A NOTE ON THE EFFECT OF BEVELLING ON THE COUPLED THICKNESS-SHEAR-FLEXURE VIBRATIONS OF CRYSTALLINE PLATES*

By E. S Rajagopal
(Department of Physics, Indian Institute of Science, Bangalore - 12)

AND

E S. Ramamurthy
(Crystal Division, Bharat Electronics Ltd, Jalahalli, Bangalore - 13)

Received on March 18, 1960

Abstract

The effect of bevelling on the thickness-shear-flexure vibrations of crystalline plates, especially AT cut quartz plates, is investigated. The theoretical calculations for infinitely long plates are based on MINDLIN'S plate theory which takes into account the rotatory inertia and transverse shear terms and the equations of the bevelled plate are obtained from the reduced equations given by MINDLIN and FORRAY for the uniform plate and for the double wedge. Whereas the thickness-shear frequency is practically unaffected by bevelling, the flexure vibrations are very seriously disturbed, the order of the flexure overtone which combines with the fundamental thickness-shear in a double wedge being almost twice that in a uniform plate. The calculations are compared with the experimental observations on long strips of AT cut quartz plates It is suggested that bevelling, together with edge clamping, will considerably reduce the flexure coupling effects, as is borne out in practice.

Introduction

Since the theory of the complete vibrational modes of crystalline plates of finite thickness presents exceptional mathematical difficulties, little progress has been made in understanding all but the simplest problems (Mindlin, 1955; Hearmon, 1956). For example, one has fairly accurate descriptions of pure modes (thickness extension and shear, face-shear, flexure, etc.) but when the modes are coupled, analysis has been very difficult. The situation is all the more unsatisfactory since in the crystal plates, used widely in science and industry, the modes are always coupled and the crystal plates can be easily excited into a number of modes (Sykes, 1946).

The technique of contouring the thickness of the crystal plate to improve the frequency quality of the resonators is widely used. Bechmann (1952) has

* Presented at the first Symposium on Solid State Physics held at Bangalore on 1st, 2nd and 3rd February 1960. An abstract is being published in the Proceedings of this Symposium.

shown that on bevelling a circular plate the spurious contour-shear frequencies are displaced away from the basic thickness-shear frequency and can thus be eliminated to result in a single response near the designed frequency. Warner (1952) has used the idea, that in a convex shaped plate the thickness-shear motion is confined to the central portion, to reduce the effect of mounting on the resonators, and this technique has subsequently been used by Vasin, Pozdyakov and Varoslavskii (1957), White (1958) and others.

Among the recent theoretical developments concerning the problem of the vibrations of contoured plates, Mindlin and Forray (1954) have developed an approximate way of solving Mindlin's equations (1951) for a plate of variable thickness. Essentially this consists in splitting the complicated fourth order equation with variable coefficients representing the coupled thickness-shear-flexure modes into two pairs of equations representing the uncoupled thickness-shear and flexure modes, still retaining the thickness-shear and rotatory inertia effects. This reduction is approximate but in the region of low thickness-shear and high flexure overtones the reduced equations agree very well with the complete equations for a uniform plate. Incidentally it is in this region in which one is interested, for the fundamental thickness-shear mode is usually coupled with a high (20th-30th) even order flexure mode.

The solutions available for the low flexure modes of wedge shaped plates (Klein 1956) are based on the classical plate theory and the neglect of the rotatory shear and inertia effects renders them useless at high overtones of plates of appreciable thickness. So in this note, the flexure solutions of the Mindlin's plate theory are obtained and using in addition Mindlin and Forray's calculations one obtains the frequency spectrum of the bevelled plates

Frequency Spectrum of the Bevelled Plate

The nature of the effects brought about by bevelling are clearly brought out by suppressing the coupling in one lateral direction, *i.e.*, by considering a y faced crystal plate (thickness $2h$) of infinite length along the z axis. In this case Mindlin and Forray (1954) have obtained the thickness-shear solutions of the bevelled plates starting from the full solutions of the uniform plate and of the double wedge. Since these relations are needed both in deducing the flexure modes of the bevelled plates (Eq. 17) and in plotting the full frequency spectrum (Eqs. 7, 11, 12, 15 to 18) they are very briefly summarized for the subsequent use.

Taking the displacements as

$$u_x = y\,\psi\,(x, t)\,; \quad u_y = \eta\,(x, t)\,; \quad u_z = 0 \text{ and } h = h\,(x)$$

the reduced equations of Mindlin and Forray are:

(A) for the transverse shear vibrations,

$$\left.\begin{aligned} &24\,k_2\,h\eta_2 = -\left(8\,h^3\,\psi_2\right)'\,; \quad ' \equiv \frac{d}{dx} \\ &D_6\left[\frac{1}{2h}\left(8\,h^3\,\psi_2\right)'\right]' + k_2\left[8\,\rho h^3\,\omega^2 - 12 D_6\right]\psi_2 = 0 \end{aligned}\right\} \quad [1]$$

where

$$k_2 = g/(1+g)\,; \quad g = \pi^2\,c_{66}/12\left[c_{11} - c_{12}^2/c_{22}\right]; \quad D_6 = \pi^2 c_{66}\,h/6 \quad [2]$$

The moment of the forces M_x and the resultant of the forces Q_x are

$$M_x = D_1\,\psi_2'\,; \quad Q_x = D_6\left(\psi_2 + \eta_2'\right) \quad [3]$$

(c_{ij} are the rotated constants for the orientation of the plate).

(B) for the flexural vibrations

$$\left(D_1\,\eta_1''\right)'' - 2\,k_1\,\rho h\omega^2\,\eta_1 = 0\,; \quad \psi_1 = -\eta_1' \quad [4]$$

where
$$k_1 = (1+g)^2/g\,; \quad D_1 = \left(2\,c_{11} - c_{12}^2/c_{22}\right)h^3/3 \quad [5]$$

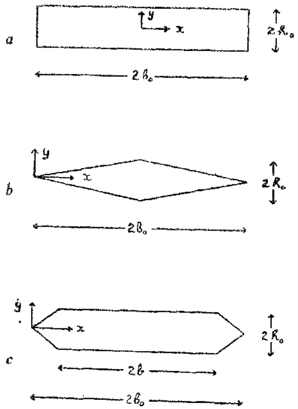

FIG. I

Co-ordinates of the infinite plates of (a) a uniform thickness (b) double wedge and (c) bevelled plate. Infinite length along z axis.

The moment and the resultant of the forces are

$$M_x = -D_1\,\eta_1'\,; \quad Q_x = -(D_1\,\eta_1'')' \qquad [6]$$

One now sees that for a uniform plate, as shown in Fig. I *a* the thickness-shear solution is obtained as

$$\eta_2 = A\sin\delta_2 x + B\cos\delta_2 x \qquad [7a]$$

and with the free plate condition, the fundamental mode (antisymmetric in x) has a frequency given by

$$\delta_2\,(2\,b_0) = n\pi \qquad (n = 1, 2, \ldots) \qquad [7b]$$

where

$$\delta_2^2 = \frac{12\,k_2}{(2\,h_0)^2}\left[\frac{2\,(2\,h_0)^2\omega^2}{\pi^2 c_{66}} - 1\right] = \frac{12\,k_2}{(2\,h_0)^2}\left[\frac{\omega^2}{\bar{\omega}^2} - 1\right] \qquad [8]$$

In the flexure waves, with the free plate conditions, the odd order modes are

$$\eta_1 = A\,[\cos\delta_1 x + \cos\delta_1 b_0\,\mathrm{sech}\,\delta_1 b_0\,\cosh\delta_1 x]$$

$$\tanh\delta_1 b_0 = -\tan\delta_1 b_0 \qquad [9a]$$

and the even order modes (anti-symmetric in x) are

$$\eta_1 = A'\,[\sin\delta_1 x + \sin\delta_1 b_0\,\mathrm{cosech}\,\delta_1 b_0\,\sinh\delta_1 x]$$

$$\tanh\delta_1 b_0 = \tan\delta_1 b_0 \qquad [9b]$$

where

$$\delta_1^4 = 12\,k_1\,\rho\,\omega^2/(2\,h_0)^3\,(c_{11} - c_{12}^2/c_{22}) \qquad [10]$$

Taking both the modes, at large r

$$\delta_1\,(2\,b_0) = (2\,r + 1)\pi/2 \qquad [11]$$

In the case of the double wedge of dimensions shown in Figure I *b*, the thickness-shear mode is

$$\psi_2 = Ax^{-2}\,J_p\,(\beta x) + \beta x^{-2}\,\gamma_p\,(\beta x) \qquad [12a]$$

and the fundamental frequency is obtained from the first root of

$$\frac{d}{d\gamma}\left[\gamma\,J_p\,(\gamma)\right] = 0 \qquad [12b]$$

where

$$\beta^2 = 12\,k_2\,\rho\omega^2/\pi^2 c_{66}\,;\; p^2 = 1 + 12\,k_2/a^2\,;\; \gamma = \beta\,b_0\,;\; a = 2\,h_0/b_0 \qquad [13]$$

The flexure solution is

$$\eta_1 = x^{-\frac{1}{2}} \left[A J_1 (\lambda x^{\frac{1}{2}}) + B J_1 (i\lambda x^{\frac{1}{2}}) + C Y_1 (\lambda x^{\frac{1}{2}}) + D Y_1 (i\lambda x^{\frac{1}{2}}) \right] \qquad [14a]$$

where

$$(\lambda/2)^4 = 12 k_1 \rho\omega^2 / a^2 \left(c_{11} - c_{12}^2/c_{22}\right) \qquad [14b]$$

Treating the symmetric and antisymmetric modes, the high overtones are given by

$$\lambda\, b_0^{\frac{1}{2}} = (2\, r + 5)\, \pi/4 \qquad [15]$$

In order to get the fundamental thickness-shear mode of the bevelled plate of cross section (Figure 1*c*),

$$2\, h = ax \text{ for } 0 \leq x \leq b_0 - b$$

$$= 2\, h_0 \text{ for } b_0 - b \leq x \leq b_0$$

$$\text{and } a = 2\, h_0 / (b_0 - b)$$

one assumes the antisymmetric solution of Eq. [8] with the conditions fixed at $x' \,(= b_0 - x) = b_0$ for the uniform portion and the solution [12 *a*] with the conditions fixed at $x = 0$ for the wedge portion. Then, demanding the continuity of ψ and η at $x = b_0 - b$ as done by Mindlin and Forray, one gets the secular equation

$$\delta_2 (b_0 - b) \tan \delta_2 b .\; J_p + (p - 1)\, J_p - \beta\, (b_0 - b)\, J_{p-1} = 0$$

$$J_p = J_p \left[\beta\, (b_0 - b)\right] \qquad (16)$$

In the case of the flexural vibrations, it is sufficient to consider the even modes, since the fundamental thickness-shear mode (antisymmetric in x) couples almost only with the even flexural modes (also antisymmetric in x) (Sykes 1946). (The method is easily applied to odd modes also and the resulting equation has $\tan x$ instead of $\cot x$ in Eq. 18).

The general antisymmetric solution for the uniform portion is

$$\eta_1 = A \sin \delta_1 x' + B \sinh \delta_1 x' , \qquad x' = b_0 + x$$

and that in the wedge portion is

$$\eta_1 = C x^{-\frac{1}{2}} J_1 (\lambda x^{\frac{1}{2}}) + D x^{-\frac{1}{2}} I_1 (\lambda x^{\frac{1}{2}})$$

one demands the continuity of the displacements η and ψ and also of the moments M_x and the stress resultants Q_x at $x = b_0 - b$. Using Eqs. [4] and [6], the secular determinant after some simplification becomes

$$\begin{aligned}
&\cot \delta_1 b \cdot \coth \delta_1 b \cdot \delta_1^2 (b_0 - b)^2 [\delta_1 (b_0 - b) - I_0/I_1 + J_0/J_1] \qquad [17]\\
&\quad + \delta_1 (b_0 - b) \cdot [1 - \delta_1 (b_0 - b) I_0/I_1] [1 - \delta_1 (b_0 - b) J_0/J_1]\\
&\quad + \coth \delta_1 b \cdot [1 - \delta_1 (b_0 - b) J_0/J_1] [1 + \delta_1^2 (b_0 - b)^2 - \delta_1 (b_0 - b) I_0/I_1]\\
&\quad - \cot \delta_1 b \cdot [1 - \delta_1 (b_0 - b) I_0/I_1] [1 - \delta_1^2 (b_0 - b)^2 - \delta_1 (b_0 - b) J_0/J_1] = 0
\end{aligned}$$

where the Bessel functions are of arguments $\lambda(b_0 - b)^{\frac{1}{2}} = 2\,\delta_1 (b_0 - b)$. In the region $\delta_1 b > 3$ and $\delta_1 (b_0 - b) > 5$ this equation can be approximated by the simpler equation

$$\begin{aligned}
&[1 - \delta_1 (b_0 - b) \cot \phi] - \cot \delta_1 b \cdot [1 - \delta_1 (b_0 - b) - \delta_1 (b_0 - b) \cot \phi] = 0\\
&\phi = 2\,\delta_1 (b_0 - b) - \pi/4 \qquad [18]
\end{aligned}$$

Thus one can evaluate the complete frequency spectrum. To solve the secular equations, the method of alternating signs was used and the roots were estimated to 1 % accuracy only in view of the experimental complications.

Discussion

The numerical computations have been made for AT cut quartz plates $2\,h_0 = 0.1$ cm. in thickness and of widths $2\,b_0 = 2.0$ cm. and $2\,b_0 = 0.5$ cm. A b_0/h_0 ratio of 20 is common for the plates widely used in industry and science while $b_0/h_0 \sim 5$ was found reasonably convenient as discussed below to simulate experimentally the conditions of an infinitely long plate. The values of the rotated elastic constants are the same as those used by MINDLIN (1951)

$$[c_{66} = 29.34\,;\ c_{22} = 129.9\,;\ c_{12} = -10.49\,;\ c_{11} = 86.05 \times 10^{10}\ \text{dynes/cm}^2\,;\ \rho = 2.660\ \text{gm/cm}^3]$$

which are based on Mason's observations.

Taking the case where $b_0/h_0 = 20$, the frequency spectrum near the fundamental thickness-shear vibration is plotted in Fig. II for various values of b/b_0. The thickness-shear frequencies were computed from Eq. [16]. The flexure frequencies in the range $0.1 < b/b_0 < 0.7$ were computed from Eq. [18] and then Eqs. [11, 15] were used to fix up the orders of the various overtones. Fig. II shows immediately two features A slight bevelling say $b/b_0 \sim 0.8$, does not cause any perceptible change in the thickness-shear frequency, but completely changes the position of the flexure frequency. It is obvious that by giving controlled amounts of small bevelling the troublesome flexure frequencies can be moved away from the thickness-shear mode so as to cause the least disturbance.

Similar calculations for $b_0/h_0 = 5$ are given in Fig. III. In addition to the general features mentioned above, two other features appear. One is that already the effect of small lateral dimension on the thickness-shear is felt both in the shift of the frequency from that of an infinite plate and in the dispersion of the frequency with b/b_0. Secondly the order of the flexure overtone which combines with the fundamental thickness-shear is reduced considerably, thus leading to a greater disturbances of the thickness-shear mode.

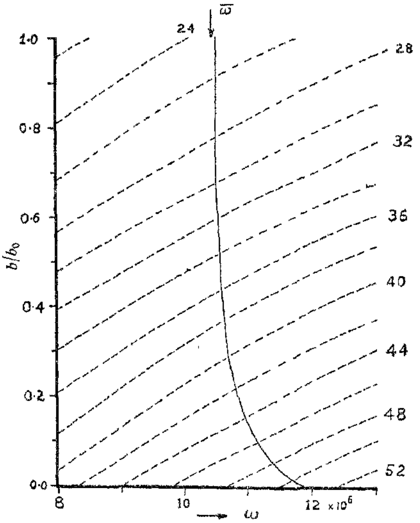

FIG. II

Frequencies of the fundamental thickness-shear (full line) and the flexure overtones (dashed lines) for various b/b_0 ($2h_0=0.1$; $2b_0=2.0$ cm.) The thickness-shear freequency $\bar{\omega}$ of the infinitely wide plate is also indicated.

The causes of the different sensitivities of the two types of modes to the bevelling has been already indicated by Mindlin and Forray (1954). Taking the thickness-shear of a double wedge (Eq. 12 a), in the case treated above ($2h_0 = 0.1$; $2b_0 = 2.0$ cm.), $p \sim 16$, βx varies from 0 to 15 and $x^{-2} J_p(\beta x)$ is negligible at small x near the edge and becomes appreciable only near the centre. (Numerically $1^{-2}J_{16}(1) \sim 1.10^{-18}$, $10^{-2}J_{16}(10) \sim 2.10^{-5}$, $15^{-2}J_{16}(15) \sim 4.10^{-3}$).

So the edge portions do not at all participate in the thickness-shear motion and only the central portion vibrates appreciably. Hence the bevelling of the edges has only a minor influence on the thickness-shear vibrations. In fact it is this absence of motion at the edges which enables one to reduce the mounting losses in high Q resonators. For flexure vibrations [Eq. 14 a], $\lambda \sim 30$-40. So $x^{-\frac{1}{2}}J_1(\lambda x^{\frac{1}{2}})$ is large at the edges and becomes small at the centre of the double wedge. (Numerically $1^{-\frac{1}{2}}J_1(1) \sim 4.10^{-1}$, $10^{-\frac{1}{2}}J_1(10) \sim 1\ 10^{-2}$, $30^{-\frac{1}{2}}J_1(30) \sim 3.10^{-3}$). So any alteration of the conditions at the edges has a pronounced bearing on the flexure modes.

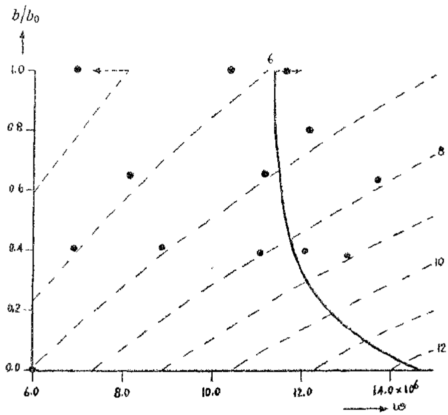

FIG. III

Fundamental thickness-shear (full line) and flexure overtones (dashed lines) for a bevelled plate $2h_0 = 0.1$; $2b_0 = 0.5$ cm. Experimental points are indicated by dots. Corrections to the plotted values are also shown in the case of the uniform plate.

This consideration suggests that edge clamping in addition to bevelling will still further reduce the flexure coupling effects Of course, the solutions derived above are for free plates and must be extended to the required case. But it is obvious that the damping of the flexure vibrations where they are most pronounced will minimize the coupling effects.

The experimental studies to check some aspects of the calculations were made on AT cut quartz plates of size $2h_0 = 0.1$, $2b_0 = 0.5$ cm. and the largest dimension was ~ 8 cm. so that the ratio of the thickness to the breadth is smaller than that of the breadth to the length. Though the coupling to the length will still persist to some extent, this effectively simulates the conditions of an infinite plate within reasonable dimensions. In actual practice several frequencies are obtained which are due to coupling with this length and with the face shear modes. They are easily recognised by altering the clamping conditions at the edges of the largest dimension or by altering this size, when the unwanted frequencies are changed considerably.

The experiments were made by the usual Π-network transmission method and are compared with the theoretical results given in Fig. III. Unfortunately, as already pointed out, the method of computation has serious errors in the region of low flexure overtones and in the case of the uniform plates the correct values can be obtained from the exact solutions of Mindlin (1951). The corrections are indicated in Figure III, where it is seen that the thickness-shear values should be increased and the flexure values should be lowered. Also, in the simplification of the exact equations, the "pulling" of the frequencies due to the coupling is lost. On the experimental side, the finite value of the length is a departure from the idealised model considered. In view of these, nothing more than a qualitative agreement between the experimental results and the theoretical computations can be claimed.

Some experiments were tried with edge clamping of the bevelled plate. A considerable reduction in the intensity of the various side overtones was observed while the thickness-shear mode is only slightly damped. In the absence of any theoretical value for the amplitude ratios, a quantitative comparison was not attempted. All the same, this bevelled edge clamping technique is already widely used in Bharat Elecronics Ltd., Bangalore, for improving the performance of several types of crystals.

ACKNOWLEDGMENT

The authors thank Shri B. V. Baliga, Managing Director, Shri C. B. Cariappa, Deputy General Manager and the other members of the Management of Bharat Electronics Ltd., and Professor R. S. Krishnan, Head of the Department of Physics, Indian Institute of Science, for their enthusiastic encouragement to this co-operative work. One of the authors (E. S. Rajagopal) thanks the Council of Scientific and Industrial Research for the award of a Senior Research Fellowship and Professor R. D. Mindlin for making available a copy of his somewhat inaccessible book.

REFERENCES

1. Bechmann, R. .. *J. Sci. Instr.*, 1952, **29**, 73.
2. Hearmon, R. F. S. .. *Advances in Physics*, 1956, **5**, 322.
3. Klein, B. .. *J. Acoust. Soc. Amer.*, 1956, **28**, 1177.

4. Mindlin, R. D. .. *J. Appl. Phys.*, 1951, **22**, 316.
5. ——— . *Introduction to the Mathematical Theory of Vibrations of Elastic Plates*, (Fort Monmouth U S Army Signal Corps), 1955.
6. ——— and Forray, M. *J. Appl. Phys.*, 1954, **25**, 12.
7. Sykes, R. A. . *Quartz Crystals for Electrical Circuits*, ed. R. A. Heising (New York: Van Nostrand), 1946, Chap. VI.
8. Vasin I. G., Pozdyakov, P. G. and Yaroslavskii, M. I. *Doklady Akad. Nauk SSSR*, 1958, **119**, 481.
9. Warner, A. W. .. *Proc. I. R. E.*, 1952, **40**, 1030.
10. White, D. L. .. *J. Appl. Phys.*, 1958, **29**, 856.