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Uniqueness of limit cycles in a generalized Liénard system

XUNCHENG HUANG* AND LEMIN ZHU
Yangzhou Polytechnic University, Yangzhou, Jiangsu 225 002, China
email: xh311@yahoo.com

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Abstract

In this paper, the conditions that guarantee the uniqueness of limit cycles in a generalized Liénard system have been studied. These conditions are different from the previous results. Several examples are given to show the applicability of theorems.

Keywords: Limit cycles, Liénard system, uniqueness.

1. Introduction

Limit cycles of plane autonomous differential systems appeared in a very famous paper of Poincaré (1881). Later, in the 1930s, van der Pol and Andronov showed that the closed orbit in the phase plane of a self-sustained oscillation occurring in a vacuum tube circuit was a limit cycle as considered by Poincaré. After that, the existence, nonexistence, uniqueness and other properties of limit cycles have been studied extensively by mathematicians and scientists (see, for example, Arrowsmith and Place [1], Ye *et al*. [2] and Perko [3]).

The van der Pol equation

$$\frac{d^2x}{dt^2}+e(x^2-1)\frac{dx}{dt}+x=0, \quad (e>0) \tag{1}$$

can be extended to the Liénard equation

$$\frac{d^2x}{dt^2}+f(x)\frac{dx}{dt}+g(x)=0. \tag{2}$$

Let $G(x)=\int_0^x g(u)du$, $F(x)=\int_0^x f(u)du$. This Liénard transformation shows that eqn (2) is equivalent to the following system of equations:

$$\frac{dx}{dt}=y-F(x), \quad \frac{dy}{dt}=-g(x), \tag{3}$$

or

*Author for correspondence.

$$\frac{dx}{dt} = -y - F(x), \quad \frac{dy}{dt} = g(x), \tag{4}$$

(see, for example, Arrowsmith and Place [1], or Ye, *et al*. [2]).

The limit cycles for Liénard equation (2) or Liénard systems (3) and (4) were studied by many authors; for example, Ye, *et al*. [2] and Zhang [4]. It is easy to see that systems (3) and (4) can be generalized to the following Liénard-type differential systems:

$$\frac{dx}{dt} = h(y) - F(x), \quad \frac{dy}{dt} = -g(x) \tag{5}$$

and

$$\frac{dx}{dt} = -h(y) - F(x), \quad \frac{dy}{dt} = g(x), \tag{6}$$

respectively.

Systems (5) and (6) are not equivalent to each other if $h(y)$ is not odd. The conditions for the uniqueness of limit cycles for system (6) was first obtained by Zhang [4], and then improved by Cherkas and Zhilevich [5]. As to system (5), the uniqueness conditions for limit cycles was proposed by Huang and Sun [6]. Recently, Zhou *et al*. [7] also reported some uniqueness theorems for system (5), which smoothed some conditions of Zhang's theorem. Actually, some of their results are equivalent to those of Huang and Sun [6].

Zhang's theorem has been widely employed in autonomous quadratic differential systems and ecological systems (see, for example, Ye, *et al*. [2], Chen and Sun [8], Huang and Merrill [9], Kuang and Freedman [10], Liu and Xiong [11], Liu and Zhao [12]), Xia and Tian [13], Xu and Dong [14], Xu and Fong [15], Dou and Huang [16], and Li [17]. However, conditions of Zhang's theorem are not always satisfied, so looking for an alternate approach is necessary.

The basic idea of Huang and Sun [6] is based on the fact that the integral of a total differential of a single variable differentiable function along with a closed orbit is zero. If one can find a function whose integrals along with two closed orbits around (0, 0) are different, a contradiction is obtained. In this paper, we apply the same technique as in Huang and Sun [6] to system (6), and prove some new conditions that guarantee the uniqueness of limit cycles for the generalized Liénard system (6). The results are different from those of Zhang [4] and Cherkas and Zhilevich [5]. For example, $\frac{d}{dx}\left(\frac{F'(x)}{g(x)}\right) \geq 0$ is no longer needed. The conditions obtained in this paper are simpler than those in Huang and Sun [6].

Since systems (5) and (6) are being used widely in chemical reaction, ecological, and physical oscillation systems, this work is useful in these fields as well. Our main theorem and its proof is in Section 2, and some examples are given in Section 3 to show the applicability of our results.

Before we end this introduction we would like to emphasize the concept of limit cycles. The topic of limit cycles is always attractive in mathematics since it first appeared in the

1880s. At the beginning of the 20th century, David Hilbert, at the Second International Congress of Mathematicians, Paris 1900, made the famous speech called: 'Mathematical Problems'. One of his 23 problems, the 16th, is on limit cycles finding the maximum number of limit cycles of the differential equations:

$$\frac{dx}{dt} = X_n(x, y)$$
$$\frac{dy}{dt} = Y_n(x, y),$$

where, $x_n(x, y)$ and $y_n(x, y)$ are polynomials whose degrees are not greater than n.

Usually, the study of limit cycles includes two aspects: one is the existence, stability and instability, number and relative positions of limit cycles, and the other is the creating and disappearing of limit cycles along with the varying of the parameters in the system (e.g. bifurcation). For the exact number of limit cycles and their relative positions, the known results are not many because determining the number and positions of limit cycles is not easy. That is the reason why the 16^{th} Hilbert problem still remains open even for the case when $n = 2$ after 100 years, although some important progress has been made recently [18–21]. Therefore, any new results regarding the number of limit cycles including the uniqueness of limit cycles is encouraged in the literature.

2. Main theorems

In our discussion, we assume that all the functions in (6) are continuous and satisfy Lipschitz conditions for the uniqueness of solutions for $|x| < +\infty$ and $|y| < +\infty$. We also assume that

(A_1) $h(0) = 0$, $h(y)$ is increasing, $|h(\pm\infty)| = +\infty$; $xg(x) > 0$, when $x \neq 0$; and there exist $a < 0 < b$, N sufficiently large, such that $xF(x) < 0$ on $x \in (a, b)$, $x \neq 0$ and $xF(x) > 0$, $F(x)$ is increasing on $x \in (-N, a)$ and x $\in (b, N)$.

(A_2) $\limsup_{x \to \pm\infty} (G(x) + \mathrm{sgn}x) = +\infty$.

(A_3) a) $F(x)$ is bounded below for $x > 0$ if $\limsup_{x \to +\infty} F(x) < +\infty$;

b) $F(x)$ is bounded above for $x < 0$ if $\limsup_{x \to -\infty} F(x) < -\infty$.

Note that conditions (A_2) and(A_3) are needed for the existence of the limit cycles of (6) surrounding the only equilibrium point O(0, 0), which is unstable (see Huang and Sun [6], Ye, *et al*. [2]). We now divide the xy-plane into the following zones:

$$Zone\ 1: \{(x, y) | x > 0,\ -h(y) - F(x) < 0\}$$
$$Zone\ 2: \{(x, y) | x < 0,\ -h(y) - F(x) < 0\}$$
$$Zone\ 3: \{(x, y) | x < 0,\ -h(y) - F(x) > 0\}$$
$$Zone\ 4: \{(x, y) | x > 0,\ -h(y) - F(x) > 0\}.$$

The directions of the trajectories of system (6) in these zones are shown in Fig. 1.

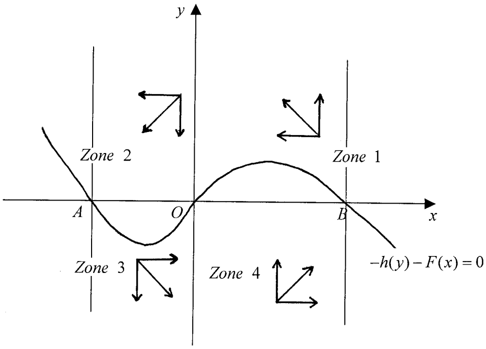

FIG. 1. The direction of the vector field of system (6).

Denote $\lambda = G(x) + H(y)$, where $H(y)=\int_0^y h(u)du$. We first prove the following lemma:

Lemma 1. Suppose R and S are distinct points lying in the vertical strip $a \le x \le b$, such that S is on the forward trajectory through R, and that the portion of this trajectory from R to S lies wholly within the same vertical strip. Then $l(R) < l(S)$.

Proof. If $(x(t), y(t))$, $r \le t \le s$ is the trajectory from R to S, then

$$\begin{aligned} l(S)-l(R) &= \int_r^s \frac{dl}{dt}dt \\ &= \int_r^s \left(\frac{\partial l}{\partial x}\frac{dx}{dt} + \frac{\partial l}{\partial y}\frac{dy}{dt} \right) dt \\ &= \int_r^s -g(x(t))F(x(t))dt \\ &> 0 \end{aligned}$$

since the integrand is positive except possibly at the endpoints and one interior point of the interval $[r, s]$.

The main theorem is as follows:

Theorem 1. If in addition to (A_1), (A_2) and (A_3), one of the following conditions is satisfied:

(i) $G(b) = G(a)$;

(ii) $G(b) > G(a)$ and there exist $x' \in (a, 0)$, $y' < 0$ such that $h(y') \ge -F(x')$, $H(y') \ge G(b)$;

(iii) $G(b) < G(a)$ and there exist $x'' \in (a, 0)$, $y'' > 0$ such that $h(y'') \le -F(x'')$, $H(y'') \ge G(a)$;

(iv) there exist $x' \in (a, 0)$, $y' < 0$ such that
$h(y') \ge -F(x')$, $H(y') \ge G(b)$; and $x'' \in (0,b)$, $y'' > 0$ such that
$h(y'') \le -F(x'')$, $H(y'') \ge G(a)$;

then system (6) has a unique limit cycle.

Proof: Let $x_l = \min\{x : (x, y) \in \Gamma\}$, $x_r = \max\{x : (x, y) \in \Gamma\}$, for Γ a limit cycle of (6). By the phase portrait analysis, it follows that x_l is the x-coordinate of the left intersection of Γ with the isocline $-h(y) - F(X) = 0$, while the x_r, the coordinate of the right intersection.

We first show that

Claim A.

$$x_l < a < b < x_r, \tag{7}$$

that is, all the limit cycles contain the line segment [a, b] on x-axis.

If not, assume that

Case 1. $a \le x_l < x_r \le b.$

Differentiating l along Γ results in

$$d\,l = G'(dx) + h(y)dy = -F(x)dy, \tag{8}$$

which is non-negative since $-F(x) < 0,\ dy < 0$ (y decreases) on $x \in (a, 0)$, and $-F(x) > 0,$ $dy > 0$ (y increases) on $x \in (0, b)$. Here the equality is valid only on $x = 0$, or a, or b. We thus have

$$\oint_\Gamma d\,l > 0.$$

But this is impossible because Γ is a closed curve and $\oint_\Gamma d\,l = 0$ by Green's formula.

Case 2. $x_l < a < x_r \le b.$

In that case Γ must cross the positive x-axis, at, say, $P(x_p, 0)$, $(0 < x_p \le b)$, and the line $x = a$, at two points: say, $P_1(a, y_{p_1})$ and $P_2(a, y_{p_2})$, where $y_{p_2} < y_{p_1}$. It is clear that $y_{P_2} < 0 < y_{P_1}$, and then $P_2(a, y_{p_2})$, is below the x-axis (see Fig. 2). Denote the curve of the trajectory from points P_2 to P as Γ_{P_2P}. It is easy to see that Γ_{P_2P} is also below the x-axis. By Lemma 1, it follows that

$$l(P) > l(P_2)$$

or

$$G(a) + H(y_{p_2}) < G(x_p) \le G(b). \tag{9}$$

Since the fact that $H(y_{p_2}) > 0$, (9) is impossible under the conditions (i) or (iii).

Case 2 is invalid also under the conditions (ii) or (iv). Let Γ_1^- be the trajectory of (6) passing through the point $B(b, 0)$ if $x_p < b$, and Γ_1^+ passing through the point $C_1(x', y')$, where y' is as in (ii) or (iv) (see Fig. 2). When t decreases, Γ_1^- crosses the negative y-axis at $B'(0, y_{B'})$. (It is left to the reader to show that point B' actually exists.).

Now it follows that on the curve $\overset{\frown}{BB'}$, $-F(x) > 0,\ dy < 0$ (y decreases from points B to B'), $d\,l\Big|_{\overset{\frown}{BB'}} < 0$.

That is

$$l(B') < l(B),$$

then

$$H(y_{B'}) < G(b). \tag{10}$$

Moreover, when t increases, Γ_1^+ crosses the negative y-axis at $C_1'(0, y_{C_1'})$. Since y is decreasing on $x < 0$, $y_{C_1'} < y'$. From (ii) or (iv), and (10),

$$H(y_{C_1'}) > H(y') \geq G(b) > H(y_{B'})$$

which implies that (because both C_1' and B' are below the x-axis)

$$y_{C_l'} < y_{B'}.$$

But this is clearly impossible by the Jordan curve theorem applied to Γ and the obvious positions of C_1 and B relative to Γ (including the case $B \in T$).

Case 3. $a \leq x_l < b < x_r$.

The proof of Case 3 is completely analogous to Case 2.

We are now in a position to prove the limit cycle is unique. If it is not, suppose there are two limit cycles: Γ and Γ′, with $\Gamma \subset \text{int}(\Gamma')$ (see Fig. 3).

Let us compute the integrals $\oint_\Gamma dl$ and $\oint_{\Gamma'} dl$. As shown in Fig. 3,

$$\Gamma = \overset{\frown}{EF} \cup \overset{\frown}{FG} \cup \overset{\frown}{GH} \cup \overset{\frown}{HE}$$

and

$$\Gamma' = \overset{\frown}{E'F''} \cup \overset{\frown}{F''F'} \cup \overset{\frown}{F'G'} \cup \overset{\frown}{G'G''} \cup \overset{\frown}{G''H''} \cup \overset{\frown}{H''H'} \cup \overset{\frown}{H'E'} \cup \overset{\frown}{E'E''}.$$

Since $\Gamma \subset \text{int}(\Gamma')$, for the same y, let $(x_1, y) \in \Gamma$, $(x_2, y) \in \Gamma'$, we have

$$|x_1| < |x_2|. \tag{11}$$

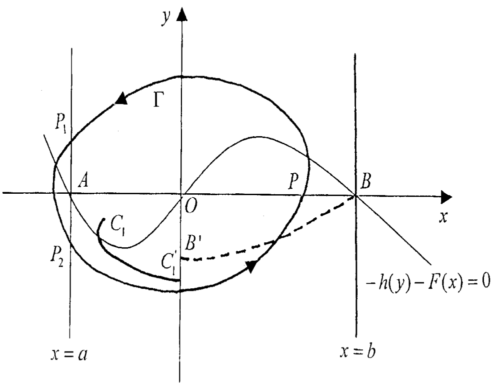

FIG. 2. It is impossible that $x_r \leq b$.

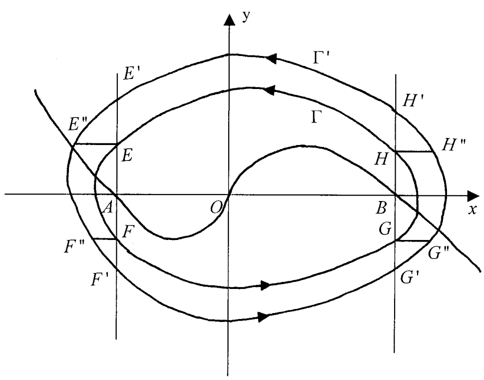

FIG. 3. $\oint_{\Gamma'} dl > \oint_\Gamma dl$ cannot be true.

By the fact that $F(x) > 0$, $dy < 0$, and $-F(x)$ is decreasing on $x \in (-N, a)$, and (2),

$$\int_{\overset{\frown}{E'F''}} dl = \int_{\overset{\frown}{E'F''}} -F(x)dy > \int_{\overset{\frown}{EF}} -F(x)dy = \int_{\overset{\frown}{EF}} dl. \tag{12}$$

Similarly,

$$\int_{\overset{\frown}{G'H''}} dl > \int_{\overset{\frown}{GH}} dl. \tag{13}$$

Also, since (a) $g(x)F(x) < 0$ on $x \neq 0$ and $x \in (a, b)$, (b) on the trajectory curves: $\Gamma'_{H'E'}$ and Γ_{HE}, we have $dx < 0$, $-h(y) - F(x) < 0$, and (c) the fact that $h(y)$ is increasing and for the same x, the y coordinate in $\Gamma'_{H'E'}$ is bigger than the one in G_{HE}.

$$\int_{\overset{\frown}{H'E'}} dl = \int_{\overset{\frown}{H'E'}} \frac{-F(x)g(x)}{-h(y) - F(x)} dx$$

$$> \int_{\overset{\frown}{HE}} \frac{-F(x)g(x)}{-h(y) - F(x)} dx \tag{14}$$

$$= \int_{\overset{\frown}{HE}} dl.$$

Similarly, we can prove

$$\int_{\overset{\frown}{F'G'}} dl > \int_{\overset{\frown}{FG}} dl. \tag{15}$$

Considering the fact that integrals of dl along with $\overset{\frown}{E'E''}$, $\overset{\frown}{F'F'}$, $\overset{\frown}{G'G''}$, and $\overset{\frown}{H'H''}$ are all positive, we have

$$\int_{\Gamma'} dl > \int_{\Gamma} dl. \tag{16}$$

This is impossible because both $\oint_{\Gamma'} dl$ and $\oint_{\Gamma} dl$ are zero. This proves that there is at most one limit cycle in system (6) if one of the conditions of Theorem 1 is satisfied. By the fact that (0, 0) is an unstable equilibrium, the limit cycle is stable if it exists. The proof of Theorem 1 is complete. $\blacksquare$

When $h(y) = y$ system (6) is reduced to the Liénard system (3) or (4). The above results can be summarized as

Theorem 2. If, in addition to the existence conditions (A_2) and (A_3),

(H_1) $xg(x) > 0$, $(x \neq 0)$; and there exist $a < 0 < b$, N sufficiently large, such that $xF(x) < 0$ for $x \in (a, b)$, $x \neq 0$, and $xF(x) < 0$, $F(x)$ is increasing for $x \in (-N, a)$ and $x \in (b, N)$;

(H_2) one of the following holds
1) $G(b) = G(a)$,

2) $G(b) > G(a)$ and there exists $x' \in (a, 0)$ such that $F(x') \geq \sqrt{2G(b)}$,

3) $G(b) < G(a)$ and there exists $x'' \in (0, b)$ such that $F(x'') \leq -\sqrt{2G(a)}$,

4) there exists $x' \in (a, 0)$ and $x'' \in (0, b)$ such that $F(x') \geq \sqrt{2G(b)}$ and $F(x'') \leq -\sqrt{2G(a)}$;

then the Liénard system (3) or (4) has a unique limit cycle.

3. Examples

Let us use some examples to show the applicability of our results.

Example 1. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= -y - F(x) \\ \frac{dy}{dt} &= 2x \end{aligned} \tag{17}$$

where

$$F(x) = \begin{cases} x^2(x-1) & \text{if } x \geq 0 \\ x^4(x+1) & \text{if } x < 0. \end{cases}$$

It is not difficult to see that the conditions (H_1), $(H_2 - 1)$ ($a = -1$, $b = 1$) in Theorem 2 are satisfied, and hence there is a unique limit cycle in (20).

However,

$$\frac{F'(x)}{g(x)} = \frac{5}{2}x^3 + 2x^2, \quad \text{if} \quad x < 0. \tag{18}$$

Therefore, $\frac{F'(x)}{g(x)}$ is decreasing on $-\frac{5}{8} < x < 0$, and hence Zhang's theorem [4] and Cherkas and Zhilevich's theorem [5] are not applicable in system (17).

Example 2.

$$\begin{aligned} \frac{dx}{dy} &= -\frac{3y^2}{1+y^2} - x\left(x + \frac{1}{3}\right)(x-1) \\ \frac{dy}{dt} &= \frac{2x}{1+x^4}. \end{aligned} \tag{19}$$

System (19) has a unique limit cycle because the condition (ii) of Theorem 1 (with $a = -\frac{1}{3}$, $x' = -\frac{1}{4}$, $b = y' = 1$) is satisfied. But

$$G(\pm\infty) = \frac{p}{2} < +\infty,$$

and hence Zhang's theorem [4], and Cherkas and Zhilevich's theorem [5] cannot be employed here either.

Example 3.

$$\begin{aligned} \frac{dx}{dt} &= -\frac{1}{5}y - x(x-1)(x+1) \\ \frac{dy}{dt} &= x^3. \end{aligned} \tag{20}$$

System (20) has a unique limit cycle since the assumptions (A_1), (A_2), (A_3) and (i) in Theorem 1 are satisfied. However, the condition $\frac{d}{dx}\left(\frac{F'(x)}{g(x)}\right) \geq 0$ fails on $x \in (-\infty, 0)$ and $x \in (0, +\infty)$, and thus we cannot use Zhang's theorem in proving the uniqueness of limit cycle of system (20).

Regarding the existence conditions of limit cycles of systems (3) and (4), there are more results in the monograph of Chen and Chen [8].

4. Conclusions and discussion

Looking for the number of limit cycles for polynomial differential systems has been considered an outstanding problem in ordinary differential equation since the Hilbert 23 problems in 1900 (see Ye, *et al*. [2]). And, in mathematical modeling of ecological systems, after the papers of May [22], and Albrecht *et al.* [23], determining the conditions that guarantee the uniqueness of limit cycles in predator-prey and related systems becomes a primary problem. Over the last century, the uniqueness of limit cycles in quadratic and ecological systems has been investigated very thoroughly (see, for example, [2–16, 22, 23]). Most of the results regarding this topic are based on the uniqueness of limit cycles for Liénard-type systems, which was first obtained by Zhang in her PhD dissertation in Russian in 1958 and then in an English paper in 1986 (Zhang [4]). The basic idea of Zhang is the comparison of integrals of the divergences. Since in the applications, conditions of Zhang's theorem are not always satisfied, so looking for an alternate approach is necessary.

As mentioned in Section 1, Liénard equation (2) can be written as Liénard systems (3) and (4), and can be generalized to Liénard-type systems (5) and (6). Systems (3) and (4) are equivalent but systems (5) and (6) are not since $h(y)$ may not be an odd function. Huang and Sun [6] proposed some uniqueness conditions for system (5). In this paper, we applied the same technique as the one for system (5) in Huang and Sun [6] to system (6). We obtained some new conditions that guarantee the uniqueness of limit cycles for the generalized Liénard system (6). The results in this paper are for a different generalized Liénard system as studied in Huang and Sun [6], and are different from those of Zhang [4] and Cherkas and Zhilevich [5].

Oscillation phenomena in chemical reaction, ecological, and physical systems, etc. is very complicated, and all the existing theorems for the uniqueness of limit cycles have their limitations. Therefore, finding new criteria or new methods in this direction is always useful in both theory and applications.

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