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ELASTIC BEHAVIOUR OF MATTER UNDER VERY HIGH PRESSURES

General Deformation

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ABSTRACT

Expressions have been derived for the effective elastic constants in respect of a substance possessing initial cubic symmetry and subjected to a general type of finite deformation. Some special cases of particular interest are deduced therefrom by introducing suitable relationships between the various parameters. Results for the simple case of Uniform Compression dealt with in an earlier paper, easily follow.

INTRODUCTION

A method of evaluating the effective elastic constants from the expression for the strain energy, utilizing the theory of non-linear elasticity, has been given in detail in a previous paper by us (1960). In that paper, the special case of uniform hydrostatic stress applied to a substance with cubic symmetry was considered. In the present paper, the initial finite deformation will be assumed to be of a general type, and a general infinitesimal deformation will be superposed on the same. The effective elastic constants for such a case are derived in terms of the second and third order elastic constants of the substance in the stress free state, so as to include up to the second powers of the initial finite strain components. By giving particular values to these components, the effective elastic constants appropriate to (1) a triaxial strain, (2) a uniaxial strain and (3) a shear are derived. It has also been indicated that several other types of strains or combinations of strains could be easily dealt with as particular cases of this general treatment.

STRAIN MATRIX FOR A GENERAL DEFORMATION

For the notation and other details of the method employed, reference should be made to the paper cited earlier. Here, the application of the method to the general case is directly worked out

As a result of the general finite deformation let a point with co-ordinates a, b, c referred to a convenient space fixed axes be carried over to the position x_0, y_0, z_0 referred, for convenience to the same axes, so that

$$x_0 = J_0\, a\,; \quad x_0 = x_0, y_0, z_0\,; \quad a = a, b, c \qquad [1]$$

and J_0 is the Jacobian of the transformation with six independent components as given by [2].

$$J_0 = \begin{vmatrix} 1+\theta_1 & \theta_6 & \theta_5 \\ \theta_6 & 1+\theta_2 & \theta_4 \\ \theta_5 & \theta_4 & 1+\theta_3 \end{vmatrix} \qquad [2]$$

This entails that

$$\begin{aligned} x_0 &= (1+\theta_1)\, a + \theta_6\, b + \theta_5\, c \\ y_0 &= \theta_6\, a + (1+\theta_2)\, b + \theta_4\, c \\ z_0 &= \theta_5\, a + \theta_4\, b + (1+\theta_3)\, c \end{aligned} \qquad [3]$$

We have assumed that J_0 is symmetric. In that form, it is not the most general displacement, as the case of a simple shear is not covered. The separation of the rotational or antisymmetric part of J_0 will be valid so long as infinitesimal deformations are considered. In the finite theory, the strain matrix elements are given by $\eta = \frac{1}{2}(J^* J - E_3)$ where J^* is the transpose of J and E_3 is the unit matrix of dimension 3. The rotational part cannot be removed from J itself, but will get so removed after $J^* J$ is formed and E_3 is subtracted therefrom. To begin with therefore, one has to start with an unsymmetrical J with nine independent components if one wants to cover the most general cases. However, the resulting calculations become rather unwieldy and it does not seem worthwhile to undertake such calculations for our present purpose. The case of a simple shear alone will be treated separately in another paper as a particular case.

The strain matrix η corresponding to J_0 will have elements given by

$$\begin{aligned} \eta_1 &= \theta_1 + \tfrac{1}{2}(\theta_1^2 + \theta_5^2 + \theta_6^2) & \eta_4 &= \theta_4 + \tfrac{1}{2}(\theta_2\theta_4 + \theta_3\theta_4 + \theta_5\theta_6) \\ \eta_2 &= \theta_2 + \tfrac{1}{2}(\theta_2^2 + \theta_6^2 + \theta_4^2) & \eta_5 &= \theta_5 + \tfrac{1}{2}(\theta_1\theta_5 + \theta_3\theta_5 + \theta_4\theta_6) \\ \eta_3 &= \theta_3 + \tfrac{1}{2}(\theta_3^2 + \theta_4^2 + \theta_5^2) & \eta_6 &= \theta_6 + \tfrac{1}{2}(\theta_1\theta_6 + \theta_2\theta_6 + \theta_4\theta_5) \end{aligned} \qquad [4]$$

As a result of the further infinitesimal deformation let the point (x_0, y_0, z_0) move to x, y, z where

$$\begin{aligned} x &= (1+\delta_1)\,x_0 + \delta_6 y_0 + \delta_5 z_0 \\ y &= \delta_6 x_0 + (1+\delta_2)\,y_0 + \delta_4 z_0 \quad ; \quad J_\delta = \begin{vmatrix} 1+\delta_1 & \delta_6 & \delta_5 \\ \delta_6 & 1+\delta_2 & \delta_4 \\ \delta_5 & \delta_4 & 1+\delta_3 \end{vmatrix} \\ z &= \delta_5 x_0 + \delta_4 y_0 + (1+\delta_3)\,z_0 \end{aligned} \qquad [5]$$

so that

$$x = J_\delta x_0\,, \quad \begin{pmatrix} x = x,\ y,\ z \\ x_0 = x_0,\ y_0,\ z_0 \end{pmatrix}$$

The $\delta's$ express the usual changes in length and changes in angles, etc , all referred to the already finitely strained state as the base state which we may refer to as the η state. As a result of these two deformations J_0 and J_δ, the point a, b, c will finally go over to (x, y, z) where

$$x = Ja\,, \quad J = J_\delta J_0 \qquad [6]$$

The strain matrix elements of this combined deformation will be given by $\eta = \frac{1}{2}(J^* J - E_3)$ Noting that

$$(J_\delta J_0)^* = J_0^* J_\delta^*\ ; \quad \text{we get} \quad \eta = \tfrac{1}{2}\,[J_0^* J_\delta^* J_\delta J_0 - E_3] \qquad [7]$$

$\frac{1}{2}(J_\delta^* J_\delta - E_3)$ gives the elements of the strain matrix corresponding to a general infinitesimal deformation. Either from the form of [4] or by direct working, we obtain its components as

$$\begin{aligned} \delta\eta_1 &= \delta_1 + \tfrac{1}{2}(\delta_1^2 + \delta_5^2 + \delta_6^2) & \delta\eta_4 &= \delta_4 + \tfrac{1}{2}(\delta_2\delta_4 + \delta_3\delta_4 + \delta_5\delta_6) \\ \delta\eta_2 &= \delta_2 + \tfrac{1}{2}(\delta_2^2 + \delta_6^2 + \delta_4^2) & \delta\eta_5 &= \delta_5 + \tfrac{1}{2}(\delta_1\delta_5 + \delta_3\delta_5 + \delta_4\delta_6) \\ \delta\eta_3 &= \delta_3 + \tfrac{1}{2}(\delta_3^2 + \delta_4^2 + \delta_5^2) & \delta\eta_6 &= \delta_6 + \tfrac{1}{2}(\delta_1\delta_6 + \delta_2\delta_6 + \delta_4\delta_5) \end{aligned} \qquad [8]$$

In terms of these components, the strain matrix [7] corresponding to the entire deformation, (*i e.*, finite plus infinitesimal) will be given by

$$\eta = \begin{vmatrix} 1+\theta_1 & \theta_6 & \theta_5 \\ \theta_6 & 1+\theta_2 & \theta_4 \\ \theta_5 & \theta_4 & 1+\theta_3 \end{vmatrix} \begin{Vmatrix} \delta\eta_1 + \frac{1}{2} & \delta\eta_6 & \delta\eta_5 \\ \delta\eta_6 & \delta\eta_2 + \frac{1}{2} & \delta\eta_4 \\ \delta\eta_5 & \delta\eta_4 & \delta\eta_3 + \frac{1}{2} \end{Vmatrix} \begin{vmatrix} 1+\theta_1 & \theta_6 & \theta_5 \\ \theta_6 & 1+\theta_2 & \theta_4 \\ \theta_5 & \theta_4 & 1+\theta_3 \end{vmatrix} - \frac{E_3}{2} \qquad [9]$$

Consequent on the δ deformation therefore, each of the strain elements $\eta_1 \ldots \eta_6$ in [4] has increased by an amount $\Delta\eta_1, \ldots \Delta\eta_6$, whose values are given from [9] by

$$\Delta\eta_1 = \delta\eta_1(1+\theta_1)^2 + \delta\eta_2\theta_6^2 + \delta\eta_3\theta_5^2 + 2\delta\eta_4\theta_5\theta_6 + 2\delta\eta_5(1+\theta_1)\,\theta_5 + 2\delta\eta_6(1+\theta_1)\theta_6$$

$$\Delta\eta_2 = \delta\eta_2(1+\theta_2)^2 + \delta\eta_3\theta_4^2 + \delta\eta_1\theta_6^2 + 2\delta\eta_5\theta_6\theta_4 + 2\delta\eta_6(1+\theta_2)\,\theta_6 + 2\delta\eta_4(1+\theta_2)\theta_4$$

$$\Delta\eta_3 = \delta\eta_3(1+\theta_3)^2 + \delta\eta_1\theta_5^2 + \delta\eta_2\theta_4^2 + 2\delta\eta_6\theta_4\theta_5 + 2\delta\eta_4(1+\theta_3)\,\theta_4 + 2\delta\eta_5(1+\theta_3)\theta_5$$

$$\Delta\eta_4 = \delta\eta_4(1+\theta_2+\theta_3+\theta_2\theta_3+\theta_4^2) + \delta\eta_5(\theta_6+\theta_3\theta_6+\theta_4\theta_5) + \delta\eta_6(\theta_5+\theta_2\theta_5+\theta_4\theta_6)$$
$$+\,\delta\eta_1\theta_5\theta_6 + \delta\eta_2(1+\theta_2)\theta_4 + \delta\eta_3(1+\theta_3)\theta_4$$

$$\Delta\eta_5 = \delta\eta_5(1+\theta_3+\theta_1+\theta_3\theta_1+\theta_5^2) + \delta\eta_6(\theta_4+\theta_1\theta_4+\theta_5\theta_6) + \delta\eta_4(\theta_6+\theta_3\theta_6+\theta_5\theta_4)$$
$$+\,\delta\eta_2\theta_6\theta_4 + \delta\eta_3(1+\theta_3)\theta_5 + \delta\eta_1(1+\theta_1)\theta_5$$

$$\Delta\eta_6 = \delta\eta_6(1+\theta_1+\theta_2+\theta_1\theta_2+\theta_6^2) + \delta\eta_4(\theta_5+\theta_2\theta_5+\theta_6\theta_4) + \delta\eta_5(\theta_4+\theta_1\theta_4+\theta_6\theta_5)$$
$$+\,\delta\eta_3\theta_4\theta_5 + \delta\eta_1(1+\theta_1)\theta_6 + \delta\eta_2(1+\theta_2)\theta_6 \qquad [10]$$

It may be noted here that $\Delta\eta_2$ can be obtained from $\Delta\eta_1$, and $\Delta\eta_5$ from $\Delta\eta_4$ by the mere cyclic permutation $S = (1\ 2\ 3)\ (4\ 5\ 6)$ of the various suffixes attached to θ and $\delta\eta$. Subsequently, the same permutation can be used to give $\Delta\eta_3$ from $\Delta\eta_2$ and $\Delta\eta_6$ from $\Delta\eta_5$. This will be found a useful device for simplifying the calculations.

STRAIN ENERGY AND THE EFFECTIVE ELASTIC CONSTANTS

The general expression for the strain energy for a substance with cubic symmetry, inclusive of second and third order terms is given, when referred to a unit volume of the initial stress free state, by

$$\phi = \tfrac{1}{2}c_{11}(\eta_1^2+\eta_2^2+\eta_3^2) + c_{12}(\eta_1\eta_2+\eta_2\eta_3+\eta_3\eta_1) + 2c_{44}(\eta_4^2+\eta_5^2+\eta_6^2)$$
$$+\,C_{111}(\eta_1^3+\eta_2^3+\eta_3^3) + C_{112}\{\eta_1\eta_2(\eta_1+\eta_2)+\eta_2\eta_3(\eta_2+\eta_3)+\eta_3\eta_1(\eta_3+\eta_1)\}$$
$$+\,C_{123}\eta_1\eta_2\eta_3 + C_{456}\eta_4\eta_5\eta_6 + C_{144}(\eta_1\eta_4^2+\eta_2\eta_5^2+\eta_3\eta_6^2)$$
$$+\,C_{155}\{\eta_1(\eta_5^2+\eta_6^2)+\eta_2(\eta_4^2+\eta_6^2)+\eta_3(\eta_4^2+\eta_5^2)\} \qquad [11]$$

If as a result of the infinetesimal deformation, the original finite strains $\eta_1 \ldots \eta_6$ have become $\eta_1+\Delta\eta_1 \ldots \eta_6+\Delta\eta_6$, the increase in energy due to the same, is given by the expression $\phi(\eta+\Delta\eta)-\phi(\eta)$, which can be developed as a power series in the $\Delta\eta$ using Taylor's theorem.

$$\phi(\eta+\Delta\eta)-\phi(\eta) = \sum\frac{\partial\phi}{\partial\eta_r}\Delta\eta_r + \frac{1}{2}\sum\frac{\partial^2\phi}{\partial\eta_r\partial\eta_s}\Delta\eta_r\Delta\eta_s + \ldots \qquad [12]$$

We retain up to quadratic terms only in the $\Delta\eta$, as they are treated as infinitesimal. Here the derivatives

$$\frac{\partial\phi}{\partial\eta_r},\ \frac{\partial^2\phi}{\partial\eta_r\partial\eta_s}$$

can be easily obtained from [11] and they are to be evaluated at $\Delta\eta = 0$. The values are given in [15]. With the use of equations [10], which express $\Delta\eta$ as a linear function of $\delta\eta$, we can express $\phi(\eta+\Delta\eta)-\phi(\eta)$ given above as a quadratic function in $\delta\eta$. This difference in energy however refers to a unit volume of the stress free state Dividing it by the determinant of J_0 expresses it per unit volume of the η state. Equating finally with ϕ', where ϕ' is the total energy of the infinitesimal deformation, we get expressions for all the coefficients in ϕ' by a comparison of the coefficients of similar powers. Thus

$$\frac{1}{\text{Det } J_0}\left[\phi(\eta+\Delta\eta)-\phi(\eta)\right]=\phi'(\delta\eta) \qquad [13]$$

where

$$\phi'(\delta\eta)=\Sigma c_1'\delta\eta_1+2\,\Sigma c_4'\,\delta\eta_4+\tfrac{1}{2}\,\Sigma b_{11}\delta\eta_1^2+\Sigma b_{12}\delta\eta_1\delta\eta_2+2\Sigma b_{44}\delta\eta_4^2$$
$$+\,2\Sigma b_{45}\delta\eta_4\delta\eta_5+2\Sigma b_{14}\delta\eta_1\delta\eta_4$$

$$\frac{1}{\text{Det } J_0}=1-\theta_1-\theta_2-\theta_3+\theta_1^2+\theta_2^2+\theta_3^2+\theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_1+\theta_4^2+\theta_5^2+\theta_6^2 \qquad [14]$$

It will be noted here that c_1', c_4', etc , are coefficients of the linear terms, while b_{11}, etc., are the coefficients of the quadratic terms in ϕ'. They can be represented by b_{rs} $(r\leq s;\ r, s=1, 2 \ldots 6)$ and are thus 21 in all. It will also be observed that except for b_{11}, b_{22}, b_{33}, b_{12}, b_{23}, b_{13}, every other b_{rs} occurs with a multiplying factor of 2. In ϕ', we retain only upto second powers of $\delta\eta$. On the left side, we retain all terms up to the squares and products of the finite strain derivatives $\theta_1 \ldots \theta_6$ (i e) the cubic powers of these are neglected. The values of

$$\frac{\partial\phi}{\partial\eta_r},\ \frac{\partial^2\phi}{\partial\eta_r\,\partial\eta_s}$$

used in [12] are given below.

$$\frac{\partial\phi}{\partial\eta_1}=c_{11}\eta_1+c_{12}(\eta_2+\eta_3)+3C_{111}\eta_1^2+C_{112}(\eta_2^2+\eta_3^2)+2C_{112}\eta_1(\eta_2+\eta_3)$$
$$+\,C_{123}\eta_2\eta_3+C_{144}\eta_4^2+C_{155}(\eta_5^2+\eta_6^2)$$

$$\frac{\partial\phi}{\partial\eta_2}=c_{11}\eta_2+c_{12}(\eta_3+\eta_1)+3C_{111}\eta_2^2+C_{112}(\eta_3^2+\eta_1^2)+2C_{112}\eta_2(\eta_3+\eta_1)$$
$$+\,C_{123}\eta_3\eta_1+C_{144}\eta_5^2+C_{155}(\eta_6^2+\eta_4^2)$$

$$\frac{\partial\phi}{\partial\eta_3}=c_{11}\eta_3+c_{12}(\eta_1+\eta_2)+3C_{111}\eta_3^2+C_{112}(\eta_1^2+\eta_2^2)+2C_{112}\eta_3(\eta_1+\eta_2)$$
$$+\,C_{123}\eta_1\eta_2+C_{144}\eta_6^2+C_{155}(\eta_4^2+\eta_5^2)$$

$$\frac{\partial\phi}{\partial\eta_4}=4c_{44}\eta_4+C_{456}\eta_5\eta_6+2C_{144}\eta_1\eta_4+2C_{155}(\eta_2+\eta_3)\eta_4$$

$$\frac{\partial\phi}{\partial\eta_5}=4c_{44}\eta_5+C_{456}\eta_6\eta_4+2C_{144}\eta_2\eta_5+2C_{155}(\eta_3+\eta_1)\eta_5$$

$$\frac{\partial\phi}{\partial\eta_6} = 4c_{44}\eta_6 + C_{456}\eta_4\eta_5 + 2C_{144}\eta_3\eta_6 + 2C_{155}\left(\eta_1 + \eta_2\right)\eta_6$$

$$\frac{\partial^2\phi}{\partial\eta_1^2} = c_{11} + 6C_{111}\eta_1 + 2C_{112}\left(\eta_2 + \eta_3\right) \qquad \frac{\partial^2\phi}{\partial\eta_4^2} = 4c_{44} + 2C_{144}\eta_1 + 2C_{155}\left(\eta_2 + \eta_3\right)$$

$$\frac{\partial^2\phi}{\partial\eta_2^2} = c_{11} + 6C_{111}\eta_2 + 2C_{112}\left(\eta_3 + \eta_1\right) \qquad \frac{\partial^2\phi}{\partial\eta_5^2} = 4c_{44} + 2C_{144}\eta_2 + 2C_{155}\left(\eta_3 + \eta_1\right)$$

$$\frac{\partial^2\phi}{\partial\eta_3^2} = c_{11} + 6C_{111}\eta_3 + 2C_{112}\left(\eta_1 + \eta_2\right) \qquad \frac{\partial^2\phi}{\partial\eta_6^2} = 4c_{44} + 2C_{144}\eta_3 + 2C_{155}\left(\eta_1 + \eta_2\right)$$

$$\frac{\partial^2\phi}{\partial\eta_4\partial\eta_5} = C_{456}\eta_6\,; \qquad \frac{\partial^2\phi}{\partial\eta_5\partial\eta_6} = C_{456}\eta_4\,; \qquad \frac{\partial^2\phi}{\partial\eta_6\partial\eta_4} = C_{456}\eta_5$$

$$\frac{\partial^2\phi}{\partial\eta_1\partial\eta_2} = c_{12} + 2C_{112}\left(\eta_1 + \eta_2\right) + C_{123}\eta_3\,; \qquad \frac{\partial^2\phi}{\partial\eta_1\partial\eta_4} = 2C_{144}\eta_4$$

$$\frac{\partial^2\phi}{\partial\eta_2\partial\eta_4} = \frac{\partial^2\phi}{\partial\eta_3\partial\eta_4} = 2C_{155}\eta_4\,; \qquad \frac{\partial^2\phi}{\partial\eta_2\partial\eta_3} = c_{12} + 2C_{112}\left(\eta_2 + \eta_3\right) + C_{123}\eta_1$$

$$\frac{\partial^2\phi}{\partial\eta_2\partial\eta_5} = 2C_{144}\eta_5\,; \qquad \frac{\partial^2\phi}{\partial\eta_1\partial\eta_5} = \frac{\partial^2\phi}{\partial\eta_3\partial\eta_5} = 2C_{155}\eta_5$$

$$\frac{\partial^2\phi}{\partial\eta_3\partial\eta_1} = c_{12} + 2C_{112}\left(\eta_3 + \eta_1\right) + C_{123}\eta_2\,; \qquad \frac{\partial^2\phi}{\partial\eta_3\partial\eta_6} = 2C_{144}\eta_6\,,$$

$$\frac{\partial^2\phi}{\partial\eta_1\partial\eta_6} = \frac{\partial^2\phi}{\partial\eta_2\partial\eta_6} = 2C_{155}\eta_6 \qquad [15]$$

Here $\eta_1 \ldots \eta_6$ are elements of the strain matrix up to the initial finite strains. They are functions of $\theta_1 \ldots \theta_6$ as given in [4]. Substituting these values in [13], and equating powers of $\delta\eta_1$, $\delta\eta_1^2$, $\delta\eta_1\,\delta\eta_2$, etc., we get the values of the first order coefficients c_1', $c_2' \ldots c_6'$ as also the values of all the b_{rs} ($r, s = 1$ to 6).

ϕ' is so far expressed in terms of the strain components $\delta\eta$. It can now be expressed in terms of the displacement derivatives $\delta_1 \ldots \delta_6$ by using relations [8]. Retaining up to the quadratic terms in the δ's, we can write ϕ' as

$$\phi' = \Delta W + \phi_e \qquad [16]$$

where $\phi_e = \frac{1}{2}\,\Sigma c_{11}'\,\delta_1^2 + \Sigma c_{12}'\,\delta_1\delta_2 + 2\Sigma c_{44}'\,\delta_4^2 + 2\Sigma c_{45}'\,\delta_4\delta_5 + 2\Sigma c_{14}'\,\delta_1\delta_4$ [17]

and $\Delta W = c_1'\,\delta_1 + c_2'\,\delta_2 + c_3'\,\delta_3 + 2c_4'\,\delta_4 + 2c_5'\,\delta_5 + 2c_6'\,\delta_6$ [18]

and c_{11}', c_{12}', etc., are new constants connected by simple relationships with b_{11}, b_{12}, etc., and with the first order constants $c_1' \ldots c_6'$. It has been shown in the previous paper of the authors referred to earlier that c_1', $c_2' \ldots c_6'$ represent the components T_1, $T_2 \ldots T_6$ of the initial stress. ΔW therefore represents the work done by these initially present stresses during the infinitesimal displacement $\delta_1 \ldots \delta_6$. ϕ' represents the total energy required for causing the additional infinitesimal deformation, while ΔW indicates the part already available for this purpose in the shape of work done by the existing initial stresses. The difference $\phi' - \Delta W$ therefore represents the effective extra energy required for causing the infinitesimal deformation and this quantity has been denoted by ϕ_e and referred to as the 'effective elastic energy' in our earlier paper. The c'_{11}, c'_{12}, etc, occurring in ϕ_e are called effective elastic constants and they completely specify the elastic response of the substance to infinitesimal deformations superposed on the existing state of a general finite strain. It has also been indicated in the earlier paper that ϕ_e and the effective elastic constants play an important role in determining the elastic stability of the substance. We now proceed to give expressions for these effective constants. The coefficients $c'_1 \ldots c'_6$ and b_{11}, b_{12}, etc, occurring in ϕ' are already determined as mentioned in an earlier paragraph. On writing ϕ' in terms of $\delta_1 \ldots \delta_6$ and comparing with ϕ_e we get simple relationships between the effective elastic constants c'_{rs} and the constants b_{rs} and c'_r. The results, in respect of c'_r and c'_{rs} are given in Table I. Although b_{rs} are first directly determined and c'_{rs} are subsequently deduced therefrom, we give here, for the sake of simplicity values of c'_r and c'_{rs} only. It will be noted that all these coefficients are developed up to the second powers of the initial strain components. The task of deriving all the coefficients is simplified when we observe as before that the permutation $S = (\theta_1 \theta_2 \theta_3)(\theta_4 \theta_5 \theta_6)$ and S^2 can be employed to derive two constants similar to each b_{rs}, c'_r and c'_{rs}. Thus one needs to evaluate directly only 7 of the b_{rs} and 2 of the c'_r.

The values of the effective elastic constants in the special case of a triaxial strain are given in Table II. In Table III, we give sub-divisions of this case, which correspond to [1] a uniaxial compression and [2] uniform compression In Table IV, we consider the case of a pure shear.

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Reference

1. Bhagavantam S. and Chelam E V. .. *Proc. Ind Acad Sci.*, 1960, 52, Sec. A, 1-19.

TABLE I GENERAL DEFORMATION

(A) *Expressions for initial stress*:

$$T_1 = c_1' = \theta_1 c_{11} + (\theta_2 + \theta_3) c_{12} + \theta_1^2 (3C_{111} + \tfrac{3}{2} c_{11}) + (\theta_2^2 + \theta_3^2)(C_{112} - \tfrac{1}{2} c_{12})$$
$$+ \theta_4^2 (c_{12} + C_{144}) + (\theta_5^2 + \theta_6^2)(\tfrac{1}{2} c_{11} + \tfrac{1}{2} c_{12} + 4c_{44} + C_{155})$$
$$+ (\theta_1\theta_2 + \theta_1\theta_3)(2C_{112} - c_{11} + c_{12}) + \theta_2\theta_3 (C_{123} - 2c_{12})$$

$T_2 = c_2'$ obtained from $T_1 = c_1'$ by the cyclic permutation $S \cdot (\theta_1\theta_2\theta_3)(\theta_4\theta_5\theta_6)$
Thus $T_2 = ST_1$; $T_3 = ST_2 = S^2 T_1$

$$T_4 = c_4' = \theta_4 2c_{44} + \theta_1\theta_4 (2c_{12} - 2c_{44} + C_{144}) + (\theta_2\theta_4 + \theta_3\theta_4)(c_{11} + c_{12} + c_{44} + C_{155})$$
$$+ \theta_5\theta_6 (\tfrac{1}{2} C_{456} + 5c_{44})$$

$$T_5 = c_5' = ST_4 ; \quad T_6 = c_6' = ST_5 = S^2 T_4$$

(B) *Effective elastic constants*:

$$c_{11}' = b_{11} + c_1' = c_{11} + \theta_1 (4c_{11} + 6C_{111}) + (\theta_2 + \theta_3)(c_{12} - c_{11} + 2C_{112})$$
$$+ \theta_1^2 (\tfrac{9}{2} c_{11} + 24C_{111}) + (\theta_2^2 + \theta_3^2)(c_{11} - \tfrac{1}{2} c_{12}) + \theta_4^2 (c_{11} + c_{12} + 2C_{112} + C_{144})$$
$$+ (\theta_5^2 + \theta_6^2)(\tfrac{3}{2} c_{11} + \tfrac{5}{2} c_{12} + 8c_{44} + C_{112} + 5C_{155} + 3C_{111})$$
$$+ (\theta_1\theta_2 + \theta_1\theta_3)(c_{12} - 4c_{11} - 6C_{111} + 8C_{112}) + \theta_2\theta_3 (c_{11} - 2c_{12} + C_{123} - 4C_{112}$$

$$c_{22}' = Sc_{11}'; \quad c_{33}' = S^2 c_{11}'$$

$$c_{44}' = b_{44} + \tfrac{1}{4} c_2' + \tfrac{1}{4} c_3' = c_{44} + \theta_1 (\tfrac{1}{2} c_{12} - c_{44} + \tfrac{1}{2} C_{144})$$
$$+ (\theta_2 + \theta_3)(\tfrac{1}{4} c_{11} + \tfrac{3}{4} c_{12} + c_{44} + \tfrac{1}{2} C_{155}) + \theta_1^2 (c_{44} - \tfrac{1}{4} c_{12} + \tfrac{1}{2} C_{112} - \tfrac{1}{4} C_{144})$$
$$+ (\theta_2^2 + \theta_3^2)(\tfrac{3}{8} c_{11} - \tfrac{1}{8} c_{12} + \tfrac{3}{4} C_{111} + \tfrac{1}{4} C_{112} + \tfrac{3}{4} C_{155})$$
$$+ \theta_4^2 (\tfrac{9}{4} c_{11} + \tfrac{9}{4} c_{12} + 5c_{44} + 5C_{155})$$
$$+ (\theta_5^2 + \theta_6^2)(\tfrac{1}{8} c_{11} + \tfrac{3}{8} c_{12} + 3c_{44} + \tfrac{1}{2} C_{144} + \tfrac{1}{2} C_{155} + \tfrac{1}{2} C_{456})$$
$$+ (\theta_1\theta_2 + \theta_1\theta_3)(\tfrac{1}{2} C_{112} + \tfrac{1}{4} C_{123} + \tfrac{1}{2} C_{144} - \tfrac{1}{2} C_{155} - \tfrac{1}{4} c_{11} - \tfrac{1}{4} c_{12} - c_{44})$$
$$+ \theta_2\theta_3 (C_{112} + C_{155} - \tfrac{1}{2} c_{11} + \tfrac{1}{2} c_{12} + c_{44})$$

$$c_{55}' = Sc_{44}'; \quad c_{66}' = S^2 c_{44}'$$

$$c_{12}' = b_{12} = c_{12} + (\theta_1 + \theta_2)(c_{12} + 2C_{112}) + \theta_3 (C_{123} - c_{12})$$
$$+ (\theta_1^2 + \theta_2^2) 3C_{112} + \theta_3^2 (c_{12} - \tfrac{1}{2} C_{123}) + (\theta_4^2 + \theta_5^2)(2c_{12} + C_{112} + 2C_{144} + \tfrac{1}{2} C_{123}$$
$$+ \theta_6^2 (2c_{11} + c_{12} + 4c_{44} + 2C_{112} + 4C_{155}) + \theta_1\theta_2 (c_{12} + 4C_{112})$$
$$+ (\theta_2\theta_3 + \theta_1\theta_3)(C_{123} - c_{12} - 2C_{112})$$

$$c'_{23} = Sc'_{12}; \quad c'_{13} = S^2 c'_{12}$$

$$\begin{aligned} c'_{45} = b_{45} + \tfrac{1}{2} c'_6 = \theta_6 (5c_{44} + \tfrac{1}{2} C_{456}) \\ + (\theta_1\theta_6 + \theta_2\theta_6)(\tfrac{1}{2} c_{11} + \tfrac{1}{2} c_{12} - \tfrac{3}{2} c_{44} + C_{144} + \tfrac{3}{2} C_{155} + \tfrac{1}{4} C_{456}) \\ + \theta_3\theta_6 (c_{12} + 3c_{44} + \tfrac{1}{2} C_{144} + 2C_{155} + \tfrac{1}{2} C_{456}) \\ + \theta_4\theta_5 (2c_{11} + 6c_{12} + \tfrac{17}{2} c_{44} + 4C_{144} + 4C_{155} + \tfrac{1}{2} C_{456}) \end{aligned}$$

$$c'_{56} = Sc'_{45}; \quad c'_{46} = S^2 c'_{45}$$

$$\begin{aligned} c'_{14} = b_{14} = \theta_4 (2 c_{12} + C_{144}) + \theta_1\theta_4 (2c_{12} + 4C_{112} + C_{144}) \\ + (\theta_2\theta_4 + \theta_3\theta_4)(2C_{112} + C_{123} + \tfrac{1}{2} C_{144} - c_{12}) \\ + \theta_5\theta_6 (c_{11} + 6c_{44} + \tfrac{1}{3} C_{144} + C_{456} + 2C_{155}) \end{aligned}$$

$$c'_{25} = Sc'_{14}; \quad c'_{36} = S^2 c'_{14}$$

$$\begin{aligned} c'_{15} = b_{15} + \tfrac{1}{2} c'_5 = \theta_5 (c_{11} + c_{12} + 3c_{44} + C_{155}) \\ + \theta_1\theta_5 (\tfrac{1}{2} c_{11} + \tfrac{3}{2} c_{12} + \tfrac{5}{2} c_{44} + 6C_{111} + 2C_{112} + 4C_{155}) \\ + \theta_2\theta_5 (2C_{112} + C_{123} + \tfrac{3}{2} C_{144} - C_{155} - c_{12} - 3c_{44}) \\ + \theta_3\theta_5 (\tfrac{1}{2} c_{12} + \tfrac{1}{2} c_{44} - \tfrac{1}{2} c_{11} + 4C_{112} + 2C_{155}) \\ + \theta_4\theta_6 (c_{12} + \tfrac{9}{2} c_{44} + \tfrac{3}{2} C_{155} + C_{144} + \tfrac{3}{4} C_{456}) \end{aligned}$$

$$c'_{26} = Sc'_{15}; \quad c'_{14} = S^2 c'_{15}$$

$$\begin{aligned} c'_{16} = b_{16} + \tfrac{1}{2} c'_6 = \theta_6 (c_{11} + c_{12} + 3c_{44} + C_{155}) \\ + \theta_1\theta_6 (\tfrac{1}{2} c_{11} + \tfrac{3}{2} c_{12} + \tfrac{5}{2} c_{44} + 6C_{111} + 2C_{112} + 4C_{155}) \\ + \theta_3\theta_6 (2C_{112} + C_{123} + \tfrac{3}{2} C_{144} - C_{155} - c_{11} - 3c_{44}) \\ + \theta_2\theta_6 (\tfrac{1}{2} c_{12} + \tfrac{1}{2} c_{44} - \tfrac{1}{2} c_{11} + 4C_{112} + 2C_{155}) \\ + \theta_4\theta_5 (c_{12} + \tfrac{9}{2} c_{44} + \tfrac{3}{2} C_{155} + C_{144} + \tfrac{3}{4} C_{456}) \end{aligned}$$

$$c'_{24} = S c'_{16}; \quad c'_{35} = S^2 c'_{16}$$

TABLE II

TRIAXIAL STRAIN $\quad \theta_1 \neq \theta_2 \neq \theta_3 \neq 0; \quad \theta_4 = \theta_5 = \theta_6 = 0$

(*A*) *Expressions for initial stress*

$$\begin{aligned} T_1 = c'_1 = \theta_1 c_{11} + (\theta_2 + \theta_3) c_{12} + \theta_1^2 (3C_{111} + \tfrac{3}{2} c_{11}) + (\theta_2^2 + \theta_3^2)(C_{112} - \tfrac{1}{2} c_{12}) \\ + (\theta_1\theta_2 + \theta_1\theta_3)(2 C_{112} - c_{11} + c_{12}) + \theta_2\theta_3 (C_{123} - 2c_{12}) \end{aligned}$$

$$T_2 = c'_2 = S c'_1; \quad T_3 = c'_3 = S^2 T_1; \quad T_4 = T_5 = T_6 = 0$$

(*B*) *Effective elastic constants*.

$$c'_{11} = c_{11} + \theta_1 (4c_{11} + 6\, C_{111}) + (\theta_2 + \theta_3)(c_{12} - c_{11} + 2C_{112}) + \theta_1^2 (\tfrac{9}{2}\, c_{11} + 24 C_{111})$$
$$+ (\theta_2^2 + \theta_3^2)(c_{11} - \tfrac{1}{2}\, c_{12}) + (\theta_1\theta_2 + \theta_1\theta_3)(c_{12} - 4c_{11} - 6\, C_{111} + 8\, C_{112})$$
$$+ \theta_2\theta_3 (c_{11} - 2\, c_{12} + C_{123} - 4\, C_{112})$$

$$c'_{22} = S\, c'_{11}\, ; \quad c'_{33} = S^2\, c'_{11}$$

$$c'_{44} = c_{44} + \theta_1 (\tfrac{1}{2}\, c_{12} - c_{44} + \tfrac{1}{2}\, C_{144}) + (\theta_2 + \theta_3)(\tfrac{1}{4}\, c_{11} + \tfrac{1}{4}\, c_{12} + c_{44} + \tfrac{1}{2}\, C_{155})$$
$$+ \theta_1^2 (c_{44} - \tfrac{1}{4}\, c_{12} + \tfrac{1}{2}\, C_{112} - \tfrac{1}{4}\, C_{144})$$
$$+ (\theta_2^2 + \theta_3^2)(\tfrac{3}{8}\, c_{11} - \tfrac{1}{8}\, c_{12} + \tfrac{3}{4}\, C_{111} + \tfrac{1}{4}\, C_{112} + \tfrac{3}{4}\, C_{155})$$
$$+ (\theta_1\theta_2 + \theta_1\theta_3)(\tfrac{1}{2}\, C_{112} + \tfrac{1}{4}\, C_{123} + \tfrac{1}{2}\, C_{144} - \tfrac{1}{2}\, C_{155} - \tfrac{1}{4}\, c_{11} - \tfrac{1}{4}\, c_{12} - c_{44})$$
$$+ \theta_2\theta_3 (C_{112} + C_{155} - \tfrac{1}{2}\, c_{11} + \tfrac{1}{2}\, c_{12} + c_{44})$$

$$c'_{55} = Sc'_{44}\, ; \quad c'_{66} = S^2 c'_{44}$$

$$c'_{12} = c_{12} + (\theta_1 + \theta_2)(c_{12} + 2C_{112}) + \theta_3 (C_{123} - c_{12}) + (\theta_1^2 + \theta_2^2)\, 3C_{112} + \theta_3^2 (c_{12} - \tfrac{1}{2} C_{123})$$
$$+ \theta_1\theta_2 (c_{12} + 4C_{112}) + (\theta_2\theta_3 + \theta_1\theta_3)(C_{123} - c_{12} - 2C_{112})$$

$$c'_{23} = Sc'_{12}\, ; \quad c'_{13} = S^2 c'_{12}$$

All other $c'_{rs} = 0$; 9 independent elastic constants

TABLE III

SPECIAL CASES OF TRIAXIAL STRAIN

(1) *Uniaxial compression*: $\theta_1 = \theta_2 = 0\, ; \quad \theta_3 = \eta$

(A) *Expressions for initial stress*:

$$T_1 = c'_1 = \eta\, c_{12} + \eta^2 (C_{112} - \tfrac{1}{2}\, c_{12})\, ; \quad T_2 = T_1$$
$$T_3 = c'_3 = \eta\, c_{11} + \eta^2 (3\, C_{111} + \tfrac{3}{2}\, c_{11})\, ; \quad T_4 = T_5 = T_6 = 0$$

(B) *Effective elastic constants*:

$$c'_{11} = c_{11} + \eta\, (c_{12} - c_{11} + 2C_{112}) + \eta^2 (c_{11} - \tfrac{1}{2}\, c_{12})$$
$$c'_{22} = c'_{11}$$
$$c'_{33} = c_{11} + \eta\, (4c_{11} + 6C_{111}) + \eta^2 (\tfrac{9}{2}\, c_{11} + 24\, C_{111})$$
$$c'_{44} = c_{44} + \eta\, (\tfrac{1}{4}\, c_{11} + \tfrac{1}{4}\, c_{12} + c_{44} + \tfrac{1}{2}\, C_{155})$$
$$+ \eta^2 (\tfrac{3}{8}\, c_{11} - \tfrac{1}{8}\, c_{12} + \tfrac{3}{4}\, C_{111} + \tfrac{1}{4}\, C_{112} + \tfrac{3}{4}\, C_{155})$$

$$c'_{55} = c'_{44}$$

$$c'_{66} = c_{44} + \eta\left(\tfrac{1}{2}\, c_{12} - c_{44} + \tfrac{1}{2}\, C_{144}\right) + \eta^2\left(c_{44} - \tfrac{1}{4}\, c_{12} + \tfrac{1}{2} C_{112} - \tfrac{1}{4}\, C_{144}\right)$$

$$c'_{12} = c_{12} + \eta\left(C_{123} - c_{12}\right) + \eta^2\left(c_{12} - \tfrac{1}{2}\, C_{123}\right)$$

$$c'_{23} = c_{12} + \eta\left(c_{12} + 2C_{112}\right) + \eta^2\, 3C_{112}$$

$$c'_{13} = c'_{23}$$

All other $c'_{rs} = 0$; 6 independent elastic constants.

(2) *Uniform compression* : $\theta_1 = \theta_2 = \theta_3 = \eta$; $\theta_4 = \theta_5 = \theta_6 = 0$

(A) *Expressions for initial stress* :

$$T_1 = c'_1 = \eta\left(c_{11} + 2c_{12}\right) + \eta^2\left(3C_{111} + 6\, C_{112} + C_{123} - \tfrac{1}{2}\, c_{11} - c_{12}\right)$$

$T_2 = T_3 = T_1 = -P$ where P is the hydrostatic pressure

(B) *Effective elastic constants* :

$$c'_{11} = c_{11} + \eta\left(6\, C_{111} + 4\, C_{112} + 2\, c_{11} + 2\, c_{12}\right)$$
$$+ \eta^2\left(12\, C_{111} + 12\, C_{112} + C_{123} - \tfrac{1}{2}\, c_{11} - c_{12}\right)$$

$$c'_{22} = c'_{33} = c'_{11}$$

$$c'_{44} = c_{44} + \eta\left(c_{44} + c_{12} + \tfrac{1}{2}\, c_{11} + \tfrac{1}{2}\, C_{144} + C_{155}\right)$$
$$+ \eta^2\left(\tfrac{3}{2}\, C_{111} + 3C_{112} + \tfrac{1}{2}\, C_{123} + \tfrac{3}{4}\, C_{144} + \tfrac{3}{2}\, C_{155} - \tfrac{1}{4}\, c_{11} - \tfrac{1}{2}\, c_{12}\right)$$

$$c'_{55} = c'_{66} = c'_{44}$$

$$c'_{12} = c_{12} + \eta\left(c_{12} + 4\, C_{112} + C_{123}\right) + \eta^2\left(6\, C_{112} + \tfrac{3}{2}\, C_{123}\right)$$

$$c'_{23} = c'_{13} = c'_{12}$$

All other $c'_{rs} = 0$; 3 independent elastic constants.

TABLE IV

PURE SHEAR : $x_0 = a + \eta b$; $y_0 = \eta a + b$; $z_0 = c$; $\theta_1 = \theta_2 = \theta_3 = 0$; $\theta_4 = \theta_5 = 0$; $\theta_6 = \eta$

(A) *Expressions for initial stress* :

$$T_1 = c'_1 = \eta^2\left(\tfrac{1}{2}\, c_{11} + \tfrac{1}{2}\, c_{12} + 4c_{44} + C_{155}\right); \quad T_2 = T_1$$

$$T_3 = c'_3 = \eta^2\left(c_{12} + C_{144}\right); \quad T_4 = T_5 = 0; \quad T_6 = 2c_{44}\eta$$

(B) *Effective elastic constants* :

$$c'_{11} = c_{11} + \eta^2 \left(\tfrac{3}{2}\, c_{11} + \tfrac{5}{2}\, c_{12} + 8c_{44} + C_{112} + 5C_{155} + 3C_{111} \right)$$

$$c'_{22} = c'_{11}$$

$$c'_{33} = c_{11} + \eta^2 \left(c_{11} + c_{12} + 2C_{112} + C_{144} \right)$$

$$c'_{44} = c_{44} + \eta^2 \left(\tfrac{3}{8}\, c_{11} + \tfrac{3}{8}\, c_{12} + 3c_{44} + \tfrac{1}{2}\, C_{144} + \tfrac{1}{2}\, C_{155} + \tfrac{1}{2}\, C_{456} \right)$$

$$c'_{55} = c'_{44}$$

$$c'_{66} = c_{44} + \eta^2 \left(\tfrac{9}{4}\, c_{11} + \tfrac{9}{4}\, c_{12} + 5c_{44} + 5C_{155} \right)$$

$$c'_{12} = c_{12} + \eta^2 \left(2c_{11} + 4c_{44} + c_{12} + 2C_{112} + 4C_{155} \right)$$

$$c'_{23} = c_{12} + \eta^2 \left(2c_{12} + C_{112} + 2C_{144} + \tfrac{1}{2}\, C_{123} \right)$$

$$c'_{13} = c'_{23}$$

$$c'_{45} = \eta \left(5c_{44} + \tfrac{1}{2}\, C_{456} \right)$$

$$c'_{36} = \eta \left(2c_{12} + C_{144} \right)$$

$$c'_{16} = \eta \left(c_{11} + c_{12} + 3c_{44} + C_{155} \right)$$

$$c'_{26} = c'_{16}$$

All other $c'_{ik} = 0$; 9 independent elastic constants.