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On the stability of solutions of a certain system of second-order differential equation

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Abstract

The paper deals with the stability of solutions along with their derivatives of a certain system of second-order differential equation with respect to certain perturbation. We consider the system

$$d^2y(x)/dx^2 + A(x)y(x) = 0, \tag{i}$$

where

$$y(x) = \begin{pmatrix} Y_1(x) \\ Y_2(x) \end{pmatrix}, \quad A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{12}(x) & a_{22}(x) \end{pmatrix} = (a_{ij}(x))$$

and $a_{ij}(x)$, $i, j = 1, 2$ is real-valued continuous function of x, $x \in [0, \infty)$.

Let $B(x) = (b_{ij}(x))$, $i, j = 1, 2$ ($b_{ij}(x)$s being real-valued continuous function of $x \in [0, \infty)$) be a set of perturbations which changes (1) to

$$d^2y(x)/dx^2 + (A(x) + B(x))y(x) = 0. \tag{ii}$$

In this paper, certain results on the stability of solutions of the system (i) along with their derivatives, which are either bounded or tend to zero as the independent variable x tends to infinity with respect to the perturbation $B(x)$ satisfying some conditions, are achieved.

Keywords: Bounded solutions, stable solutions.

1. Introduction

The basic problems of study of the qualitative theory of differential equations deal with their solutions which are either periodic or approaches some known functions asymptotically or are stable with respect to certain perturbations satisfying some specific conditions.

The study of stability theory associated with the solutions of second-order differential equations is very old [1, 2]. Later, many mathematicians and physicists like Cesari [3] and Knowles [4] worked on the problem extensively. Hence, a study of the methods of mathematical physics is required to know the specific character or behavior of the solutions of the differential equations (i.e. equation of motions) considered. Attention was paid in the past to the study of the stability properties of solutions of second-order differential equations

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which are either bounded or tend to zero or belong to some L-classes.

In this paper we consider the system of second-order differential equation

$$d^2y(x)/dx^2 + A(x)y(x) = 0, \quad (1)$$

where
$$y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \{y_1(x),\ y_2(x)\},$$

$A(x) = (a_{ij}(x))$, $i, j = 1, 2$; and $a_{ij}(x)$s are assumed to be real-valued continuous functions of x, $x \in [0, \infty)$.

Let $B(x) = (b_{ij}(x))$ $i, j = 1, 2$ be a set of perturbations which changes (1) to

$$d^2y(x)/dx^2 + [A(x) + B(x)]y(x) = 0, \quad (2)$$

where $b_{ij}(x)$s are assumed to be real-valued continuous functions of x, $x \in [0, x)$.

Certain results on the stability of solution of the system (1) which belong to suitable L-classes are available in Chakravarty and Sengupta [5], where the perturbed matrix $B(x)$ satisfies certain conditions.

In this paper, some more results on the stability of solution of the system (1), which are either bounded or tend to zero as the independent variable x tends to ∞, are available with respect to the perturbation $B(x)$ satisfying some specific conditions. The stability of the differential coefficients of the first order of the solutions of eqn (1) are also discussed. The definition of the stability of the solutions of the system (1) is defined in Chakravarty and Sengupta [5]. We only define the stability of the derivatives of the solution of the first order of the system (1) with respect to some property P (i.e. the solutions are either bounded or tend to zero or belong to some L-classes) and perturbation $B(x) = (b_{ij}(x))$, $i, j = 1, 2$ in the following way:

Definition: The derivatives of the solution of the first order of the system (1) which satisfy a certain property P are said to be stable with respect to the set of perturbation $B(x)$ and the property P, if the solutions of (1) and the solutions of (2) with their derivatives of the first order also satisfy the same property P for all $B(x)$ satisfying certain specified conditions.

2. Notations

We use the following notations:

(i) The boundedness, the differentiability or the integrability of a matrix means that all the elements of the matrix are bounded differentiable or integrable, respectively. We use for the matrix $A = (a_{ij})$, $i, j = 1, 2$ the symbol $\| A \| = \sum_{i,j} | a_{ij} |$. In particular, for the vector $v = (v_1, v_2)^T$, norm of v defined by $\|v\| = |v_1| + |v_2|$.

(ii) (a, b) represents the scalar product. $(a,\ b) = \sum_{j=1}^{m} a_j b_j$, where $a = (a_1, a_2, \ldots, a_m)$, $b = (b_1, b_2, \ldots, b_m)$.

(iii) The determinant $\det(a_{ij})$ of order n (≥ 2) is represented in terms of the diagonal elements a_{ij} by $|a_{11}, a_{22}, \ldots, a_{nn}|$.

(iv) $[f_i, f_j] = |u_i, u_j'| + |v_i, v_j'|$ is the bilinear concomitant of two vectors $f_i = (u_i, v_i)^T$ and $f_j = (u_j, v_j)^T$.

(v) $z(t) = a_{11}(t)a_{22}(t)$.

(vi) $M(t) = -5/16a_{11}^{-9/4}(t)a_{22}^{-1/4}(t)a_{11}'^{2}(t) - 1/8a_{11}^{-5/4}(t)a_{22}^{-5/4}(t)a_{11}'(t)a_{22}'(t)$

$$-5/16a_{11}^{-1/4}(t)a_{22}^{-9/4}(t)a_{22}'^{2}(t) + 1/4a_{11}^{-5/4}(t)a_{22}^{-1/4}(t)a_{11}''(t) + 1/4a_{11}^{-1/4}(t)a_{22}^{-5/4}(t)a_{22}''(t).$$

(vii) $x_1(t) = (z(t))^{1/2} - a_{11}^{1/2}(t)a_{22}^{-1/2}(t) + M(t)(z(t))^{1/4}$.

(viii) $x_2(t) = (z(t))^{1/2} - a_{11}^{-1/2}(t)a_{22}^{1/2}(t) + M(t)(z(t))^{-1/4}$.

(ix) $h_1(t) = -a_{12}(t)(z(t))^{-1/2}, \; h_2(t) = -a_{21}(t)(z(t))^{-1/2}$.

(x) $N_1(t) = \begin{pmatrix} x_1(t) & h_1(t) \\ h_2(t) & x_2(t) \end{pmatrix}$.

(xi) $N(0) = \begin{pmatrix} h(0) & h'(0)(z(0))^{-1/2} \\ r(0) & r'(0)(z(0))^{-1/2} \end{pmatrix}$.

3. Preliminaries

Let $f_i = (u_i, v_i)^T$, $i = 1, 2$; $q_j = (u_s, v_s)^T$, $j = 1.$ $s = 3$; $j = 2$, $s = 4$ be the four solution vectors of the system (3) satisfying, at $x = a \geq 0$, the conditions:

(i) $[f_1, f_2] = [q_1, q_2] = 0$

(ii) $[f_1, f_j] = d_{ij}$, $i, j = 1, 2$, d_{ij} being the Kronecker delta.

Let $y(x) = (y_1(x), y_2(x))^T$ be a solution of the system (2).

Then, following Chakravarty and Sengupta [5], one obtains

$$y_1(x) = \sum_{i=1}^{4} c_i u_i(x) + \int_0^x (B_2, y(t))V_1 dt - \int_0^x (B_1, y(t))U_1 dt \tag{3}$$

where

$$U_1 = |u_1(x), u_3(t)| + |u_2(x), u_4(t)|$$

$$V_1 = |u_1(x), v_3(t)| + |u_2(x), v_4(t)|$$

$$B_1 = B_1(t) = (b_{11}(t), b_{12}(t))^T$$

$$B_2 = B_2(t) = (b_{21}(t), b_{22}(t))^T$$

and c_i, $I = 1, 2, 3, 4$ are constants.

Differentiating (3) with respect to x once, the first order derivative of $y_1(x)$ is represented by

$$y_1'(x) = \sum_{i=1}^{4} c_i u_i'(x) + \int_0^x (B_2,\ y(t))V_1'dt - \int_0^x (B_1, y(t))U_1'dt + B_2, y(x))V_3 \tag{4}$$

where

$$U_1' = |u_1'(x),\ u_3(t)| + |u_2'(x),\ u_4(t)|$$

$$V_1' = |u_1'(x),\ v_3(t)| + |u_2'(x),\ v_4(t)|$$

$$V_3 = |u_1(x),\ v_3(x)| + |u_2(x),\ v_4(x)|.$$

A similar expression for $y_2(x)$ and $y_2'(x)$ involving

$$U_2 = |v_1(x), u_3(t)| + v_2(x), u_4(t)|$$

$$V_2 = |v_1(x), v_3(t)| + v_2(x), v_4(t)|$$

and

$$U_2' = |v_1'(x),\ u_3(t)| + |v_2'(x),\ u_4(t)|$$

$$V_2' = |v_1'(x),\ v_3(t)| + |v_2'(x),\ v_4(t)|$$

are also obtained.

4. The Liouville transformation

Applying the transform

$$x(x) = \int_0^x a_{11}^{1/2}(t) a_{22}^{1/2}(t)dt,$$

$$\Omega(x) = \{h(x),\ r(x)\} = a_{11}^{1/4}(x) a_{22}^{1/4}(x) Y(x),$$

to the system of eqn (1) and then following Titchmarsh [6], it easily follows that

$$\Omega(x) = N(0)S(x) + \int_0^x \sin(x(x) - x(t))N_1(t)\Omega(t)dt, \tag{5}$$

where $S(x) = \{\cos x(x), \sin x(x)\}$ and $N(0)$, $N_1(t)$ are those given in Section 2.

5. Lemmas

Lemma 1. Let $y(x) = \{y_1(x), y_2(x)\}$ be any solution of (1). Then

$$y_1(x) = \sum_{i=1}^{4} d_i\, p_i(x) + \int_0^x [\{(a_{21}(t)) - a_{21})y_1(t) + (a_{22}(t)) - a_{22})y_2(t)\}H_1$$

$$- (a_{11}(t) - a_{11})y_1(t) + (a_{12}(t) - a_{12})y_2(t)\}H_2]dt, \tag{6}$$

where (i) $A = (a_{ij})$, $i, j = 1, 2$ be a constant matrix (a_{12}, a_{21} are not zero simultaneously).

(ii) $P_i = \{p_i, q_i\})$, $Q_j = \{p_s, q_s\}$, $I = 1, 2$; $j = 1$, $s = 3$; $j = 2$, $s = 4$ are the fundamental set of solutions of the system

$$y''(x) - Ay(x) = 0 \tag{7}$$

such that at $x = a \geq 0$, the conditions

$[P_1, P_2] = [Q_1, Q_2] = 0$ and $[P_i, Q_j] = d_{ij}$ (Kronecker delta), $i, j = 1, 2$ are satisfied;

(iii) $H_1 = |p_1(x), q_3(t)| + |p_2(x), q_4(t)|$

$H_2 = |p_1(x), p_3(t)| + |p_2(x), p_4(t)|$

(iv) $d_i s$ ($i = 1, 2, 3, 4$) are constants.

A similar expression for $y_2(x)$ involving G_1, G_2 is also obtained, where G_1, G_2 are the H_1, H_2, respectively, with p_i, q_i interchanged.

Proof: Let $y(x) = \{y_1(x), y_2(x)\}$ be any solution of (1). Writing (1) in the form

$$y''(x) - Ay(x) = (A(x) - A)y(x),$$

$A = (a_{ij})$, $i, j = 1, 2$ being a constant matrix (a_{12}, a_{21} are not zero simultaneously) and then applying the variation of the constant method the proof follows easily.

Lemma 2: Let (i) $a_{ij}(x)$ be positive real-valued, $a_{ij}(x) \geq d > 0$, $x \geq 0$ and $a_{ij}(x) \in L[0, \infty)$ for $i, j = 1, 2$; (ii) $a''_{ij}(x) \in L[0, \infty)$ and $a_{ij}(x) \in L_2[0, \infty)$ for $i = j$, then

$$\int_0^x [|x_1(t)| + |x_2(t)| + |h_1(t)| + |h_2(t)| \, dt] < \infty, \tag{8}$$

where $a_{ij}(x)$s ($i, j = 1, 2$) are those given in (1) and $x_1(x)$, $x_2(x)$, $h_1(x)$, $h_2(x)$ are given in Section 2.

Proof:

$$\int_0^x |x_1(t)| \, dt \leq \int_0^x |(zt))^{1/2}| \, dt + \int_0^x |a_{11}^{1/2}(t).(a_{22}^{-1/2}(t)| \, dt + \int_0^x |M(t)(z(t))^{-1/4}| \, dt \equiv J_1 + J_2 + J_3 (\text{say}).$$

Applying Cauchy–Schwarz inequality

$$J_1 \leq \left[\int_0^x |a_{11}(t)| \, dt \int_0^x |a_{22}(t)| \, dt \right]^{1/2} = 0\ (1).$$

$$J_2 \leq \int_0^x |a_{11}^{1/2}(t) a_{22}^{-1/2}(t)| \, dt \leq 1/d \int_0^x |a_{11}^{1/2}(t) a_{22}^{1/2}(t)| \, dt = 0(1), \text{ by } J_1$$

$$J_3 \leq \int_0^x |a_{11}^{-5/2}(t) a_{22}^{-1/2}(t) a_{11}'^2(t)| \, dt + \int_0^x |a_{11}^{-2/3}(t) a_{22}^{-3/2}(t) a_{11}'(t) a_{22}'(t)| \, dt$$

$$+\int_0^x |a_{11}^{-1/2}(t)a_{22}^{-5/2}(t)a_{11}'^2(t)|\,dt + \int_0^x |a_{11}^{-3/2}(t)a_{22}^{-1/2}(t)a_{11}''(t)|\,dt$$

$$+\int_0^x |a_{11}^{-1/2}(t)a_{22}^{-3/2}(t)a_{22}''(t)|\,dt$$

$$\equiv Q_1 + Q_2 + Q_3 + Q_4 + Q_5 \text{ (say).}$$

Now,

$$Q_1 \le 1/d^3 \int_0^x |a_{11}'^2(t)|\,dt = 0(1)$$

$$Q_2 \le \left[\int_0^x |a_{11}^{-3}(t)a_{11}'^2(t)|\,dt \int_0^x |a_{22}^{-3}(t)a_{22}'^2(t)|\,dt\right]^{1/2}$$

$$\le 1/d^3 \left[\int_0^x |a_{11}'^2(t)|\,dt \int_0^x |a_{22}'^2(t)|\,dt\right]^{1/2} = 0(1).$$

In the same order of ideas, it easily follows that $Q_3 = 0(1)$, $Q_4 = 0(1)$, $Q_5 = 0(1)$.

Hence,

$$\int_0^x |x_1(t)|\,dt = 0(1).$$

Similarly,

$$\int_0^x |x_2(t)|\,dt = 0(1).$$

Also,

$$\int_0^x |h_1(t)dt \le 1/d \int_0^x |a_{12}(t)|\,dt = 0(1) \text{ and } \int_0^x |h_2(t)|\,dt = 0(1).$$

Thus the lemma is proved.

6. Theorems

Theorem 1: Let $A = (a_{ij})$, i, $j = 1, 2$ be a constant matrix (a_{12}, a_{21} are not both zero) whose characteristic roots are negative and distinct. Then all the solutions of (1) are bounded if

$$\int_0^\infty |a_{ij}(t) - a_{ij}|\,dt < \infty,\ i, j = 1, 2.$$

Proof: It follows from Mirski [7] that under the stated conditions, all the solutions of the system (7) are bounded.

Let $y(x) = \{(y_1(x), y_2(x)\}$ be any solution of (1). Now following Lemma 1, it easily follows that

$$| y_1(x) |, | y_2(x) | \leq M + K\int_0^x [\{| a_{21}(t) - a_{21}| + | a_{11}(t) - a_{11} |\} | y_1(t)|$$

$$+ [\{|a_{22}(t) - a_{22})| + |a_{22} + |a_{12}(t) - a_{12}|\}|y_2(\mathrm{t})|]dt \qquad (11)$$

where

$$M = \max{}^m . \left\{ \sum_{i=1}^{4} | d_i || p_i |, \sum_{i=1}^{4} | d_i || q_i | \right\} \text{ and } K = \max{}^m \ \{| H_1 |, | H_2 |, | G_1 |, | G_2 |\}.$$

Finally, applying Gronwall's lemma [8], making $x \to \infty$ and using the conditions of the theorem, from (11) the proof follows easily.

Theorem 2: Let $A = (a_{ij})$, $i, j = 1, 2$ be a constant matrix (a_{12}, a_{21} are not both zero) whose characteristic roots are negative and distinct. Then all the solutions of (1) tend to zero if $|a_{ij}(t) - a_{ij}| \leq Me^{qt}$ for all $i, j = 1, 2$ and for all $t > t_0$, M, q, t_0 being some positive constants.

Proof: It follows from Mirsky [7] that under the stated conditions of the theorem, all the solutions of the system (7) tend to zero as $x \to \infty$.

Let $|P_i|$, $|Q_j| \leq N.e^{-qx}$, $i, j = 1, 2$; $N, q > 0$ are constants, where P_i, Q_j are those mentioned in Lemma 1. Let $M_1 = N.\sum_{i=1} | d_i |$ and let $y(x) = \{y_1(x), y_2(x)\}$ be any solution of (1).

Following Lemma 1, one obtains

$$| y_1(x) | \leq M_1 e^{-qx} + N^2 \int_0^x [\{| a_{21}(t) - a_{21} | + | a_{11}(t) - a_{11} |\} e^{-qx}.e^{-qt} | y_1(t) |$$

$$+ [\{|a_{22}(t) - a_{22}| + |a_{12}(t) - a_{12}|\}e^{-qx}.e^{-qt}|y_2(t)|]dt$$

with a similar expression for $|y_2(x)|$.

Using the conditions of the theorem, it follows that $|y_1(x)e^{qx}$, $|y_2(x)|e^{qx} \leq \infty$, as $x \to \infty$.

Hence, the theorem follows.

Theorem 3: Let (i) $a_{ij}(x)$ be positive real-valued, $a_{ij}(x) \geq d > 0$ for $x \geq 0$ and $a_{ij}(x) \in L[0, \infty)$ for $i, j = 1, 2$; (ii) $a''_{ij}(x) \in L[0, \infty)$ and $a_{ij}(x) \in L_2\ [0, \infty)$ for $i = j(a_{ij}(x)$s are those given in (1)). Then the solutions of (1) are bounded in $[0, \infty)$.

Proof: $|\sin x(x)|$, $|\cos x(x)| \leq 1$ as $a_{11}(x)$, $a_{22}(x)$ are real-valued.

By using (5), it now follows that

$$|h(x)| \leq |(z(0))^{-1/2}\,\|h'(0)|+|h(0)|+\int_0^x [|x_1(t)|h(t)|+h_1(t)\|\,r(t)|]\,dt, \text{ and}$$

$$|\,r(x)| \leq |(z(0))^{-1/2}\,|+|\,r'(0)|+|\,r(0)|+\int_0^x [|x_2(t)\|\,r(t)|+|h_2(t)\|h(t)|]\,dt$$

where $\Omega(x) = \{h(x),\ r(x)\}$ is as explained in Section 4.

Now, $|h(t)|, |r(t)| \leq \|\Omega(t)\|$

Let $K = \text{Max}^m.[(z(0))^{-1/2}\|h'(0)| + |h(0)|, |(z(0))^{-1/2}\|r'(0)| + |r(0)|]$

$\therefore\ |h(x)| + |r(x)| \leq 2K + \int_0^x [|x_1(t)|+|x_2(t)|+|h_2(t)|+h_1(t)|]\,|\Omega(t)|\,dt.$

Now applying Gronwall's lemma [8] first and then using Lemma 2, the theorem follows easily.

Theorem 4: Let S be the set of all solutions of (1) and S_1, the set formed of the derivatives of these solutions. If all the elements of S and S_1 are bounded, then all the elements of S and S_1 are stable with respect to the perturbation $B(x)$ provided

$$\text{(i)}\ \int_0^\infty \|B(t)\|\,dt < \infty, \text{ and (ii)}\ \int_0^\infty \|B'(t)\|\,dt < \infty.$$

Proof: Let $y(x) = \{y_1(x),\ y_2(x)\}$ be any solution of (2).

From eqn (3) if now easily follows that

$$|y_1(x)|,\ y_2(x)| \leq M + K\int_0^x \|((B_1+B_2),\ y(t)\|\,dt$$

where $M = \text{Max}^m.c(|f_i|, |q_j|)$, $i, j = 1, 2$; $c = \sum_{j=1}^{4}|c_j|$ and $K = \text{Max}^m.\{|U_i|, |V_i|, i = 1, 2.$

Now, using Gronwall's lemma [8], x tends to ∞ and finally using the conditions (i) it follows that the solutions of (1) are stable.

We now consider,

$$M_1 = \text{Max}^m.c\{|f_i'(x)|, |q_j'(x)|\}i, j = 1, 2$$

$$M_2 = \text{Max}^m\{|U_i'|, |V_i'|\},\ i = 1,\ 2$$

$$N_1 = |V_3|,\ N_2 = \text{Max}^m\{|y_i(x)|\},\ i = 1, 2, \text{ and } |d_{ij} = |b_{ij}(0)| \neq 0,\ i, j = 1,\ 2.$$

Hence, by using (4) it follows that

$$|y_1'|(x)|, |y_2'(x)|$$
$$\leq M_1 + M_2 N_2 \int_0^x \left(\sum_{i,j=1}^{2} |b_{ij}(t)| \right) dt + N_1 N_2 \left[\int_0^x |b_{21}'(t)|\, dt + \int_0^x |b_{22}'(t)|\, dt + |d_{21}| + |d_{22}| \right].$$

Now x tends to ∞ and using conditions (i) and (ii) the proof follows completely.

It may be noted that Theorem 4 remains true when the conditions (i) and (ii) are replaced by the single condition $\|B(x)\| < \infty$.

Theorem 5: Let S be the set of all solutions of (1) and S_1, the set formed of the derivatives of these solutions. Let all the elements of S and S_1 tend to zero as x tends to ∞, then all these elements of S and S_1 are stable with respect to the perturbation $B(x)$, provided $|b_{ij}(t) < Me^{pt}$, for any positive p, M; $i, j = 1, 2$.

Proof: Let $y(x) = (y_1(x), \{y_2(x)\}$ be any solution of (2).

Let $c\{|f_i||q_j|\} < N.e^{-qx}$, $i, j = 1, 2$, $q > 0$.

Using eqn (3), it follows that

$$|y_1(x)|e^{qx}, |y_2(x)|e^{qx}$$
$$\leq 4N + 4N \int_0^x \{|y_1(t)|(|b_{21}(t)| + |b_{11}(t)|) + |y_2(t)|(|b_{22}(t)| + b_{11}(t)|)\} e^{-qt}\, dt.$$

Applying Gronwall's Lemma [8], it now follows that

$$|y_1(x)|e^{qx}, |y_2(x)|e^{qx} \leq 4N.\exp\left(4\text{NM} \int_0^x e^{(p-qt)t} dt \right),$$

where $|b_{ij}(t)| \leq M.e^{pt}$, $i, j = 1, 2$.

Now, choosing $q > 1/2\, p$ and making x tend to infinity the proof of the first part of the theorem is concluded.

For the proof of the second part, we consider

$$\text{Max}^m.\{|f_i(x), |f_i'(x)|, |q_j(x), |q_j'(x)|,\ c|f_i(x)|,\ c(q_j(x)|, |y_i(x)|\} = K.e^{-qx},\ i,\ \ j = 1,\ 2,$$

q being a positive number.

Now using (4) in the same order of ideas as before one obtains

$$|y_1'(x)|, |y_2'(x)| \leq 4Ke^{-qx}.\int_0^x 4Me^{pt}.e^{-qt} dt + 4K.e^{-3qx}.2M.e^{px}.$$

Making x tends to infinity, the proof of the concluding part follows.

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References

1. A. M. Lyapunov, *The general problem of stability of motion*, ONTI, 1935 (Russ.).
2. R. Bellman, *Stability of differential equations*, McGraw-Hill (1953).
3. L. Cesari, *Asymptotic behavior and stability problems in ordinary differential equation*, 2nd edn, Springer-Verlag (1963).
4. I. Knowles, On stability conditions for second order linear differential equations, *J. Differential Equations*, **34**, 179–203 (1979).
5. N. K. Chakravarty and D. Sengupta, On the stability of solution of a certain differential system with respect to a perturbation, *J. Pure Math.*, **3**, 1–9 (1983).
6. E. C. Titchmarsh, *Eigenfunction expansions associated with second-order equations*, Part I, Clarendon (1962).
7. L. Mirski, *An introduction to linear algebra*, Clarendon (1972).
8. T. H. Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations. *Ann. Math. (2)*, **20**, 292–296 (1919).

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