J. Indian Inst. Sci., Nov.–Dec. 2006, **86**, 773–783
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Riemann–Stieltjes operators from Dirichlet-type space to Bloch-type space

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Received on March 7, 2006; Revised on October 29, 2006.

Abstract

Let g: $B \to \mathbb{C}^1$ be a holomorphic map of the unit ball in $\mathbb{C}^n$. We study the following operators

$$T_g f(z) = \int_0^1 f(tz)\Re g(tz)\frac{\mathrm{d}t}{t}, \quad L_g f(z) = \int_0^1 \Re f(tz) g(tz)\frac{\mathrm{d}t}{t}, \quad z \in B.$$

The boundedness and compactness of T_g and L_g from Dirichlet type spaces to Bloch type spaces on the unit ball of $\mathbb{C}^n$ are discussed in this paper. We also estimate the norm of T_g and L_g.

Keywords: Riemann–Stieltjes operator, Bloch-type space, Dirichlet-type space.

1. Introduction

Let $B = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball of $\mathbb{C}^n$. dv denotes the normalized Lebesgue measure of B, i.e. $v(B) = 1$. We denote the class of all holomorphic functions on B by $H(B)$ and the class of all bounded holomorphic functions on B by H^∞. Let $\Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_i}(z)$ stand for the radial derivative of $f \in H(B)$ (see [1]). It is easy to see that, if $f \in H(B)$, $f(z) = \sum_\alpha a_\alpha z^\alpha$, then

$$\Re f(z) = \sum_\alpha |\alpha| a_\alpha z^\alpha,$$

where $\alpha = (\alpha_1,\dots, \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n$.

The Bloch-type space $\mathcal{B}^q(B) = \mathcal{B}^q$, $q > 0$, is the space of all $f \in H(B)$ such that (see [2])

$$b_q(f) = \sup_{z\in B}(1 - |z|^2)^q |\Re f(z)| < \infty.$$

The little Bloch-type space $\mathcal{B}_0^q = \mathcal{B}_0^q(B)$ consists of all $f \in H(B)$ such that $\lim_{|z|\to 1}(1 - |z|^2)^q |\Re f(z)| = 0$. On $\mathcal{B}^q$ the norm is introduced by $\|f\|_{\mathcal{B}^q} = |f(0)| + b_q(f)$. With this norm $\mathcal{B}^q$ is a Banach space and $\mathcal{B}_0^q$ is a closed subspace of $\mathcal{B}^q$.

Let $p \in \mathbb{R}$ and $f \in H(B)$ with Taylor expansion $f(z) = \sum_{|\alpha|\ge 0} b_\alpha z^\alpha$. Recall that the Dirichlet-type spaces D_p consists of all holomorphic functions such that

$$\| f \|_{D_p}^2 = \sum_{|\alpha|\geq 0} (n+|\alpha|)^p |b_\alpha|^2 \omega_\alpha < \infty,$$

where

$$\omega_\alpha = \int_{\partial B} |\xi^\alpha|^2 \mathrm{d}\sigma(\xi) = \frac{(n-1)!\alpha!}{(n+|\alpha|-1)!}, \quad \alpha! = \alpha_1!\ldots\alpha_n!, \quad z^\alpha = z_1^{\alpha_1}\ldots z_n^{\alpha_n}.$$

It is clear that D_p is a Hilbert space with the inner product $\langle f, g\rangle$ defined by

$$\langle f, g\rangle = \sum_{|\alpha|\geq 0} (n+|\alpha|)^p \omega_\alpha a^\alpha \overline{b}_\alpha,$$

where $f(z) = \sum_{|\alpha|\geq 0} a_\alpha z^\alpha$, $g(z) = \sum_{|\alpha|\geq 0} b_\alpha z^\alpha$. The spaces D_0 and D_{-1} are just the Hardy space H^2 and the Bergman space A^2, respectively. D_n is the Dirichlet space. The Dirichlet-type space D_p has been characterized by many authors (see [3–6]).

Let g: $B \to \mathbb{C}^1$ be a holomorphic map. For $f \in H(B)$, the operator T_g is defined by

$$T_g f(z) = \int_0^1 f(tz) \frac{\mathrm{d}g(tz)}{dt} = \int_0^1 f(tz) \Re g(tz) \frac{\mathrm{d}t}{t}, \quad z \in B, \tag{1}$$

The operator T_g was called the Riemann–Stieltjes operator (see [7]) or extended Cesàro operator (see [8]), which has been studied by many authors (see, for example [7–12]).

Similarly, we define another operator L_g as following:

$$L_g f(z) = \int_0^1 \Re f(tz) g(tz) \frac{\mathrm{d}t}{t}, \quad z \in B. \tag{2}$$

In this paper, we study the boundedness and compactness of T_g and L_g from the Dirichlet-type space into the Bloch-type space on the unit ball of $\mathbb{C}^n$.

Constants are denoted by C in this paper, they are positive and may differ from one occurrence to the other. $a \preceq b$ means that there is a positive constant C such that $a \leq Cb$. Moreover, if both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2. Main result and proof

In this section, we give our main results and proofs. To this end, we need some auxiliary results which are incorporated in the following lemmas.

Lemma 1. *For every f, $g \in H(B)$ it holds*

$$\Re[T_g(f)](z) = f(z)\Re g(z), \quad \Re[L_g(f)](z) = \Re f(z) g(z).$$

Proof. The first equality was proved in [8]. The proof of the second equality is similar, so we omit the details.

The following two lemmas can be found in [13].

Lemma 2. *Let* $w \in B$, $|w| > 1/2$,

$$f_w(z) = \sum_{|\alpha|>0} \frac{|\alpha|^{|\alpha|-1/2}}{\alpha! e^{|\alpha|}} \overline{w}^{\alpha} z^{\alpha}. \tag{3}$$

Then (*a*)

$$\|f_w\|_{D^p} \asymp \begin{cases} (1-|w|^{2-\frac{p-n}{2}} & : \ p > n; \\ (\log \frac{1}{1-|w|^2})^{1/2} & : \ p = n; \\ O(1) & : \ p < n. \end{cases}$$

(*b*)

$$f_w(w) \asymp \log(1-|\mathrm{w}|^2)^{-1} \quad \textit{and} \quad (\Re f_w)(w) \asymp (1-|w|^2)^{-1}. \tag{4}$$

Lemma 3. *Let* $f \in D_p$, $½ < |z| < 1$, $i = 0, 1$. *Then*

$$|\Re^{(i)}f(z)| \leq \begin{cases} C\|f\|_{D_p} (1-|z|^2)^{-(n-p+2i)/2} & : \ p < n+2i; \\ C\|f\|_{D_p} \{\log(1-|z|^2)^{-1}\}^{1/2} & : \ p = n+2i; \\ C\|f\|_{D_p} & : \ p > n+2i. \end{cases}$$

Lemma 4. *Let* g: $B \to \mathbb{C}$ *be a holomorphic map. Then* $T_g(L_g)$: $D_p \to \mathcal{B}^q$ *is compact if and only if* $T_g(L_g)$: $D_p \to \mathcal{B}^q$ *is bounded and for any bounded sequence* $(f_k)_{k \in \mathrm{N}}$ *in* D_p *which converges to zero uniformly on compact subsets of B, we have* $\|T_g f_k\|_{\mathcal{B}^q}$ $(\|L_g f_k\|_{\mathcal{B}^q}) \to 0$, *as* $k \to \infty$.

Proof. We only prove the first case. Assume that T_g is compact and $(f_k)_{k \in \mathrm{N}}$ is a sequence in D_p with $\sup_{k \in \mathrm{N}} \|f_k\|_{D^p} < \infty$ and $f_k \to 0$ uniformly on compact subsets of B, as $k \to \infty$. By the compactness of T_g we have that $T_g(f_k)$ has a subsequence $T_g(f_{k_m})$ which converges in $\mathcal{B}^q$, say, to f. Hence we have that for any compact $K \subset B$, there is a positive constant C_K independent of f such that

$$|T_g(f_{k_m})(z) - f(z)| \leq C_K \|T_g(f_{k_m}) - f\|_{\mathcal{B}^q}, \text{ for all } z \in K.$$

This implies that $T_g(f_{k_m})(z) - f(z) \to 0$ uniformly on compact subsets of B: Since $f_{k_m} \to 0$ on compacts, by the definition of T_g, it is easy to see that for each $z \in B$, $\lim_{m\to\infty} T_g(f_{k_m})(z) = 0$. Hence the limit function f is equal to 0. Since this is true for arbitrary subsequence of (f_k), we see that $T_g(f_k) \to 0$ in $\mathcal{B}^q$, as $k \to \infty$.

Conversely, let $(h_k)_{k \in \mathrm{N}}$ be any sequence in the ball $\mathcal{K}_M = B_{D_p}(0, M)$ of the space D_p. Since $\|h_k\|_{D_p} \leq M < \infty$, by Lemma 3 $(h_k)_{k \in \mathrm{N}}$ is uniformly bounded on compact subsets of B and hence normal by Montel's theorem. Hence we may extract a subsequence $(h_{k_j})_{j \in \mathrm{N}}$ which converges uniformly on compact subsets of B to some $h \in H(B)$. It follows that $\frac{\partial h_{k_j}}{\partial z_l} \to \frac{\partial h}{\partial z_l}$ uniformly on compacts, for each $l \in \{1,\ldots, n\}$, it follows that $\frac{\partial h_{k_j}}{\partial z_l}(z) \to \frac{\partial h}{\partial z_l}(z)$ for each $z \in B$ and each $l \in \{1,\ldots, n\}$. Choosing a positive integer k such that $p < 2k$, from Lemma 4 of [3], we know that $f \in D_p$ if and only if

$$\int_B |\Re^{(k)} f(z)|^2 (1-|z|^2)^{2k-1-p} dv(z) < \infty.$$

Now, apply the Fatou lemma, and obtain that

$$\int_B \liminf_{j\to\infty} |\Re^{(k)} h_{k_j}(z)|^p (1-|z|^2)^{2k-1-p} dv(z) \le \liminf_{j\to\infty} \int_B |\Re^{(k)} h_{k_j}(z)|^p (1-|z|^2)^{2k-1-p} dv(z),$$

that is

$$\int_B |\Re^{(k)} h(z)|^p (1-|z|^2)^{2k-1-p} dv(z) \le \liminf_{j\to\infty} \int_B |\Re^{(k)} h_{k_j}(z)|^p (1-|z|^2)^{2k-1-p} dv(z)$$
$$\le \liminf_{j\to\infty} \| h_{k_j} \|_{D_p}^p \le M^p.$$

From the above statements and Lemma 4 of [3], we see that $h \in D_p$ and $||h||_{D_p} \le M$. Hence the sequence $(h_{k_j} - h)_{j \in \mathrm{N}}$ is such that $||h_{k_j} - h||_{D_p} \le 2M < \infty$, and converges to 0 on compact subsets of B. By the hypothesis we have that $T_g(h_{k_j}) \to T_g(h)$ in $\mathcal{B}^q$. Thus the set $T_g(\mathcal{K}_M)$ is relatively compact, finishing the proof.

Now, we are in a position to formulate and prove the main results of this paper.

Theorem 1. *Let* g: $B \to \mathbb{C}$ *be a holomorphic map and* $q \ge 1$. *Then* T_g: $D_n \to \mathcal{B}^q$ *is bounded if and only if*

$$I = \sup_{z\in B}(1-|z|^2)^q \left(\log \frac{1}{1-|z|^2} \right)^{1/2} |\Re g(z)| < \infty. \tag{5}$$

Moreover, the following relationship holds.

$$||T_g|| \asymp \sup_{z\in B}(1-|z|^2)^q \left(\log \frac{1}{1-|z|^2} \right)^{1/2} |\Re g(z)|. \tag{6}$$

Proof. Assume (5). Let $f \in D_n$. Note that $T_g f(0) = 0$, by Lemma 3 we have

$$\| T_g f \|_{\mathcal{B}^q} = \sup_{z\in B}(1-|z|^2)^q |\Re(T_g f(z)| = \sup_{z\in B}(1-|z|^2)^q |f(z)||\Re g(z)|$$

$$\le C \| f \|_{D_p} \sup_{z\in B}(1-|z|^2)^q \left(\log \frac{1}{1-|z|^2} \right)^{1/2} |\Re g(z)|. \tag{7}$$

Therefore (5) implies that T_g is a bounded operator from D_n to $\mathcal{B}^q$.

Conversely, suppose T_g is bounded from D_n to $\mathcal{B}^q$. For $w \in B$, $|w| > 1/2$, set

$$f_w(z) = \log \frac{1}{1-\langle z, w\rangle}. \tag{8}$$

After some computations, we see that $f_w \in D_n$ and $\| f_w \|^2_{D_n} \le C \log \frac{1}{1-|w|^2}$.

Hence,

$$(1-|z|^2)^q \, |f_w(z)\Re g(z)| \le \| T_g f_w \|_{\mathcal{B}^q} \le \| T_g \| \| f_w \|_{D_n} \le C \| T_g \| \left(\log \frac{1}{1-|w|^2} \right)^{1/2}.$$

Put $z = w$, we have

$$(1-|w|^2)^q \, |f_w(w)| |\Re g(w)| \le C \| T_g \| \left(\log \frac{1}{1-|w|^2} \right)^{1/2},$$

i.e. we get

$$\sup_{|w|>1/2} (1-|w|^2)^q \left(\log \frac{1}{1-|w|^2} \right)^{1/2} |\Re g(w)| \le C \| T_g \|. \tag{9}$$

By the Maximum modulus principle there is a positive constant C independent of $g \in H(B)$ such that

$$\sup_{|w|\le 1/2} (1-|w|^2)^q \left(\log \frac{1}{1-|w|^2} \right)^{1/2} |\Re g(w)| \le C \sup_{|w|>1/2} (1-|w|^2)^q \left(\log \frac{1}{1-|w|^2} \right)^{1/2} |\Re g(w)|, \tag{10}$$

from which the result follows. From (8), (9) and the above statements, (6) follows. □

Theorem 2. *Let* $g: B \to \mathbb{C}$ *be a holomorphic map and* $q \ge 1$. *Then* $L_g: D_n \to \mathcal{B}^q$ *is bounded if and only if*

$$\sup_{z\in B} (1-|z|^2)^{q-1} |g(z)| < \infty. \tag{11}$$

Moreover, the following relationship holds:

$$\|L_g\| \asymp \sup_{z\in B} (1-|z|^2)^{q-1} |g(z)|. \tag{12}$$

Proof. Suppose that (11) holds. Noticing that $L_g f(0) = 0$, by Lemma 3, we get

$$\| L_g f \|_{\mathcal{B}^q} = \sup_{z\in B} (1-|z|^2)^q \, |\Re(L_g f)(z)| \le C \| f \|_{D_p} \sup_{z\in B} (1-|z|^2)^{q-1} |g(z)|. \tag{13}$$

From this, (11) implies that $L_g: D_n \to \mathcal{B}^q$ is bounded.

Conversely, suppose that $L_g: D_n \to \mathcal{B}^q$ is bounded. Let $\beta(z, w)$ denote the Bergman metric between two points z and w in B. It is well known that

$$\beta(z,w) = \frac{1}{2} \log \frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}.$$

Here $\varphi_z(w)$ is the automorphisms of B that interchanges 0 and z (see [2]). For $a \in B$ and $r > 0$ the set

$$D(a,r) = \{z \in B : \beta(a,z) < r\}, \quad a \in B$$

is a Bergman metric ball at a with radius r. It is well known that (see [2])

$$\frac{(1-|a|^2)^{n+1}}{|1-\langle a,z\rangle|^{2(n+1)}} \asymp \frac{1}{(1-|z|^2)^{n+1}} \asymp \frac{1}{(1-|a|^2)^{n+1}} \asymp \frac{1}{|D(a,r)|} \tag{14}$$

when $z \in D(a, r)$. For $w \in B$, $|w| > 1/2$, let $f_w(z)$ be defined by (8). By the subharmonicity of $|\Re f_w|^2$ and $|\Re f_w(w)| = |w|^2/(1 - |w|^2)$, we get

$$(1-|w|^2)^{-2}|g(w)|^2 \asymp |\Re f_w(w)g(w)|^2 \le \frac{C}{(1-|w|^2)^{n+1}} \int_{D(w,r)} |\Re f_w(z)|^2 |g(z)|^2 \, dv(z)$$

$$\le \frac{C}{(1-|w|^2)^{n+1}} \int_{D(w,r)} |\Re f_w(z)|^2 |g(z)|^2 \, (1-|z|^2)^{2q} \frac{1}{(1-|z|^2)^{2q}} dv(z)$$

$$\le C \int_{D(w,r)} \frac{dv(z)}{(1-|z|^2)^{2q+n+1}} \sup_{z\in D(w,r)} (1-|z|^2)^{2q} |\Re f_w(z)|^2 |g(z)|^2 \le \frac{C}{(1-|w|^2)^{2q}} \| L_g f_w \|_{\mathcal{B}^q}^2,$$

i.e.

$$(1-|w|^2)^{2q-2} |g(w)|^2 \le C \| L_g f_w \|_{\mathcal{B}^q}^2.$$

Hence

$$\sup_{|w|>1/2} (1-|w|^2)^{q-1} |g(w)| \le C \| L_g f_w \|_{\mathcal{B}^q}. \tag{15}$$

By the Maximum modulus principle we get the desired result. From (13) and (15), (12) follows. □

Theorem 3. *Let* g: $B \to \mathbb{C}$ *be a holomorphic map and* $q \ge 1$. *Then* T_g: $D_n \to \mathcal{B}^q$ *is compact if and only if*

$$\lim_{|z|\to 1} (1-|z|^2)^q \left(\log \frac{1}{1-|z|^2} \right)^{1/2} |\Re g(z)| = 0. \tag{16}$$

Proof. First we assume that (16) holds. Let $\{f_k\}$ be a sequence in D_n with $\sup_{k\in\mathbb{N}} \|f_k\|_{D_n} \le K$ and $f_k \to 0$ uniformly on compact subsets of B, as $k \to \infty$. By the assumption, for any $\varepsilon > 0$, there is a constant δ, $0 < \delta < 1$, such that

$$(1-|z|^2)^q \left(\log \frac{1}{1-|z|^2} \right)^{1/2} |\Re g(z)| < \varepsilon / 2$$

whenever $\delta < |z| < 1$. Let $E = \{z \in B: |z| \le \delta\}$. Note that E is a compact subsect of B and using Lemma 3, we have

$$\begin{aligned}\| T_g f \|_{\mathcal{B}^q} &= \sup_{z\in B}(1-|z|^2)^q |\Re(T_g f)(z)| \\ &= \sup_{z\in E}(1-|z|^2)^q |\Re g(z) f_k(z)| + \sup_{z\in B\setminus E}(1-|z|^2)^q |\Re g(z) f_k(z)| \\ &\le M\left(\log\frac{1}{1-\delta^2}\right)^{-1/2} \sup_{z\in E}|f_k(z)| + CK\varepsilon/2,\end{aligned}$$

where

$$M = \sup_{z\in B}(1-|z|^2)^q\left(\log\frac{1}{1-|z|^2}\right)^{1/2} |\Re g(z)|.$$

By the assumption we obtain $\limsup_{k\to\infty} \|T_g f_k\|_{\mathcal{B}^q} \le CK\varepsilon/2$. Since ε is an arbitrary positive number we have that $\lim_{k\to\infty} \|T_g f_k\|_{\mathcal{B}^q} = 0$. By Lemma 4 we know that T_g: $D_n \to \mathcal{B}^q$ is compact.

Conversely, suppose T_g: $D_n \to \mathcal{B}^q$ is compact. Let $\{z_k\}$ be a sequence in B such that $|z_k| \to 1$ as $k \to \infty$. Take

$$f_k(z) = \left(\log\frac{1}{1-\langle z, z_k\rangle}\right)^2\left(\log\frac{1}{1-|z_k|^2}\right)^{-1}, \quad k \in \mathrm{N}. \tag{17}$$

Then $f_k \in D_n$ and f_k converges to 0 uniformly on compact subsets of B. Since T_g is compact, we have

$$\| T_g f_k \|_{\mathcal{B}^q} \to 0 \quad \text{as} \quad k \to \infty.$$

Thus

$$(1-|z_k|^2)^q \log\frac{1}{1-|z_k|^2}|\Re g(z_k)| \le \sup_{z\in B}(1-|z|^2)^q |\Re(T_g f_k)(z)| = \| T_g f_k \|_{\mathcal{B}^q} \to 0,$$

as $k \to \infty$. From which we get the desired result. □

Theorem 4. *Let g: $B \to \mathbb{C}$ be a holomorphic map and $q \ge 1$. Then L_g: $D_n \to \mathcal{B}^q$ is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)^{q-1}|g(z)| = 0. \tag{18}$$

Proof. Assume (18). Let $\{f_k\}$ be a sequence in D_n with $\sup_{k\in\mathbb{N}} \|f_k\|_{D_n} \le K$ and $f_k \to 0$ uniformly on compact subsets of B, as $k \to \infty$. By the assumption, for any $\varepsilon > 0$, there is a constant δ, $0 < \delta < 1$, such that $\delta < |z| < 1$ implies

$$(1-|z|^2)^{q-1}|g(z)| < \varepsilon/2$$

whenever $\delta < |z| < 1$. Using Lemma 3, we have

$$\begin{aligned}\| L_g f_k \|_{\mathcal{B}^q} &= \sup_{z\in B}(1-|z|^2)^q |\Re(L_g f_k)(z)| \\ &= \sup_{z\in E}(1-|z|^2)^q |g(z)\Re f_k(z)| + \sup_{z\in B\setminus E}(1-|z|^2)^q |g(z)\Re f_k(z)| \\ &\le N \sup_{z\in E} |1-|z|^2| \Re f_k(z)| + CK\varepsilon/2,\end{aligned}$$

where $N = \sup_{z \in B} (1 - |z|^2)^{q-1}|g(z)|$. By the Cauchy's estimate the condition $f_k \to 0$ as $k \to \infty$ uniformly on compact subsets of B implies that $\Re f_k \to 0$ as $k \to \infty$ uniformly on compact subsets of B. Hence, we have $\|L_g f_k\|_{\mathcal{B}^q} \to 0$ as $k \to \infty$. Therefore, L_g: $D_n \to \mathcal{B}^q$ is compact by Lemma 4.

Conversely, suppose that L_g: $D_n \to \mathcal{B}^\alpha$ is compact. Let $\{z_k\}$ be a sequence in B such that $|z_k| \to 1$ as $k \to \infty$. Let $f_k(k \in \mathrm{N})$ be defined by (17). Then $f_k \in D_n$ and f_n converges to 0 uniformly on compact subsets of B. Since L_g is compact, we have

$$(1-|z_k|^2)^{q-1} |g(z_k)| \preceq \sup_{z\in B}(1-|z|^2)^q |\Re(L_g f_k)(z)| = \| L_g f_k \|_{\mathcal{B}^q} \to 0$$

as $k \to \infty$. From which we get the desired result. □

Theorem 5. *Let $g: B \to \mathbb{C}$ be a holomorphic map and $n + 2 - 2q < p < 0$. Then T_g: $D_p \to \mathcal{B}^q$ is bounded if and only if $g \in \mathcal{B}^{q-\frac{n-p}{2}}$. Moreover, the following relationship holds:*

$$\| T_g \| \asymp \sup_{z\in B}(1-|z|^2)^{q-\frac{n-p}{2}} |\Re g(z)|. \tag{19}$$

Proof. Let $f \in D_p$. By Lemma 3,

$$\| T_g f \|_{\mathcal{B}^q} = \sup_{z\in B} |f(z)| (1-|z|^2)^q |\Re g(z)| \le C \| f \|_{D_p} \sup_{z\in B}(1-|z|^2)^{q-\frac{n-p}{2}} |\Re g(z)|. \tag{20}$$

Therefore, $g \in \mathcal{B}^{q-\frac{n-p}{2}}$ implies that T_g: $D_p \to \mathcal{B}^q$ is bounded.

Conversely, assume that T_g: $D_p \to \mathcal{B}^q$ is bounded. For any $w \in B$, $|w| > 1/2$, set

$$f_w(z) = \frac{1-|w|^2}{(1-\langle z, w\rangle)^{1+\frac{n-p}{2}}}. \tag{21}$$

From [12], we see that $f_w \in D_p$ and $\|f_w\|_{D_p} \le C$. Therefore, the boundedness of T_g: $D_p \to \mathcal{B}^q$ implies that

$$(1 - |z|^2)^q|f_w(z)\Re g(z)| \le \|T_g f_w\|_{\mathcal{B}^q} \le C\|T_g\| \, \|f_w\|_{D_p} \le C\|T_g\|.$$

Put $z = w$ in the above inequality, we have

$$(1-|w|^2)^{q-\frac{n-p}{2}} |\Re g(w)| \le C \| T_g \|. \tag{22}$$

From the above inequality and using the maximum modulus principle we get the desired results. From (20) and (22) we get (19). □

Theorem 6. *Let $g : B \to \mathbb{C}$ be a holomorphic map and $n + 2 - 2q < p < 0$. Then $L_g : D_p \to \mathcal{B}^q$ is bounded if and only if*

$$\sup_{z\in B}(1-|z|^2)^{q-\frac{n-p+2}{2}}|g(z)|<\infty. \tag{23}$$

Moreover, the following relationship holds:

$$\|L_g\| \asymp \sup_{z\in B}(1-|z|^2)^{q-\frac{n-p+2}{2}}|g(z)|. \tag{24}$$

Proof. Let $f \in D_p$. By Lemma 3,

$$\|L_g f\|_{\mathcal{B}^q} = \sup_{z\in B}(1-|z|^2)^q|\Re(L_g f)(z)| \le C\|f\|_{D_p}\sup_{z\in B}(1-|z|^2)^{q-\frac{n-p+2}{2}}|g(z)|. \tag{25}$$

Hence (23) implies that $L_g : D_p \to \mathcal{B}^q$ is bounded.

Conversely, suppose that $L_g : D_p \to \mathcal{B}^q$ is bounded. For any $w \in B$, $|w| > 1/2$. Let $f_w(z)$ be defined by (21). Then after some calculations we get

$$|\Re f_w(w)| = \frac{w^2}{(1-|w|^2)^{\frac{n-p+2}{2}}}. \tag{26}$$

Similarly to the proof of Theorem 2, we get

$$(1-|w|^2)^{2q-(n-p+2)}|g(w)|^2 \le C\|L_g f_w\|^2_{\mathcal{B}^q}. \tag{27}$$

From the above inequality and using the Maximum modulus principle we get (23). From (25) and (27) we obtain (24). □

Theorem 7. Let $g : B \to \mathbb{C}$ *be a holomorphic map and $n + 2 - 2q < p < 0$. Then $L_g : D_p \to \mathcal{B}^q$ is compact if and only if* $g \in \mathcal{B}_0^{q-\frac{n-p}{2}}$.

Proof. First assume $g \in \mathcal{B}_0^{q-\frac{n-p}{2}}$. Let $\{f_k\}$ be a sequence in D_p with $\sup_{k\in\mathbb{N}}\|f_k\|_{D_p} \le K$ and $f_k \to 0$ uniformly on compact subsets of B, as $k \to \infty$. By the assumption, for any $\varepsilon > 0$, there is a constant δ, $0 < \delta < 1$, such that

$$(1-|z|^2)^{q-\frac{n-p}{2}}|\Re g(z)| < \varepsilon/2$$

whenever $\delta < |z| < 1$. Using Lemma 3, we have

$$\begin{aligned}\|T_g f_k\|_{\mathcal{B}^q} &= \sup_{z\in B}(1-|z|^2)^q|\Re(T_g f_k)(z)|\\ &= \sup_{z\in E}(1-|z|^2)^q|\Re g(z) f_k(z)| + \sup_{z\in B\setminus E}(1-|z|^2)^q|\Re g(z) f_k(z)|\\ &\le M(1-|\delta|^2)^{-q+\frac{n-p}{2}}|\sup_{z\in E}|f_k(z)| + CK\varepsilon/2,\end{aligned}$$

where $M = \sup_{z\in B}(1-|z|^2)^{q-\frac{n-p}{2}}|\Re g(z)|$. By the assumption, $\limsup_{k\to\infty}\|T_g f_k\|_{\mathcal{B}^q} \le CK\varepsilon/2$. Since ε is an arbitrary positive number we have that $\lim_{k\to\infty}\|T_g f_k\|_{\mathcal{B}^q} = 0$. By Lemma 4 we see that $T_g : D_p \to \mathcal{B}^q$ is compact.

Conversely, suppose that $T_g : D_p \to \mathcal{B}^q$ is compact. Let $\{z_k\}$ be a sequence in B such that $|z_k| \to 1$ as $k \to \infty$. Take

$$f_k(z) = \frac{1-|z_k|^2}{(1-<z, z_k>)^{1+\frac{n-p}{2}}}. \tag{28}$$

Then $f_k \in D_p$ and f_k converges to 0 uniformly on compact subsets of B. Since T_g is compact, by Lemma 4 we have

$$\|T_g f_k\|_{\mathcal{B}^q} \to 0 \text{ as } k \to \infty.$$

Thus,

$$(1-|z_k|^2)^{q-\frac{n-p}{2}} |\Re g(z_k)| \le \sup_{z\in B}(1-|z|^2)^q |\Re(T_g f_k)(z)| = \| T_g f_k \|_{\mathcal{B}^q} \to 0,$$

as $k \to \infty$. From which we see that $g \in \mathcal{B}_0^{q-\frac{n-p}{2}}$. □

Theorem 8. *Let $g : B \to \mathbb{C}$ be a holomorphic map and $n + 2 - 2q < p < 0$. Then $L_g : D_p \to \mathcal{B}^q$ is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)^{q-\frac{(n-p+2)}{2}} |g(z)| = 0. \tag{29}$$

Proof. Assume (29). Let $\{f_k\}$ be a sequence in D_p with $\sup_{k\in\mathbb{N}}\|f_k\|_{D_p} \le K$ and $f_k \to 0$ uniformly on compact subsets of B as $k \to \infty$. By the assumption, for any $\varepsilon > 0$, there is a constant δ, $0 < \delta < 1$, such that

$$(1-|z|^2)^{q-\frac{(n-p+2)}{2}} |g(z)| < \varepsilon/2$$

whenever $\delta < |z| < 1$. Using Lemma 3, we obtain

$$\begin{aligned}\| L_g f_k \|_{\mathcal{B}^q} &= \sup_{z\in B}(1-|z|^2)^q |\Re(L_g f_k)(z)| \\ &= \sup_{z\in E}(1-|z|^2)^q |g(z)\Re f_k(z)| + \sup_{z\in B\setminus E}(1-|z|^2)^q |g(z)\Re f_k(z)| \\ &\le N \sup_{z\in E}(1-|\delta|^2)^{\frac{(n-p+2)}{2}} |\Re f_k(z)| + CK\varepsilon/2,\end{aligned}$$

where $N = \sup_{z\in B}(1-|z|^2)^{q-\frac{(n-p+2)}{2}} |g(z)|$. Similarly to the proof of Theorem 4, we see that $L_g : D_p \to \mathcal{B}^q$ is compact.

Conversely, assume that $L_g : D_p \to \mathcal{B}^q$ is compact. Let $\{z_k\}$ be a sequence in B such that $|z_k| \to 1$ as $k \to \infty$. Let $f_k(z)$ be defined by (28). Then $f_k \in D_p$ and f_n converges to 0 uniformly on compact subsets of B. Since L_g is compact, by Lemma 4 we have

$$(1-|z_k|^2)^{q-\frac{(n-p+2)}{2}} |g(z_k)| \preceq \sup_{z\in B}(1-|z|^2)^q |\Re(L_g f_k)(z)| = \| L_g f_k \|_{\mathcal{B}^q} \to 0$$

as $k \to \infty$. From which we get the desired result. □

Especially, let $p = -1$, i.e. $D_{-1} = A^2$. We get the following corollary.

Corollary 9. *Let $g : B \to \mathbb{C}$ be holomorphic and $q > \frac{n+3}{2}$. Then*

(1) $T_g : A^2 \to \mathcal{B}^q$ *is bounded if and only if* $g \in \mathcal{B}^{q-\frac{n+1}{2}}$;

(2) $L_g : A^2 \to \mathcal{B}^q$ *is bounded if and only if* $\sup_{z\in B}(1-|z|^2)^{q-\frac{n+3}{2}}|g(z)| < \infty$;

(3) $T_g : A^2 \to \mathcal{B}^q$ *is compact if and only if* $g \in \mathcal{B}_0^{q-\frac{n+1}{2}}$;

(4) $L_g : A^2 \to \mathcal{B}^q$ *is compact if and only if* $\lim_{|z|\to 1}(1-|z|^2)^{q-\frac{n+3}{2}}|g(z)| = 0.$

Acknowledgements

This author is supported partly by NNSF China (No. 10371051, No. 10671115), PHD Foundation (No. 20060560002) and NSF of Guangdong Province (No. 06105648).

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