J. Indian Inst. Sci., Jan.–Feb. 2005, **85**, 27–38

Optimal beamforming for Rayleigh-faded time-frequency varying channels using fractional Fourier transform

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Received on September 17, 2004; Revised on November 10, 2004.

Abstract

A method of optimal beamforming for Rayleigh-faded time-frequency selective channels using fractional Fourier transform (FRFT) is considered in this paper. The conventional minimum mean squared error (MMSE) beamforming in the frequency domain or spatial domain becomes a special case of optimal beamforming with FRFT. The method is especially useful in moving source problems, where Doppler effect produces frequency shift when the sinusoidal source is moving in wireless communication where the user produces frequency shift while moving. With the advent of FRFT and its related concept, it is seen that the properties and applications of the ordinary Fourier transform are special cases of those of the FRFT. It is not only richer in theory and more flexible in application but is also not costly in implementation. The advantage of the proposed optimum FRFT domain beamformer is the practical consideration that the synthesis or analysis of the FRFT can be implemented with the same complexity as the conventional fast Fourier transform (FFT).

Keywords: Beamforming, fractional Fourier transform, time-frequency varying channels.

1. Introduction

The phenomenal growth in the number of mobile subscribers and the huge demand for new broadband services, such as facsimile, electronic mail, new innovative multimedia, full motion video, and wireless teleconferencing requiring high data rates, present new technological challenges. Current 3G standards are not able to support broadband services since they limit the maximum data rate to 2 Mbps in indoor communications and less than 1 Mbps in outdoor communications. As data rates supported by 4G systems reach 100 Mbps, the price per bit should drop by at least a factor of 100. 4G is not one of solely higher data rates, but has the focus of public service. The 4th Generation Wireless Infrastructures (4GW) project has been started with the vision "Mobile multimedia to all at today's prices for fixed telephony". Thus, 4G is expected to support more multimedia services at improved quality and higher data rates compared to 3G, 2.5G and 2G. Various approaches have been suggested to achieve the above objective and the Wireless World Research Forum (WWRF) has identified the following related research items [1]:

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- Beamforming and space-time processing at the transmitter as well as the receiver;
- Realistic channel models and interference scenarios to evaluate the performance of smart antennas;
- Multiple input and multiple output (MIMO) transmission systems;
- Incorporation of MIMO techniques into multitechnology radio networks;
- Robust signal processing techniques.

From the above research items one issue is clear that there is a need for an advanced array processing technique for adaptive antenna architectures of wireless systems that have a practical implementation complexity and can achieve high performance levels in multipath environments. In wireless communication, the transmitted signal is usually modified in amplitude and phase by the multipath, which by itself might be changing with time and spatial position. In all these cases, the signals that arrive at the array can be regarded as random, and at times the physical phenomena responsible for the randomness in the signal make it plausible to assume that the signals are nonstationary random processes [2]. In such situations, the signal and noise processes can be completely specified by their second-order statistics. Since the natural causes responsible for the signal and noise are often unrelated, it is customary to assume that the signal and noise are uncorrelated with each other [3, 4]. These uncertainties, combined with the nonstationary nature of the signal and noise processes, make the fractional Fourier transformation (FRFT) potentially powerful tool in designing the optimal beamformer in the array environment. Thus the ordinary Fourier transform is suited best for analysis and processing of time-invariant signals and systems. When dealing with time-varying signals and systems, filtering in fractional Fourier domains might allow us to estimate signals with smaller minimum mean squared error (MMSE) [5]. In this paper, the nonstationary signal received at the array is processed to get the optimum weights. As mentioned by WWRF any robust signal-processing technique will definitely improve the performance of wireless communication systems. One such technique which is recognized is the FRFT. In this paper, the key result is the derivation of the optimum FRFT-based beamformer for Rayleigh-faded multipath signals over additive white Gaussian noise (AWGN) channels.

2. Beamforming

Beamforming provides an effective and versatile means of spatial filtering of signals arriving at distributed array of sensors [6]. The sensor array collects spatial samples of propagating waves, which are processed by beamformer. To find the spectral differences a beamformer performs temporal filtering along with spatial filtering. It is applicable to either reception or radiation of energy. A beamformer linearly combines the spatially sampled time series from each sensor to obtain a scalar output time series as in the case of FIR filter. Mathematically, a general beamformer process can be expressed as

$$y(t) = \sum_{m=1}^{M} \sum_{n=0}^{N-1} w_{m,n}^{*} x_m(t - nT) \tag{1}$$

where $y(t)$ is the beamformer output, $x_m(t)$s are the signals arriving at the sensors, $N-1$, the number of delays in each of sensor channels, T, the duration of single time delay, M, the number of sensors, $w_{m,n}$s are the weights of the beamformer and the superscript '*' denotes complex conjugate. The $w_{m,n}$s are chosen to achieve the required cost function. Equation (1) can be written in the vector form as

$$y(t) = w^H x(t) \tag{2}$$

where $x(t) = [x_1(t), x_1(t - T), \ldots, x_1(t - (N - 1)T, \ldots, x_2(t), \ldots, x_M(t - (N - 1)T)]^T$, and

$$w = [w_{1,0}, w_{1,1}, \ldots, w_{1,N-1}, w_{2,0}, \ldots, w_{M,N-1}]^H$$

where the superscript "T" denotes transpose and "H" Hermitian conjugate.

In optimal beamforming, the weights of an antenna array are selected according to optimization criteria which minimize a cost function. Typically, the cost function is inversely associated with the quality of signal at the array output. So, when the cost function is minimized the quality of the signal is maximized. The main criteria of optimization are MMSE, maximum signal-to-noise ratio (SNR) and minimum variance. Each criterion has its own advantages and disadvantages. The MMSE requires the reference signal and there is no need for information about the direction of arrival (DOA) of the signal. The maximum SNR requires the direction of arrival of both desired and interfering signals. In this, the weights are chosen such that the SNR at the output of the beamformer is optimized. The minimum variance method does not require knowledge of the angle of arrival of interferences but requires the angle of arrival of the desired signal. In this, the output power is minimized to ensure that the effect of noise is minimized [7]. A study in 1987 has shown that the antenna arrays based on MMSE criterion can suppress the multipath signals with large delay differences and can be employed for high-speed mobile communications [8]. The method proposed in this paper can be viewed as the generalization of MSE, which minimizes error between its output and the desired signal. This paper is divided into various sections. Section 3 deals with a brief overview of FRFT and its related concepts. In Section 4, the Rayleigh-faded time-frequency-varying channels have been discussed. Beamforming using FRFT for Rayleigh-faded signals is explained in Section 5. Experimental details are given in Section 6. The results of the simulation and conclusions have been discussed in Section 7.

3. Fractional Fourier transform

The FRFT, generalization of the classical Fourier transform (FT), depends on a parameter a and can be interpreted as a rotation by an angle a in the time-frequency plane [9]. The capability of fractional Fourier domain filtering to significantly reduce the error as compared to FT for certain types of degradation and noise (especially producing Doppler shift) is the main motivation behind this paper. The replacement of FT with FRFT improves the performance considerably in the case of array signal processing and wireless communication. However, in every area where FT and frequency domain concepts are used, there exists the potential for generalization and implementation by using FRFT. FRFT is a time-varying filter that generalizes the FT by letting it depend on a continuous parameter a where $a = 2a/p$. Mathematically, ath order FRFT is the ath power of FT operator. FRFT has many applications in space or time-variant filtering, multiplexing, signal recovery, restoration and enhancement, study of space or time-frequency distributions, solution of differential equations, optical beam propagation and spherical mirror resonators, optical diffraction theory, quantum mechanics, statistical optics, optical system design and optical signal processing, signal detectors, correlation and pattern recognition, etc. [10–14] The expression used to evaluate the *a*th order FRFT of an arbitrary signal $x(u')$ is given by the following equation

$$F^{a}[x(u')] = X(u) = \frac{\exp .i\left(\frac{p}{4} - \frac{a}{2}\right)^2}{\sqrt{2p \, \sin a}} \int_{-\infty}^{\infty} \exp\left(-\frac{i}{2}u^2 \cot a - \frac{i}{2}u'^2 \cot a - \frac{iu'u}{\sin a}\right) x(u')du' \quad (3)$$

$$= \int_{-\infty}^{\infty} K_a(u-u')x(u')du' \quad (4)$$

where $F^a[x(u')]$ gives the fractional Fourier transform of the signal represented by $x(u')$. For an angle from 0 to $2p$, the values of a lie from 0 to 4 and it can be shown that the transform kernel is periodic with a period 4. FRFT of a function is equivalent to a four-step process: multiplying the function with a chirp, taking its Fourier transform, again multiplying with a chirp, and then multiplying with an amplitude factor. There exists a fast method for computation of FRFT of order $N \log_2(N)$, where N is signal temporal length [15]. The effect of FRFT on a signal can be seen easily in a time-frequency plane. One of the most popular time-frequency distributions, the Wigner distribution, that gives an idea about the energy distribution of a signal in time-frequency can be used to illustrate this effect [9, 14, 16]. The Wigner distribution function is defined as:

$$W(u,v) = \int_{-\infty}^{\infty} x\left(u + \frac{u'}{2}\right) x\left(u - \frac{u'}{2}\right) e^{2ipu'v} du'. \quad (5)$$

The extra degree of freedom given by FRFT as parameter a improves the performance of a large class of signal-processing applications. The MMSE, which is obtained in frequency or time domain, can now be optimized over this parameter. It has been shown that FRFT beamformer improves the performance in case of sources that may be stationary or moving at a constant velocity in AWGN channel. In the above cases, it has been shown that the space or frequency domain filtering does not serve the purpose so it is required to optimize filtering in an appropriate fractional Fourier domain [16–17]. In Yetik and Nehorai [18], the FRFT beamformer has been evaluated in AWGN channel, while in this paper it is evaluated in Rayleigh-faded channels considering it to be realistic scenario for a wireless channel as suggested by WWRF.

4. Rayleigh-faded time-frequency varying channels

In most of the wireless communication systems, the height of the antenna is much lower than the surrounding structures. Thus, the existence of a direct or line-of-sight path between the transmitter and the receiver is highly unlikely. In such a case, propagation is mainly due to reflection and scattering from the buildings and by diffraction over and/or around them. So, in practice, the transmitted signal arrives at the receiver via several paths with different time delays creating a multipath situation. In a multipath environment, if the difference in the time delay of the number of paths is less than the reciprocal of the transmission bandwidth, the paths cannot be resolved individually. These paths also have random phases. They add up at the receiver according to their relative strengths and phases. The envelope of the received signal is therefore a random variable [19]. This random nature of the received signal envelope is referred to as fading. In the case of an unmodulated carrier, the transmitted signal at frequency w_c reaches the receiver via a number of paths, the ith path having an amplitude a_i, and a phase f_i. The received signal $x(t)$ can be expressed as

$$x(t) = \mathrm{Re}\left\{\sum_{i=1}^{L} a_i e^{j(w_c t + f_i)}\right\} = \sum_{i=1}^{L} a_i \cos(w_c t + f_i), \tag{6}$$

where L is the number of paths. The phase f_i depends on the varying path lengths, changing by $2p$ when the path length changes by a wavelength. Therefore, the phases are uniformly distributed over $[0, 2p]$. Let the ith reflected wave with amplitude a_i and phase f_i arrive at the receiver from an angle y_i relative to the direction of motion of the antenna. The Doppler shift of this wave is given by

$$w_{d_i} = \frac{w_c v}{c} \cos y_i, \tag{7}$$

where v is the velocity of the mobile, c, the speed of light (3×10^8 m/s), and the y_is are uniformly distributed over $[0, 2p]$. The received signal $x(t)$ can now be written as

$$x(t) = \sum_{i=1}^{L} a_i \cos(w_c t + w_{di} t + f_i). \tag{8}$$

Expressing the signal in terms of its in-phase and quadrature form, eqn (8) can be written as

$$x(t) = I(t)\cos w_c t - Q(t)\sin w_c t, \tag{9}$$

where the in-phase and quadrature components are given, respectively, as

$$I(t) = \sum_{i=1}^{L} a_i \cos(w_{d_i} t + f_i) \tag{10}$$

$$Q(t) = \sum_{i=1}^{L} a_i \sin(w_{d_i} t + f_i). \tag{11}$$

If L is sufficiently large, by virtue of the central limit theorem, the in-phase and quadrature components $I(t)$ and $Q(t)$ will be independent Gaussian processes which can be completely characterized by their mean and autocorrelation function . In this case, the means of $I(t)$ and $Q(t)$ are zero. Furthermore, $I(t)$ and $Q(t)$ will have equal variances, s^2, given by the mean-square value or the mean power. The envelope, $r(t)$, of $I(t)$ and $Q(t)$ is obtained by demodulating the signal $x(t)$. The received signal envelope is given by

$$r(t) = \sqrt{I^2(t) + Q^2(t)}, \tag{12}$$

and the phase q is given by

$$q = \arctan\left(\frac{Q(t)}{I(t)}\right). \tag{13}$$

The probability density function (pdf) of the received signal amplitude (envelope), $f(r)$, can be shown to be Rayleigh given by

$$f(r) = \frac{r}{s^2}\exp\left\{-\frac{r^2}{2s^2}\right\}, \quad r \geq 0. \tag{14}$$

The cumulative distribution function (cdf) for the envelope is given by

$$F(r) = \int_0^r f(r)dr = 1 - \exp\left\{-\frac{r^2}{2s^2}\right\}. \tag{15}$$

The mean and the variance of the Rayleigh distribution are $s\sqrt{p/2}$ and $(2 - p/2)s^2$, respectively. The phase q is uniformly distributed over $[0, 2p]$. The instantaneous power is thus exponential. The Rayleigh distribution is a commonly accepted model for small-scale amplitude fluctuations in the absence of a direct line-of-sight (LOS) path, due to its simple theoretical and empirical justifications [19].

5. Beamforming using FRFT in Rayleigh-faded channels

In this paper, the MMSE method has been used to obtain the optimum weights. The objective is to recover the desired signal free from noise and fading from the measured signal at the array output, in moving source problems. In other beamforming applications it may have different meaning. Let the array output be $y(t)$ and the reference signal $d(t)$. The weights of the array can be chosen in order to minimize the mean squared error between the array output and the reference signal.

$$J(w) = E\{\|d(t) - y(t)\|^2\}, \tag{16}$$

where $\|\bullet\|$ is the L_2 norm given by $\|y(t)\|^2 = \int_{-\infty}^{\infty} y(t)y^*(t)dt$. The optimum weights can be found by setting the derivative of $J(w)$ to w^* equal to zero. They are given as

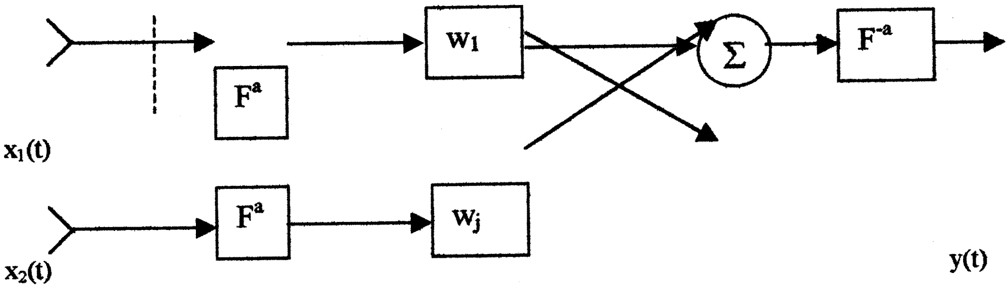
$$w_{opt} = R_x^{-1} r_{xd}, \tag{17}$$

where R_x is the covariance matrix of the measurements at the sensor and r_{xd} is the cross covariance between the measurements at the sensors and the desired signal [1, 2]. The above results, which are spatial filtering of signals arriving at the sensors, can be extended using filtering in fractional Fourier domain [5]. The structure of the proposed beamformer in fractional Fourier domain is given in Fig. 1. The methodology is to transform the signal in an optimal ath domain, then process the beamforming in this domain and again transforming back into the time domain [20].

The output of the beamformer can de expressed as:

$$y(t) = F^{-a} \| w^H (F^a\{x(t)\}) \|, \tag{18}$$

where $F^a(\cdot)$ denotes the ath order FRFT. For FRFT beamformer the cost function $J(w)$ is minimized and the optimum weights are given by:



'*a*' denotes the order of the FRFT.

FIG 1. Block diagram of optimum FRFT domain beamformer [18].

$$w_{opt} = R_{x_a}^{-1} r_{x_a d}, \tag{19}$$

where R_{x_a} is the covariance of the *a*th order FRFT of the signals arriving at sensors and $r_{x_a d}$, the cross-covariance between the *a*th order FRFT of the desired signal and the FRFT of the signal arriving at the sensor. The R_{x_a} and $r_{x_a d}$ can be computed using the covariance as follows:

$$R_{x_a} = R_{x_a}(u,u') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_a(u,u'')K_{-a}(u',u''')R_x(u'',u''')du''du'''; \tag{20}$$

$$r_{x_a d} = r_{x_a d}(u,u') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_a(u,u'')K_{-a}(u',u''')r_{xd}(u',u''')du''du'''. \tag{21}$$

In this paper, the optimum *a*th domain is found by calculating the MSE for different values of $a \in [-1,1]$ and choosing the value of *a* which gives minimum MSE (Fig. 2). The proposed method reduces to spatial domain for $a = 0$ and frequency domain for $a = 1$.

6. Experimental

In the experiment, a 10-element uniform linear array is taken. The spacing between elements is taken as $d = l/2$. The harmonic source is in far-field emitting frequency 100 kHz.

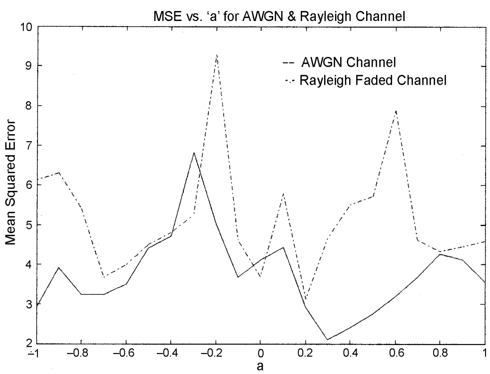


FIG. 2. Variation of MSE with *a* for AWGN and Rayleigh-faded channels.

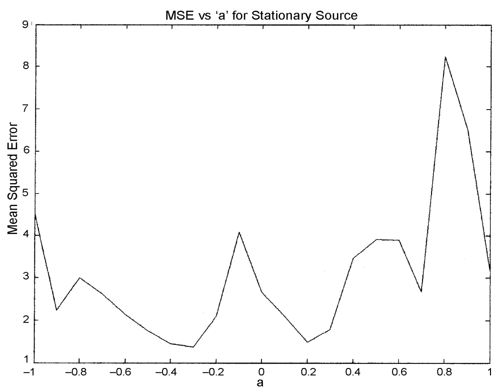


FIG. 3. Variation of MSE in various FRFT domains for a stationary source in Rayleigh-faded channel.

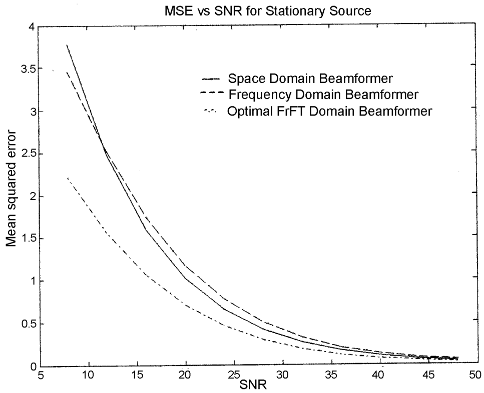


FIG. 4. Plot for comparison of MSE with varying SNR for different beamformers for a stationary source.

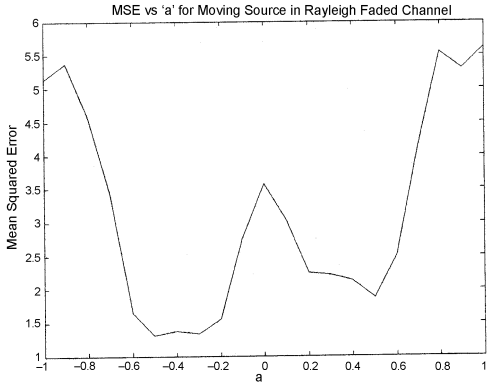


FIG. 5. Variation of MSE in various FRFT domains for a moving source in Rayleigh-faded channel.

The first element of beamformer is taken as reference element. The source is considered as stationary as well as moving with a constant velocity of 120 m/s. For finding the variation in MSE with SNR, the minimum MSE is found for each value of SNR varying from 8 to 48 dB.

7. Results and conclusions

The proposed beamformer is used for stationary and moving sources. The channel is assumed to be Rayleigh-faded with additive Gaussian noise. The measurements used in simulation are instantaneous without any delay. For finding optimum a, the experiment was repeated for values of $a \in [-1,1]$ with a step of 0.1. The value of a that gave the minimum MSE is taken as optimum value. Figure 2 shows the variation of MSE with a in Rayleigh-faded as well as in AWGN channels. It is obvious that MSE is higher in the case of Rayleigh-faded signals. However, in both the cases, the MSE is smaller in optimum FRFT domain ($a = a_{opt}$) as compared to space domain ($a = 0$) and frequency domain ($a = 1$). In Fig. 3, MSE is shown as a function of FRFT domain a in the case of stationary source in Rayleigh-faded channels. It is clear that the optimum FRFT domain for stationary source is $a = -0.3$. Variation of MSE with SNR in the case of space, frequency and optimum

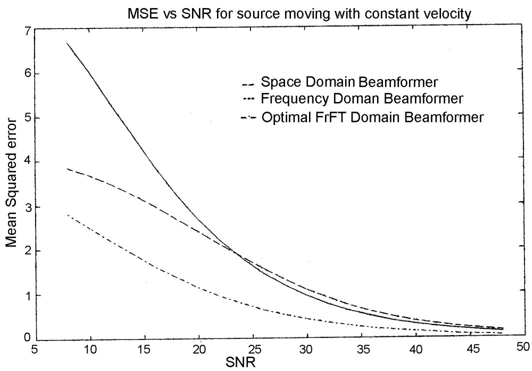


FIG. 6. Plot for comparison of MSE with varying SNR for different beamformers for a moving source.

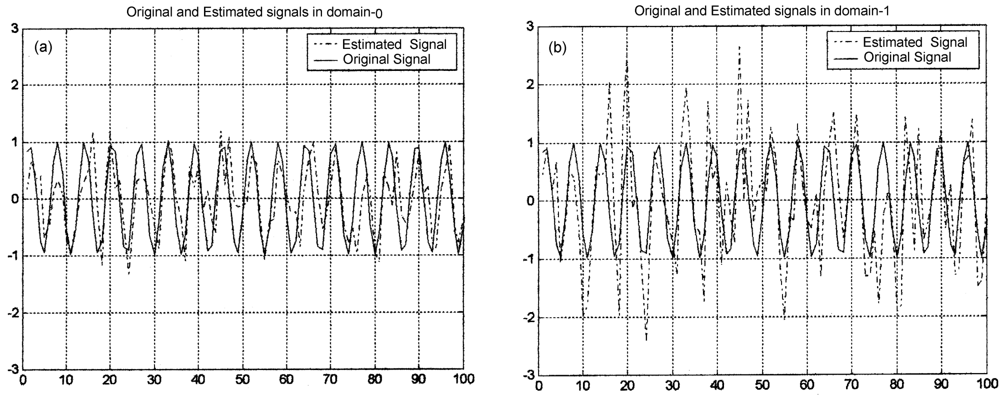


FIG. 7. Estimated signal in (a) time and (b) frequency domains using MMSE beamformer.

FRFT domain is shown in Fig. 4. Figures 5 and 6 give MSE as a function of a and SNR respectively for moving sources with a constant velocity of 120 m/s. For a moving source the optimum FRFT domain is $a = -0.5$. It is observed that the optimum domain is different from 0 and 1 in both the cases. From these plots it is clear that there is a significant improvement in the performance of beamformer for a Rayleigh-faded signal in optimum FRFT domain as compared to space and frequency domains. The results for stationary and moving source with constant velocity in Rayleigh-faded channel, with SNR = 30 dB are summarized in Table I.

Table I shows that in the case of moving source, the MSE is less than that of stationary source in optimum FRFT domain. This is due to the fact that FRFT can handle frequency varying signal (produced because of Doppler shift generated by moving source) more effectively. It is also clear that there is improvement of 76.7% compared with frequency domain and 68.3% compared with space domain in the case of moving source with constant velocity. The proposed optimum FRFT domain beamformer can be used for yielding small errors

Table I
MSE for stationary as well as moving source

Domain	MSE	
	Stationary source	Moving source
$a = a_{opt}$	1.368	1.311
$a = 0$	2.653	3.625
$a = 1$	3.146	5.643

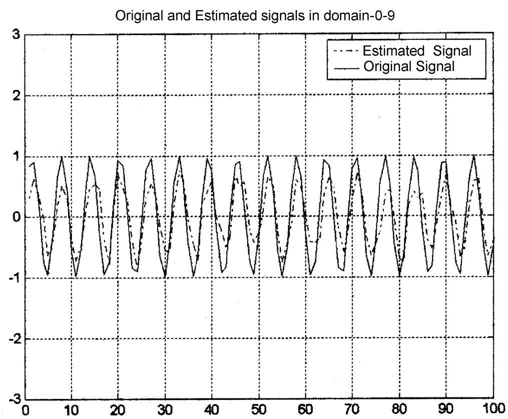


FIG. 8. Estimated signal in optimum domain using MMSE beamformer.

in the case of accelerating source problems. The proposed method of obtaining optimum *a* is based on frame-by-frame basis. In practice, the optimum *a* that gives minimum MSE requires an efficient online procedure for its computation. In this paper, the experiment is performed on a known transmitted signal which is received by the array as Rayleigh-faded over AWGN channel. At the output of the array the original signal is estimated back by using MMSE criterion. So in this paper the beamformer is used to estimate the transmitted signal. The plots of the estimated signals in time ($a = 0$, Fig. 7(a)), frequency ($a = 1$, Fig. 7(b)) and optimum domain (Fig. 8) are shown. The Wigner distribution for Rayleigh case is also given in Figs 9 and 10, which also shows that in optimum FRFT domain the signal energy is more concentrated as compared to space or frequency domain. The class of fractional Fourier domain filters is a subclass of the class of linear filters [17]. The optimal filter found in this fractional Fourier domain is not the most optimal among all filters. However, the class of fractional Fourier domain filters is much broader than the class of ordinary domain filters, so that in general the optimal filter found in FRFT domain results in much smaller errors as compared to ordinary time or Fourier domain filters. This reduction in error comes at no additional cost because FRFT can be implemented with the same cost as ordinary Fourier transform [15]. For time-invariant degradation models and stationary

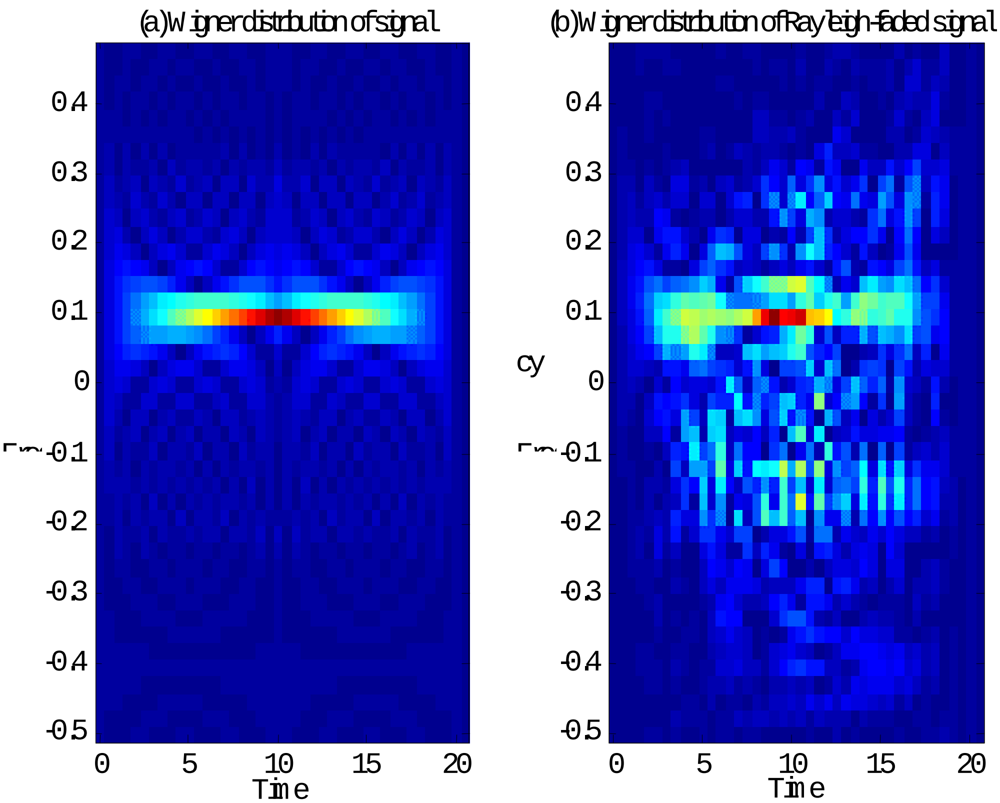


FIG. 9(a). Wigner distribution of signal, and (b) Rayleigh-faded signal (initial velocity = 10 m/s and acceleration 2 m/s^2).

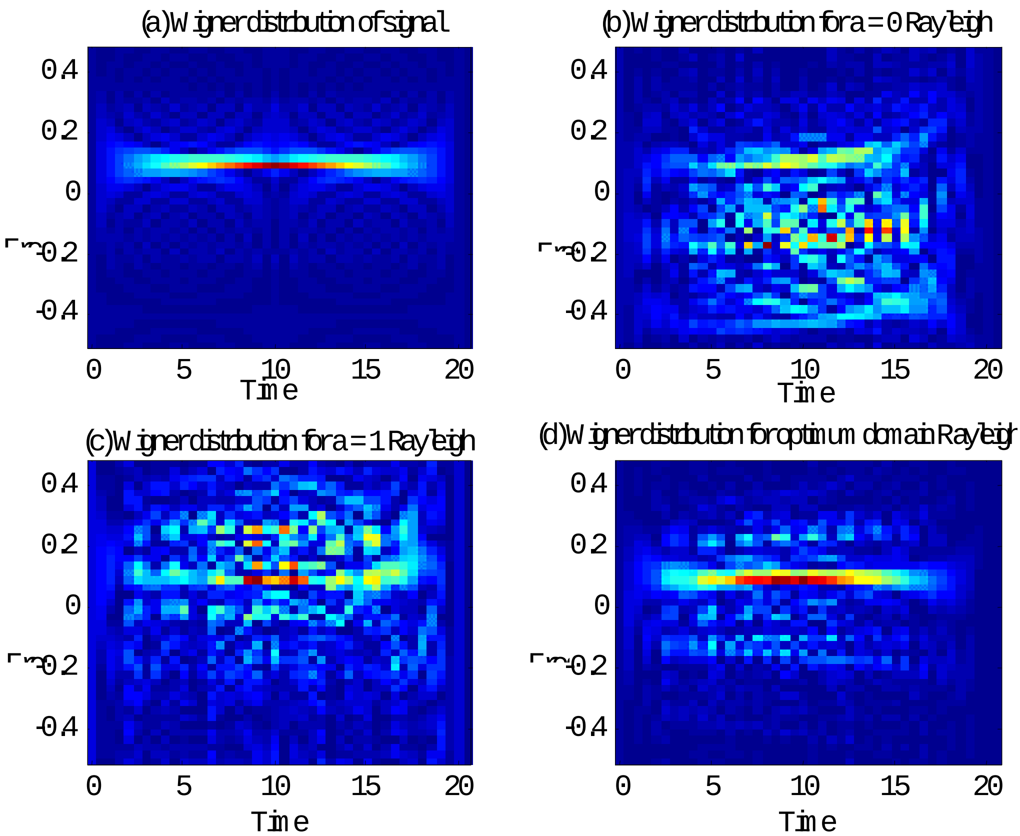


FIG. 10. Wigner distribution (a) of signal, (b) in time domain, (c) in frequency domain, and (d) in optimum FRFT domain (Rayleigh-faded channel).

signals and noise, the classical Fourier domain Wiener beamformer, which can be implemented in O (N $\log_2$ N) time, gives the minimum MSE estimate of the original undistorted signal. For time-varying degradations and nonstationary processes, however, the optimal linear estimate requires O(N^2) time for implementation. Beamforming in fractional Fourier domains, which enables significant reduction of the error compared with ordinary Fourier domain beamforming for certain types of degradation and noise (especially of chirped nature), requires only O (N $\log_2$ N) implementation time. Thus, improved performance is achieved at no additional cost.

Acknowledgements

The authors would like to thank Dr I. S. Yetik for useful suggestions.

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