tag{44}$$

where D_1, D_2, D_3 are arbitrary constants. As $y \to \infty$, D_3 becomes zero and as $y \to 0$, D_2 equals $-(A_1/2K)$

Then the dual integral equations (43) and (44) can be rewritten as

$$\int_0^\infty kC(k)\sin ky dk = -\frac{iA_1}{2}\exp(-Ky) - \exp(Ky)\int_a^y f(t)\exp(-Kt)dt + D_1\exp(Ky),\ \ a<y<b, \tag{45}$$

and

$$\int_0^\infty C(k)\sin ky\, dk = \begin{cases} -\frac{A_1}{K}\sinh Ky,\ \ 0<y<a, \\ \frac{A_1}{2K}\exp(-Ky),\ \ b<y<\infty. \end{cases} \tag{46}$$

To solve the integral equation (45) and (46) we take

$$\int_0^\infty kC(k)\sin ky dk = g(y),\ \ y\in G \tag{47}$$

where $g(u)$ is an unknown function.

Now, from (45) and (47) we get by Fourier sine inversion

$$kC(k) = \frac{2}{p}\int_0^\infty M(t)\sin ktdt, \tag{48}$$

where

$$M(y) = \begin{cases} -\frac{iA_1}{2}\exp(-Ky) - \exp(Ky)\int_a^y f(t)\exp(-Kt)dt + D_1\exp(Ky), & a < y < b, \\ g(y), & y \in G. \end{cases} \tag{49}$$

Substituting $C(k)$ into (46) and simplifying we get,

$$\frac{1}{p}\int_G g(u)\ln\left|\frac{u+x}{u-x}\right|du = h(x), \quad \text{for} \quad x \in (G), \tag{50}$$

where

$$h(x) = \begin{cases} -\frac{A_1}{K}\sinh Kx - \frac{1}{p}\int_a^b M(t)\ln\left|\frac{x+t}{x-t}\right|dt, & 0 < x < a, \\ \frac{A_1}{2K}\exp(-Kx) - \frac{1}{p}\int_a^b M(t)\ln\left|\frac{x+t}{x-t}\right|dt, & b < x < \infty. \end{cases} \tag{51}$$

It is now important to know the behaviour of $g(y)$ as y tends to a and b. Noting (32), let us write

$$y_x(0, y) = \begin{cases} w(y), & y \in G, \\ f(y), & y \in B. \end{cases}$$

Using (40) we get,

$$\left(\frac{d}{dy} - K\right)\int_0^\infty kC(k)\sin kydk = iKA_1\exp(-Ky) - w(y), \quad y \in G,$$

which can be written alternatively as

$$\int_0^\infty kC(k)\sin(ky)dk = \begin{cases} -\frac{iA_1}{2}\exp(-Ky) - \exp(Ky)\int_a^y w(t)\exp(-Kt)dt + E_1\exp(Ky), & 0 < y < a, \\ -\frac{iA_1}{2}\exp(-Ky) - \exp(Ky)\int_a^y w(t)\exp(-Kt)dt + E_2\exp(Ky), & b < y < \infty. \end{cases}$$

Comparing with (47), we get

$$w(y) = \frac{dg}{dy} + Kg(y) + iKA_1\exp(-Ky).$$

Noting (34) we observe that

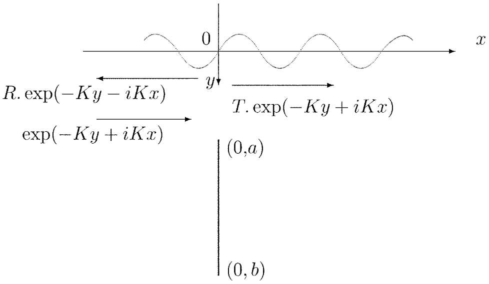

FIG. 1. Scattering problem.

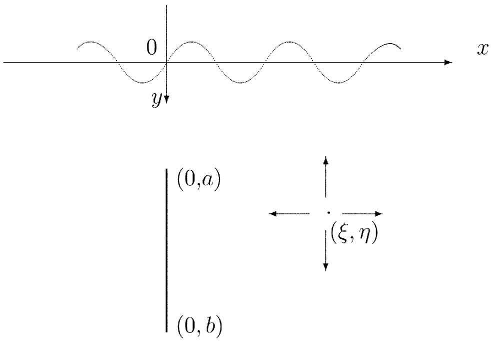

FIG. 2. Generation of waves due to a source.

$$g(y) \sim O(|y-t|)^{\frac{1}{2}} \text{as } y \to t,\ \ t = a \ \text{ or } \ b. \tag{52}$$

Now, in (50) $g(y)$ satisfies (52). So in order that the solution of (50) exists the following two solvability conditions are to be satisfied. Thus from (26) and (27) we have

(i) $$\int_0^a \frac{h'(t)}{R(t)} dt - \int_b^\infty \frac{h'(t)}{R(t)} dt = 0, \tag{53}$$

and

(ii) $$\int_0^a h'(t) \frac{\sqrt{b^2-t^2}}{\sqrt{a^2-t^2}} dt + \int_b^\infty h'(t) \frac{\sqrt{t^2-b^2}}{\sqrt{t^2-a^2}} dt = 0. \tag{54}$$

We now use (53) and (54) to obtain wave amplitude at infinity for the following problems.

4. Problems

In this section, we consider three well-known water-wave scattering and radiation problems. These problems are described below.

4.1. *Scattering by a vertical plate*

We consider the problem of scattering of water waves by a vertical plate $x = 0$, $y \in B$ (cf Fig. 1). Here we substitute $h(t)$ from (51) into the solvability conditions (53) and (54), after using (49) and (38) to get

$$D_4 \int_a^b \frac{\exp(Ku)}{R(u)} du = -\frac{A_1}{2} \int_{-a}^a \frac{\exp(-Kt)}{R(t)} dt$$

$$+\frac{A_1}{2} \int_b^\infty \frac{\exp(-Kt)}{R(t)} dt + \frac{i}{2}(A_1+1) \int_a^b \frac{\exp(-Ku)}{R(u)} du \tag{55}$$

and

$$D_4 \int_a^b \frac{b^2 - a^2}{R(u)} \exp(Ku)du = -\frac{A_1}{2} \int_{-a}^{a} \frac{(b^2 - t^2)\exp(-Kt)}{R(t)} dt - \frac{A_1}{2} \int_b^{\infty} \frac{(t^2 - b^2)\exp(-Kt)}{R(t)} dt$$

$$+\frac{i}{2}(A_1 + 1)\int_a^b \frac{(b^2 - u^2)\exp(-Ku)}{R(u)} du \tag{56}$$

where,

$$D_4 = D_1 + \frac{i}{2}\exp(-2Ka).$$

(55) and (56) are two equations in A_1 and D_4 which can be solved.

We divide (56) by (55) and take

$$d_0^2 = \frac{\int_a^b \frac{u^2 \exp(Ku)}{R(u)} du}{\int_a^b \frac{\exp(Ku)}{R(u)} du}$$

to get after simplification

$$A_1 = \frac{ig_0}{\Delta_0}.$$

Hence from (39)

$$A_1 = T - 1,$$

so,

$$T = \frac{a_0 - b_0}{\Delta_0} \tag{57}$$

and

$$R = 1 - T = -\frac{ig_0}{\Delta_0} \tag{58}$$

where

$$\Delta_0 = a_0 - b_0 - ig_0,$$

$$a_0 = \int_{-a}^{a} \frac{d_0^2 - u^2}{R(u)} \exp(-Ku)du,$$

$$b_0 = \int_b^{\infty} \frac{d_0^2 - u^2}{R(u)} \exp(-Ku)du,$$

$$g_0 = \int_a^b \frac{d_0^2 - u^2}{R(u)} \exp(-Ku)du.$$

Thus, knowing A_1, one can be obtain D_4. Equations (57) and (58) coincide with the results in [7]. Thus, knowing A_1 and D_4, $g(u)$ can be obtained from (28) and hence $C(k)$ from (48). Thus y can be obtained from (40).

The limiting case $a \to 0$ represents the problem of scattering of water waves by a vertical barrier partially immersed in deep water which was considered by Ursell [6]. Making $a \to 0$, (53) and (54) reduce to single solvability criterion

$$\int_b^\infty \frac{t h'(t)}{\sqrt{t^2 - b^2}}\, dt = 0, \tag{59}$$

where

$$h(t) = -\frac{R}{2K} \exp\,(-Kx) - \frac{i}{p}(1 - R) \int_0^b \sinh Kt \ln\left|\frac{x+t}{x-t}\right| dt.$$

Thus, (59) gives

$$R = \frac{p\, I_1(Kb)}{p\, I_1(Kb) + iK_1(Kb)}$$

which coincides with Ursell [6].

4.2. *Generation of waves by a source in front of a submerged vertical plate*

A source $G(x, y; x, h)$ is situated at a point (x, h) in front of a vertical plate $x = 0$, $a < y < b$ submerged in deep water (cf Fig. 2). The motion is described by the velocity potential Re(Φ $(x, y)\exp(-ist)$), where Φ satisfies the following boundary value problem:

$$\nabla^2 \Phi = 0 \quad \text{in the fluid region except at } x(x, h),$$

$$K\Phi + \Phi_y = 0 \qquad \text{on } y = 0.$$

where, $K = \frac{s^2}{g}$, g being gravity, s being circular frequency,

$$\Phi_x = 0 \quad \text{for} \quad x = 0, y \in B,$$

$$\Phi \sim \ln r \text{ as } r \to 0 \text{ where } r = \{(x - x)^2 + (y - h)^2\}^{1/2},$$

$$r^{1/2}\nabla\Phi \text{ is bounded as } r \to 0,\ r = \{(x)^2 + (y - c)^2\}^{1/2};\ c = a \text{ or } b$$

$$\nabla\Phi \to 0 \quad \text{as} \quad y \to \infty,$$

$$\Phi \sim \begin{cases} B_+ \exp(-Ky + iKx) & \text{as} \quad x \to \infty, \\ B_- \exp(-Ky - iKx) & \text{as} \quad x \to -\infty, \end{cases}$$

where B_+ and B_- are amplitudes of radiated waves at infinity on either side of the wall.

Let

$$\Phi = G + y,$$

where, $G(x, y, x, h)$ is the velocity potential due to the presence of a line source at (x, h) in the absence of a barrier. The form of G is given by [cf [10]]

$$G(x,y,x,h) = -2\int_0^\infty \frac{L(k,h)L(k,y)\exp(-k(x-x))}{k(k^2+K^2)}\,dk.$$

$$-\,2pi\exp(-K(y+h)+iK|x-x|).$$

Thus, y is the correction to G and it satisfies (30), (31), (32), (33), (34) and

$$y_x(0,y) = f(y) = -G_x(0,y;x,h), \quad y \in B.$$

Knowing $f(y)$, one can obtain $h(t)$ from (51). Substituting $h(t)$ into the solvability conditions (53) and (54) we get,

$$A_1 = \frac{2}{\Delta_0}\left(\int_a^b \frac{d_0^2-u^2}{R(u)}\exp(Ku)\,du\int_a^u h(t)\exp(-Kt)\,dt\right). \tag{60}$$

The result in (60) coincides with [8].

4.3. *Generation of waves due to rolling of a submerged vertical plate*

In this case, the vertical plate is hinged at the point $(0,c)$, $a \le c \le b$ and performs a small rolling oscillation of amplitude q_0 (Fig. 3). In this case, $f(y)$ in (32) can be written as (cf [5], [8])

$$f(y) = isq_0(c-y). \tag{61}$$

Using (61), $h(t)$ can be calculated from (51). Substituting $h'(t)$ into the solvability conditions (53) and (54), we get,

$$A_1 = \frac{2isq_0}{\Delta_0K^2}\left[\frac{1-Kc}{b}\left(d_0^2K(q)-b^2E(q)\right)+K\frac{p}{2}\left(d_0^2-\frac{a^2+b^2}{2}\right)\right], \tag{62}$$

where

$$K(q) = \int_0^{\frac{p}{2}}(1-q^2\sin^2 b)^{-\frac{1}{2}}\,db,$$

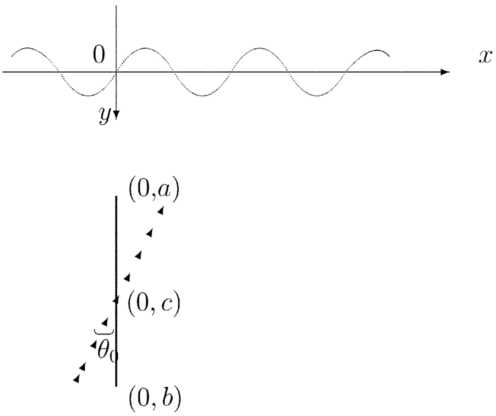

FIG. 3. Waves due to rolling of a submerged plate.

$$E(q) = \int_0^{\frac{p}{2}} (1 - q^2 \sin^2 b)^{\frac{1}{2}} d b$$

and

$$q = \frac{\sqrt{b^2 - a^2}}{b} < 1.$$

The result given by (62) coincides with [8].

5. Conclusion

The solution of (1) is obtained here for $G = (0, a) \cup (b, \infty)$ by reducing it to a Riemann–Hilbert problem when $g(u)$ has the behaviour given by (2) and (3) as it tends to a and b. The integral equation (1) involves weakly singular kernel. As the integrals with weak singularity are more amenable to numerical methods, so it is always advantageous to solve (1) without converting it to an integral equation with strong singularity. The solution of (1) for $g(u)$ satisfying (3) exists when two solvability conditions (26) and (27) are satisfied. From these two solvability conditions, the wave amplitude at infinity for some classical scattering and radiation problems in the linearized water wave theory are recovered in a very simple manner.

Acknowledgement

This work is partially supported by the Department of Science and Technology (DST), Government of India, through the research project no. SR/S4/MS:263/05. BD is thankful to Council of Scientific and Industrial Research (CSIR) for supporting this work.

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