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New second-order optimality conditions in multiobjective optimization problems: Differentiable case

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Abstract

To get positive Lagrange multipliers associated with each of the objective function, Maeda [Constraint qualification in multiobjective optimization problems: Differentiable case, *J. Optimization Theory Appl.*, **80**, 483–500 (1994)], gave some special sets and derived some generalized regularity conditions for first-order Karush–Kuhn–Tucker (KKT)-type necessary conditions of multiobjective optimization problems. Basing on Maeda's set, Bigi and Castellani [Second order optimality conditions for differentiable multiobjective problems, *RAIRO, Op. Res.*, **34**, 411–426 (2000)], tried to get the same result for second-order optimality conditions but their treatment was not convincing. In this paper, we have generalized these regularity conditions for second-order optimality conditions under different sets and obtained positive Lagrange multipliers for the objective function.

Keywords: Multiobjective optimization, local vector minimum point, regularity conditions, second-order necessary conditions.

1. Introduction

Investigation of optimality conditions has been one of the most interesting topics in the theory of multiobjective optimization problems. Many authors have derived the first- and second-order necessary conditions for vector minimum solution under the same constraint qualification as used in scalar-valued objective function [1], but none could obtain positive Lagrange multipliers associated with the vector-valued objective function. So it is possible that due to some zero multipliers the corresponding components of the vector valued objective functions have no role in the necessary conditions of multiobjective problem. To avoid this undesirable situation getting positive Lagrange multipliers, Maeda [2] gave some special sets and derived some generalized regularity conditions for the first-order KKT-type necessary conditions that ensure the existence of positive Lagrange multipliers for first-order multiobjective optimality conditions. For getting positive Lagrange multipliers, some authors analyzed these conditions for second-order KKT-type necessary conditions [3–5]. In particular, basing on Maeda's sets, Bigi and Castellani [5] generalized these regularity conditions for second-order optimality conditions, but their treatment is not convincing.

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In this paper, we have also generalized Maeda-type regularity conditions for second-order KKT-type necessary conditions, but have generalized these conditions under more general sets, later called “Proposed sets”. As a result, we have ensured positive Lagrange multipliers, associated with the objective function and derived second-order KKT-type necessary conditions for both equality and inequality constraints.

Some notations, definitions, and preliminary results are given in Section 2. In Section 3, we have given the comparison between Maeda’s sets and proposed sets, generalization of Maeda’s regularity conditions and also derived KKT-type necessary conditions for second-order optimality conditions with an important remark.

2. Preliminaries

In this section, we introduce some notations and definitions, which are used throughout the paper [6]. Let E_n be n-dimensional Euclidean space.

For $\mathbf{x}, \mathbf{y} \in E_n$, we use the following conventions.

$$\mathbf{x} \geqq \mathbf{y}, \quad \text{iff} \quad x_i \geqq y_i, \quad i = 1, \ldots, n,$$

$$\mathbf{x} \geq \mathbf{y}, \quad \text{iff} \quad \mathbf{x} \geqq \mathbf{y} \quad \text{and} \quad \mathbf{x} \neq \mathbf{y},$$

$$\mathbf{x} > \mathbf{y}, \quad \text{iff} \quad x_i > y_i \quad i = 1, \ldots, n.$$

Now, we consider the following multiobjective optimization problem P:

$$\min \mathbf{f}(\mathbf{x}), \text{ subject to the set X: } \bar{\mathbf{x}} \in \mathrm{X} = \{\mathbf{x} \in E_n | \mathbf{g}(\mathbf{x}) \leqq 0, \mathbf{h}(\mathbf{x}) = 0\}$$

Let, $f : E_n \to E_l$, $g : E_n \to E_m$ and $h : E_n \to E_k$ be twice continuously differentiable vector-valued functions. Assume that $I(\bar{\mathbf{x}}) = \{ j : g_j(\bar{\mathbf{x}}) = 0\}$ for $j = 1, \ldots, m$.

For any twice continuously differentiable function $\mathbf{g} : E_n \to E_m$ and for any vector $\mathbf{y} \in E_m$, we denote by $\nabla \mathbf{g}(\bar{\mathbf{x}})$ and $\nabla^2 \mathbf{g}(\bar{\mathbf{x}})(\mathbf{y}, \mathbf{y})$, respectively, the $m \times n$ Jacobian matrix and the m-dimensional vector whose ith component is $\mathbf{y}^T \nabla^2 g_i(\bar{\mathbf{x}})\mathbf{y}$.

Now, we shall define the nonempty sets M^i and M by

$$M^i \equiv \{\mathbf{x} \in E_n | \mathbf{x} \in \mathrm{X},\ f_i(\mathbf{x}) \leqq f_i(\bar{\mathbf{x}})\}, i = 1, 2, \ldots, l$$

and $M \equiv \{\mathbf{x} \in E_n | \mathbf{x} \in \mathrm{X},\ \mathbf{f}(\mathbf{x}) \leqq \mathbf{f}(\bar{\mathbf{x}})\} = \bigcap_{i=1}^{l} M^i$ = Set of vector minimum point.

For any two vectors $\mathbf{x} = (x_1, x_2)^T$ and $\mathbf{y} = (y_1, y_2)^T$ in E_2, we use the following conventions:

$$\mathbf{x} \leqq_{lex} \mathbf{y} \text{ means that } x_1 < y_1 \text{ holds or } x_1 = y_1 \text{ and } x_2 \leqq y_2 \text{ hold.}$$

$$\mathbf{x} <_{lex} \mathbf{y} \text{ means that } x_1 < y_1 \text{ holds or } x_1 = y_1 \text{ and } x_2 < y_2 \text{ hold.}$$

The subscript *lex* means lexicographic order.

Due to the conflicting nature of the objectives, an optimal solution that simultaneously minimizes all the objectives is usually not obtainable. Thus, for problem P, the solution is defined in terms of a local vector minimum point [3].

Definition 2.1. A point $\bar{\mathbf{x}} \in X$ is said to be a local vector minimum point of P if and only if there exists a neighbourhood N of $\bar{\mathbf{x}}$, such that no $\mathbf{x} \in X \cap N$ satisfies $\mathbf{f}(\bar{\mathbf{x}}) - \mathbf{f}(\mathbf{x}) > \mathbf{0}$, that is $f_i(\bar{\mathbf{x}}) > f_i(\mathbf{x})$ for all i.

Now we define two kinds of second-order approximation sets to the feasible region.

Definition 2.2. The second-order tangent set to X at $\bar{\mathbf{x}} \in X$ is the set defined by

$$T^2(X;\bar{\mathbf{x}}) \equiv \{(\mathbf{y},\mathbf{z}) \in E_{2n} \,|\, \exists \mathbf{x_n} \in X,\ \exists\, t_n \to +0 \text{ such that } \mathbf{x_n} = \bar{\mathbf{x}} + t_n\mathbf{y} + \frac{1}{2}t_n^2\mathbf{z} + o(t_n^2)\}$$

where $o(t_n^2)$ is a vector satisfying $\dfrac{\| o(t_n^2) \|}{t_n^2} \to 0.$

Definition 2.3. The second-order linearizing set to M at $\bar{\mathbf{x}} \in M$ is the set defined by

$$L^2(M;\bar{\mathbf{x}}) = \left\{ \begin{array}{l} (\mathbf{y},\mathbf{z}) \in E_{2n} \,|\, (\nabla f_i(\bar{\mathbf{x}})^T\mathbf{y},\ \nabla f_i(\bar{\mathbf{x}})^T\mathbf{z} + \nabla^2 f_i(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y}))^T \leqq_{lex} (0,0)^T,\ \ i = 1,\dots,l \\ (\nabla g_j(\bar{\mathbf{x}})^T\mathbf{y},\ \nabla g_j(\bar{\mathbf{x}})^T\mathbf{z} + \nabla^2 g_j(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y}))^T \leqq_{lex} (0,0)^T,\ j \in I(\bar{x}) \\ \text{and } (\nabla h_p(\bar{\mathbf{x}})^T\mathbf{y}, \nabla h_p(\bar{\mathbf{x}})^T\mathbf{z} + \nabla^2 h_p(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y}))^T = (0,0)^T \quad p = 1,\dots,k \end{array} \right\}$$

A first-order sufficient conditions for vector minimum point is that the following system has no nonzero solution $\mathbf{y}$:

$$\nabla\mathbf{f}(\bar{\mathbf{x}})^T\mathbf{y} \leqq 0, \quad \nabla\mathbf{g}_I(\bar{\mathbf{x}})^T\mathbf{y} \leqq 0, \quad \nabla\mathbf{h}(\bar{\mathbf{x}})^T\mathbf{y} = 0 \tag{1}$$

The Kuhn–Tucker-type condition for optimality is equivalent to the inconsistency of the following system:

$$\nabla\mathbf{f}(\bar{\mathbf{x}})^T\mathbf{y} < 0, \quad \nabla\mathbf{g}_I(\bar{\mathbf{x}})^T\mathbf{y} \leqq 0, \quad \nabla\mathbf{h}(\bar{\mathbf{x}})^T\mathbf{y} = 0. \tag{2}$$

The gap between (1) and (2) is caused by the following directions:

$$\nabla\mathbf{f}(\bar{\mathbf{x}})^T\mathbf{y} \leqq 0,\ \nabla f_i(\bar{\mathbf{x}})^T\mathbf{y} = 0 \text{ at least one } i,\ \nabla\mathbf{g}_I(\bar{\mathbf{x}})^T\mathbf{y} \leqq 0,\ \nabla\mathbf{h}(\bar{\mathbf{x}})^T\mathbf{y} = 0. \tag{3}$$

A direction $\mathbf{y}$ that satisfies (3) is called a critical direction.

For the sake of simplicity, we use the following notations:

$$F_i(\mathbf{y},\mathbf{z}) = (\nabla f_i(\bar{\mathbf{x}})^T\mathbf{y},\ \nabla f_i(\bar{\mathbf{x}})^T\mathbf{z} + \nabla^2 f_i(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y}))^T$$

$$G_j(\mathbf{y},\mathbf{z}) = (\nabla g_j(\bar{\mathbf{x}})^T\mathbf{y},\ \nabla g_j(\bar{\mathbf{x}})^T\mathbf{z} + \nabla^2 g_j(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y}))^T$$

$$H_p(\mathbf{y},\mathbf{z}) = (\nabla h_p(\bar{\mathbf{x}})^T\mathbf{y},\ \nabla h_p(\bar{\mathbf{x}})^T\mathbf{z} + \nabla^2 h_p(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y}))^T$$

3. Generalized regularity conditions

To get positive Lagrange multipliers for each of the objective function, Maeda [2] gave the following generalized Guignard regularity conditions (GGRC) for the first-order KKT-type necessary conditions under the set Q^i;

$$\Omega(M;\bar{\mathbf{x}}) \subseteq \bigcap_{i=1}^{l} cl\ convT(Q^i;\bar{\mathbf{x}})\,,$$

where $T(X,\bar{x})$ is Bouligand tangent cone and $\Omega(X;\bar{\mathbf{x}})$ is a first-order linearizing cone.

In this section, we have generalized these conditions for the second-order KKT-type necessary conditions under different sets M^i.

Comparison between Maeda's sets and the proposed sets:

Maeda's sets:

$$Q^i \equiv \{\ \in E_n \big|\ \in X,\ \ f_k(\ ) \leqq f_k(^{-}),\ \ k=1,2,...,l \ \text{ and } \ k \neq i\}$$

Proposed sets:

$$M^i \quad \{\mathbf{x} \quad E_n \big| \mathbf{x} \quad \text{X},\ f_i(\mathbf{x}) \leqq f_i(\bar{\mathbf{x}})\}\,,\ i=1,2,...,l$$

The relationship between the two types of sets is

$$Q^i = \bigcap_{\substack{k=1\\k\neq i}}^{l} M^k,\quad i=1,...,l.$$

For generalizing the Maeda's [2] regularity conditions, we first show that the relationship between the tangent sets $T^2(M^i;\bar{\mathbf{x}})$ and linearzing set $L^2(M;\bar{\mathbf{x}})$.

Lemma 3.1: We assume that $\bar{\mathbf{x}}$ is a feasible solution to problem P then we have

$$\bigcap_{i=1}^{l} T^2(M^i;\bar{\mathbf{x}}) \subseteq L^2(M;\bar{\mathbf{x}})\,.$$

Proof: Let (**y**, **z**) be any element in $T^2(M^i;\bar{\mathbf{x}})$ then there exist $\mathbf{x_n} \in M^i$ and $t_n \to +0$ such that $\mathbf{x_n} = \bar{\mathbf{x}} + t_n\mathbf{y} + \frac{1}{2}t_n^2\mathbf{z} + o(t_n^2)$.

By the Taylor expansion,

$$f_i(\mathbf{x}_n) - f_i(\bar{\mathbf{x}}) = t_n \nabla f_i(\bar{\mathbf{x}})^T\mathbf{y} + \tfrac{1}{2}t_n^2(\nabla f_i(\bar{\mathbf{x}})^T\mathbf{z} + \nabla^2 f_i(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y})) + o(t_n^2)\,,\ i = 1,\ 2,\ \ldots\ ,\ l$$

$$\mathbf{g}_I(\mathbf{x}_n) - \mathbf{g}_I(\bar{\mathbf{x}}) = t_n \nabla \mathbf{g}_I(\bar{\mathbf{x}})^T\mathbf{y} + \tfrac{1}{2}t_n^2(\nabla \mathbf{g}_I(\bar{\mathbf{x}})^T\mathbf{z} + \nabla^2 \mathbf{g}_I(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y})) + o(t_n^2)$$

$$\mathbf{h}(\mathbf{x}_n) - \mathbf{h}(\bar{\mathbf{x}}) = t_n \nabla \mathbf{h}(\bar{\mathbf{x}})^T\mathbf{y} + \tfrac{1}{2}t_n^2(\nabla \mathbf{h}(\bar{\mathbf{x}})^T\mathbf{z} + \nabla^2 \mathbf{h}(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y})) + o(t_n^2)\,.$$

Then for all n we have,

$$f_i(\mathbf{x}_n) = f_i\left(\bar{\mathbf{x}} + t_n\mathbf{y} + \frac{1}{2}t_n^2\mathbf{z}\right) \leqq f_i(\bar{\mathbf{x}})\,,\quad i = 1,2,\ \ldots,\ l \tag{4}$$

[Since $M^i \equiv \{\mathbf{x} \in E_n \mid \mathbf{x} \in X,\ \ f_i(\mathbf{x}) \leqq f_i(\bar{\mathbf{x}})\}$, i = 1, 2, …, l]

$$\mathbf{g}_I(\mathbf{x}_n) = \mathbf{g}_I\left(\bar{\mathbf{x}} + t_n\mathbf{y} + \frac{1}{2}t_n^2\mathbf{z}\right) \leqq 0 = \mathbf{g}_I(\bar{\mathbf{x}}) \tag{5}$$

and $$\mathbf{h}(\mathbf{x}_n) = \mathbf{h}\left(\bar{\mathbf{x}} + t_n\mathbf{y} + \frac{1}{2}t_n^2\mathbf{z}\right) = 0 = \mathbf{h}(\bar{\mathbf{x}}). \tag{6}$$

Now, from (4), we have

$$t_n \nabla f_i(\bar{\mathbf{x}})^T \mathbf{y} + \tfrac{1}{2}t_n^2(\nabla f_i(\bar{\mathbf{x}})^T \mathbf{z} + \nabla^2 f_i(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y})) + o(t_n^2) \leqq 0,\ i = 1, 2, \ldots, l \tag{7}$$

Now, if $\nabla f_i(\bar{\mathbf{x}})^T \mathbf{y} = 0$, i = 1, 2, …, l then, from (7), we have

$$\frac{1}{2}t_n^2\left\{(\nabla f_i(\bar{\mathbf{x}})^T \mathbf{z} + \nabla^2 f_i(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y})) + \frac{2o(t_n^2)}{t_n^2}\right\} \leqq 0$$

as $n \to \infty$, we have

$$\lim_{n\to\infty}\left\{(\nabla f_i(\bar{\mathbf{x}})^T \mathbf{z} + \nabla^2 f_i(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y})) + \frac{2o(t_n^2)}{t_n^2}\right\} = (\nabla f_i(\bar{\mathbf{x}})^T \mathbf{z} + \nabla^2 f_i(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y})) \leqq 0.$$

Also, from (7), we have

$$\frac{1}{2}t_n^2\left\{\frac{2}{t_n}\nabla f_i(\bar{\mathbf{x}})^T \mathbf{y} + (\nabla f_i(\bar{\mathbf{x}})^T \mathbf{z} + \nabla^2 f_i(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y})) + \frac{2o(t_n^2)}{t_n^2}\right\} \leqq 0.$$

Now, if $\nabla f_i(\bar{\mathbf{x}})^T \mathbf{y} < 0$, i = 1, 2, …, l and as $n \to \infty$ we have

$$(\nabla f_i(\bar{\mathbf{x}})^T \mathbf{z} + \nabla^2 f_i(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y})) \leqq 0,$$

i.e. $$(\nabla f_i(\bar{\mathbf{x}})^T \mathbf{y},\ \nabla f_i(\bar{\mathbf{x}})^T \mathbf{z} + \nabla^2 f_i(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y}))^T \leqq_{lex} (0,0)^T,\ i = 1, 2, \ldots, l$$

Similarly, $$(\nabla g_j(\bar{\mathbf{x}})^T \mathbf{y},\ \nabla g_j(\bar{\mathbf{x}})^T \mathbf{z} + \nabla^2 g_j(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y}))^T \leqq_{lex} (0,0)^T,\ j \in I(\bar{\mathbf{x}}),$$

and $$(\nabla h_p(\bar{\mathbf{x}})^T \mathbf{y}, \nabla h_p(\bar{\mathbf{x}})^T \mathbf{z} + \nabla^2 h_p(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y}))^T = (0,0)^{\mathrm{T}}\ \ p = 1, \ldots, k,$$

which implies that $(\mathbf{y},\mathbf{z}) \in L^2(M^i;\bar{\mathbf{x}}) \Rightarrow T^2(M^i;\bar{\mathbf{x}}) \subseteq L^2(M^i;\bar{\mathbf{x}})$, $\forall i$.

Since $L^2(M^i;\bar{\mathbf{x}})$ is closed convex set and i is arbitrary, we have

$$\bigcap_{i=1}^{l} T^2(M^i;\bar{\mathbf{x}}) \subseteq \bigcap_{i=1}^{l} L^2(M^i;\bar{\mathbf{x}}) = L^2(M;\bar{\mathbf{x}}).$$

For closeness of $L^2(M^i;\bar{\mathbf{x}})$ we can also write

$$\bigcap_{i=1}^{l} cl\ conv T^2(M^i;\bar{\mathbf{x}}) \subseteq L^2(M;\bar{\mathbf{x}})$$

where $cl\ conv T^2(M^i;\bar{\mathbf{x}})$ denotes the closure of convex hull of $T^2(M^i;\bar{\mathbf{x}})$.

Remark: 3.1. In general, the converse inclusion in lemma 3.1 does not hold. So to obtain the necessary conditions for a feasible point to problem P be a local vector minimum point, it is reasonable to assume that

$$L^2(M;\bar{\mathbf{x}}) \subseteq \bigcap_{i=1}^{l} T^2(M^i;\bar{\mathbf{x}}) \tag{8}$$

and

$$L^2(M;\bar{\mathbf{x}}) \subseteq \bigcap_{i=1}^{l} cl\ conv T^2(M^i;\bar{\mathbf{x}}) . \tag{9}$$

Conditions (8) and (9) are considered, respectively, as a generalized Abadie second-order regularity condition (GASORC) and generalized Guignard second-order regularity condition (GGSORC).

For checking optimality along the corresponding curves, we achieve the impossibility of a family of nonhomogeneous system as necessary optimality conditions. These systems depend upon a descent direction for f at the considered optimal point $\bar{\mathbf{x}} \in X$ and involve only the components of f, for which this direction is stationary at $\bar{\mathbf{x}}$; therefore, given any direction $\mathbf{y} \in E_n$, let $P(\mathbf{y}) = \{i \in \{1,\dots,l\}: \nabla f_i(\bar{\mathbf{x}})^T \mathbf{y} = 0\}$.

Now, we are in a position to state the primal form of our second-order necessary conditions.

Theorem 3.1. Let $\bar{\mathbf{x}}$ be a local vector minimum point to Problem P, and assume that the second-order (GASORC) holds at $\bar{\mathbf{x}} \in X$. Then, the following system has no solution ($\mathbf{y}$, $\mathbf{z}$):

$$F_i(\mathbf{y},\mathbf{z}) \leqq_{lex} 0 \quad \forall i, \tag{10}$$

$$F_i(\mathbf{y},\mathbf{z}) <_{lex} 0 \quad \text{at least one } i \text{ where } i \in P(\mathbf{y}) \tag{11}$$

$$G_j(\mathbf{y},\mathbf{z}) \leqq_{lex} 0 \quad \forall j \in I(\bar{\mathbf{x}}), \tag{12}$$

$$H_p(\mathbf{y},\mathbf{z}) = 0. \quad \forall p \tag{13}$$

where $P(\mathbf{y}) = \{i \in \{1,\dots,l\}: \nabla f_i(\bar{\mathbf{x}})^T \ \mathbf{y} = 0\}$.

Proof: Let ($\mathbf{y}$, $\mathbf{z}$) be the solution of (10)–(13). Then, $(\mathbf{y},\mathbf{z}) \in L^2(M;\bar{\mathbf{x}})$.

By the assumption of GASORC we have, $(\mathbf{y},\mathbf{z}) \in T^2(M^i;\bar{\mathbf{x}})$. Then, there exist $\mathbf{x}_n \in M^i$ and $t_n \to +0$ such that $\mathbf{x}_n = \bar{\mathbf{x}} + t_n\mathbf{y} + \frac{1}{2}t_n^2\mathbf{z} + o(t_n^2)$.

By the Taylor expansion,

$$f_i(\mathbf{x}_n) - f_i(\bar{\mathbf{x}}) = t_n \nabla f_i(\bar{\mathbf{x}})^T \mathbf{y} + \tfrac{1}{2}t_n^2(\nabla f_i(\bar{\mathbf{x}})^T \mathbf{z} + \nabla^2 f_i(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y})) + o(t_n^2).$$

Then for all n we have

$$f_i(\mathbf{x}_n) = f_i(\bar{\mathbf{x}} + t_n\mathbf{y} + \tfrac{1}{2}t_n^2\mathbf{z}) \leqq f_i(\bar{\mathbf{x}}) , \quad \forall i. \tag{14}$$

From (14), we have

$$t_n \nabla f_i(\bar{\mathbf{x}})^T \mathbf{y} + \tfrac{1}{2}t_n^2(\nabla f_i(\bar{\mathbf{x}})^T \mathbf{z} + \nabla^2 f_i(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y})) + o(t_n^2) \leqq 0 , \ \forall i .$$

If

$$i \in P(\mathbf{y}), \text{ then } \nabla f_i(\bar{\mathbf{x}})^T \mathbf{z} + \nabla^2 f_i(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y}) \leqq 0. \tag{15}$$

Hence, $(\mathbf{y},\mathbf{z})\in T^2(M^i;\bar{\mathbf{x}}) \Rightarrow F_i(\mathbf{y},\mathbf{z}) \leqq_{\text{lex}} 0,\ \forall i\in P(\mathbf{y})$.

Since T^2 is closed and isotone and $\bigcap_{i=1}^{l} M^i \subseteq X$, we can write

$$\bigcap_{i=1}^{l} T^2(M^i;\bar{\mathbf{x}}) \subseteq T^2(X;\bar{\mathbf{x}}). \Rightarrow (\mathbf{y},\mathbf{z})\in T^2(X;\bar{\mathbf{x}}).$$

Then, there exist $\mathbf{x}_p \in X$ and $t_p \to +0$ such that $\mathbf{x}_p = \bar{\mathbf{x}} + t_p\mathbf{y} + \frac{1}{2}t_p^2\mathbf{z} + o(t_p^2)$.

Since $\bar{\mathbf{x}}$ be a local vector minimum point to problem *P*, there is no point $\mathbf{x}_p \in X$, where $\mathbf{f}(\mathbf{x}_p) < \mathbf{f}(\bar{\mathbf{x}})$.

i.e. $(t_p\nabla f_i(\bar{\mathbf{x}})^T\mathbf{y} + \frac{1}{2}t_p^2(\nabla f_i(\bar{\mathbf{x}})^T\mathbf{z} + \nabla^2 f_i(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y})) + o(t_p^2)) < \mathbf{0}$, $i = 1, 2, \ldots, l$.

Now, if $i\in P(\mathbf{y})$ and as $p \to \infty$ then, we have,

$$(\nabla f_i(\bar{\mathbf{x}})^T\mathbf{z} + \nabla^2 f_i(\bar{\mathbf{x}})(\mathbf{y},\mathbf{y})) < \mathbf{0},\quad \forall i\in P(\mathbf{y}). \tag{16}$$

It means that if $\bar{\mathbf{x}}$ is a local vector minimum point, then we do not get (16), which is a contradiction with (11). So the system has no solution.

In order to obtain KKT-type necessary conditions, we need the following nonhomogeneous form of the Tuckers theorem of the alternative [6].

Lemma 3.2: Let A, B and C be $m\times n$, $p\times n$ and $q\times n$ real matrices and $b_1 \in E_m$, $b_2 \in E_p$ and $b_3 \in E_q$ be real vectors. Then either

$$\left\langle \begin{array}{l} \mathbf{Ax}+\mathbf{b_1} \le \mathbf{0} \\ \mathbf{Bx}+\mathbf{b_2} \leqq \mathbf{0} \\ \mathbf{Cx}+\mathbf{b_3} = \mathbf{0} \end{array} \right\rangle \text{ has a solution } \mathbf{x}\in E_n.$$

or

$$\left\langle \begin{array}{l} \mathbf{A^T y_1} + \mathbf{B^T y_2} + \mathbf{C^T y_3} = \mathbf{0} \\ \mathbf{b_1^T y_1} + \mathbf{b_2^T y_2} + \mathbf{b_3^T y_3} \geqq \mathbf{0} \\ \mathbf{y_1} > \mathbf{0},\quad \mathbf{y_2} \geqq \mathbf{0} \end{array} \right\rangle \text{ has a solution } \mathbf{y}_1, \mathbf{y}_2, \text{ and } \mathbf{y}_3,$$

but never both.

The proof is identical with [3,6].

Remark 3.1: To get the positive Lagrange multipliers $w_i > 0$, $i\in P(\mathbf{y})$ in [5], Bigi and Castellani gave the lemma 2.1. Using this lemma, they tried to prove their theorem 5.5 for getting w_i positive, but it is not possible, because the lemma provides semi-positive $\mathbf{w}$, i.e. ($\mathbf{w} \geq 0$, $\mathbf{w} \neq 0$). In a recent paper [3], they establish theorem 3.2 by using SMFRC conditions. They restrict $\mathbf{w}$ by $\|\mathbf{w}\| = 1$, but $\|\mathbf{w}\| = 1$ does not necessarily imply that all the components of $\mathbf{w}$ are not equal to zero.

Applying Lemma 3.2 and Theorem 3.1, we have deduced the following KKT-type necessary conditions, which ensure the existence of positive Lagrange multipliers of the objective functions. Here, we consider those components of *f*, for which the direction $\mathbf{y}\in E_n$ is stationary at $\bar{\mathbf{x}}$; i.e. $P(\mathbf{y}) = \{i\in\{1,\ldots,l\}: \nabla f_i(\bar{\mathbf{x}})^T\mathbf{y} = 0\}$.

Theorem 3.2: Let $\bar{\mathbf{x}}$ satisfy the assumptions made in Theorem 3.1. Then, for each critical direction $\mathbf{y}$, there exist multipliers $\mathbf{w} \in E_l$, $\mathbf{u} \in E_m$ and $\mathbf{v} \in E_k$ such that

$$\sum_{i=1}^{l} w_i \nabla f_i(\bar{\mathbf{x}}) + \sum_{j=1}^{m} u_j \nabla g_j(\bar{\mathbf{x}}) + \sum_{p=1}^{k} v_p \nabla h_p(\bar{\mathbf{x}}) = 0$$

$$\left(\sum_{i=1}^{l} w_i \nabla^2 f_i(\bar{\mathbf{x}}) + \sum_{j=1}^{m} u_j \nabla^2 g_j(\bar{\mathbf{x}}) + \sum_{p=1}^{k} v_p \nabla^2 h_p(\bar{\mathbf{x}}) \right)(\mathbf{y}, \mathbf{y}) \geqq 0$$

$w_i > 0 \;\; i \in P(\mathbf{y})$, $w_i = 0$ for all $i \notin P(\mathbf{y})$, $u_j \geqq 0 \;\; j \in I(\mathbf{y})$, $u_j = 0$ for all $j \notin I(\mathbf{y})$

$$P(\mathbf{y}) = \{ i \in \{1, \dots, l\} : \nabla f_{\mathrm{i}}(\bar{\mathbf{x}})^T \mathbf{y} = 0 \}, \quad I(\mathbf{y}) = \{ j \in \{1, \dots, m\} : g_j(\bar{\mathbf{x}}) = 0, \; \nabla \mathrm{g}_j(\bar{\mathbf{x}})^T \mathbf{y} = 0 \}$$

Proof: Let $\mathbf{y}$ be a critical direction. Then, the system

$$\nabla \mathbf{f}_{P(y)}(\bar{\mathbf{x}})^T \mathbf{z} + \nabla^2 \mathbf{f}_{P(y)}(\bar{\mathbf{x}})(\mathbf{y}, \; \mathbf{y}) \leq 0$$

$$\nabla \mathbf{g}_{I(y)}(\bar{\mathbf{x}})^T \mathbf{z} + \nabla^2 \mathbf{g}_{I(y)}(\bar{\mathbf{x}})(\mathbf{y}, \; \mathbf{y}) \leqq 0$$

$$\nabla \mathbf{h}(\bar{\mathbf{x}})^T \mathbf{z} + \nabla^2 \mathbf{h}(\bar{\mathbf{x}})(\mathbf{y}, \; \mathbf{y}) = 0$$

has no solution $\mathbf{z}$.

By lemma 3.2, there exist multipliers $\mathbf{w} \in E_l$, $\mathbf{u} \in E_m$ and $\mathbf{v} \in E_k$ such that,

$$\sum_{i=1}^{l} w_i \nabla f_i(\bar{\mathbf{x}}) + \sum_{j=1}^{m} u_j \nabla g_j(\bar{\mathbf{x}}) + \sum_{p=1}^{k} v_p \nabla h_p(\bar{\mathbf{x}}) = 0$$

$$\left(\sum_{i=1}^{l} w_i \nabla^2 f_i(\bar{\mathbf{x}}) + \sum_{j=1}^{m} u_j \nabla^2 g_j(\bar{\mathbf{x}}) + \sum_{p=1}^{k} v_p \nabla^2 h_p(\bar{\mathbf{x}}) \right)(\mathbf{y}, \mathbf{y}) \geqq 0$$

$w_i > 0 \;\; i \in P(\mathbf{y})$, $w_i = 0$ for all $i \notin P(\mathbf{y})$, $u_j \geqq 0 \;\; j \in I(\mathbf{y})$, $u_j = 0$ for all $j \notin I(\mathbf{y})$.

This completes the proof.

Since GASORC $\Rightarrow$ GGSORC, theorems 3.1 and 3.2 hold for GGSORC also.

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