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ELASTIC BEHAVIOUR OF MATTER UNDER VERY HIGH PRESSURES

Considerations of Instability

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ABSTRACT

Basing on earlier work of Bhagavantam and Chelam on the elastic behaviour of matter under high pressures wherein an effective elastic energy expression was introduced, it has been shown that the positive definiteness of this expression signifies elastic stability. Since the coefficients in this energy expression have been worked out, it may be expected that this would provide a convenient approach for determining the elastic instability in any system subjected to large strains.

INTRODUCTION

The subject of elastic instability has found application in a wide range of engineering problems such as structural stability, buckling of beams, columns and plates, stability of small vibrations, the design of dams, bridges, ships and aircraft, etc. It has also a major role to play in geophysical phenomena such as earthquakes and rockbursts where presumably the accumulated strains reach a critical stage, beyond which elastic instability accompanied by release of tremendous amounts of energy sets in.

A survey of the basic problem of elastic instability will be found in books by Timoshenko (1936), Bleich (1952), Shanley (1957) and in other recent contributions by Langharr (1958), Argyris (1954, 1955), Zeigler (1956), Hoff (1954), Pearson (1956), Goodier and Plass (1952), Lime (1952), Hoff (1951) and Prager (1947). Barring some exceptions when the forces are gyroscopic, nonconservative, largely dissipative etc. when the definition of elastic stability itself has to be given on a dynamic basis (Zeigler 1956), the general criterion for an elastic system to remain stable in any equilibrium configuration is that its total potential energy in that configuration should be an absolute minimum. This criterion, proposed by Bryan in 1889, has now received extensive applications. The total potential energy in this context is the sum of the strain energy (which is also referred to as the potential energy of the internal forces by Zeigler) and the potential energy of the external forces such as gravity, externally applied loads etc. Moreover, it is the potential energy of the body as a whole (*i.e.*) integrated over the entire volume of the body. This criterion therefore appears as the minimization of a certain integral representing the potential energy. The criterion of equilibrium as stated above is best studied by considering the *change* in such potential energy (say V) brought about when an infinitesimally small virtual displacement from the configuration is considered. This small displacement could be specified by suitable parameters such as the infinitesimal strain components, and the change in potential energy could be developed as a power series in these components. The energy minimum would then require that the linear terms in this development vanish, ($\delta V = 0$, corresponding to the stationary value of the energy) while the second order terms comprise a positive definite quadratic form ($\delta^2 V > 0$ corresponding to the minimum value of V). If this quadratic, thrown into its 'normal' form (*i.e.*) as a sum of squares with no cross products, has any of its terms with a negative coefficient, then instability sets in. From the theory of small oscillations, it is clear that the corresponding original small disturbance will, under such conditions, grow with time and become unbounded (Zeigler 1956).

CRITERION FOR INSTABILITY IN A STATE OF FINITE STRAIN

At this stage, we note the important point that the equilibrium configuration referred to could be either the natural or undeformed state of the body or a state of initial strain, caused by the presence of initial stresses. This second type is of particular importance in practical applications as we face here the question whether a substance, which is stable in its undeformed state, could be rendered unstable when it is subjected to a specified degree of finite strain. In dealing with such cases, a mere statement about the criterion of minimum potential energy has only a limited practical value. One should therefrom deduce criteria which relate the onset of instability with the magnitude and nature of the initial finite strain.

In previous papers (Bhagavantam and Chelam 1960), the problem of the elastic behaviour of substances, which are already under a state of finite strain, was investigated. It is the purpose of this paper to show that the positive definiteness of the quadratic form ϕ_e(the effective elastic energy as defined in those papers), in *any portion* of the elastic medium or body, is the criterion for the elastic stability of that portion of the medium or body. This criterion is therefore to be shown as equivalent to the criterion of minimum potential energy computed over the appropriate portion of the medium. Presentation of the criterion in this form, rather than as the minimum value of an integral taken over the whole body, has the additional advantage that we could consider the presence or absence of elastic instability in any portion, or what amounts to the same thing, in the small neighbourhood around any point within the body, by investigating the positive definiteness of the function ϕ_e at that point. Of course this gives us only the magnitude of initial strain, or equivalently, the magnitude of initial stress at each point, the exceeding of which results in instability. Whether this magnitude is reached at any point within the body under given boundary conditions and applied external loads has to be separately examined for each individual problem, using the finite deformation theory.

Strain Energy, Potential Energy and Equilibrium Conditions

When an elastic body is strained, internal forces (Stresses) develop within the body which tend to take the body back to its natural or undeformed state. If therefore a body has to be maintained in equilibrium in such a strained condition, external forces have to be applied and maintained, to balance these internal forces. In every such equilibrium state, we have thus to take into account both the external and internal forces and obtain a balance between them. Instead of directly balancing the forces, one can also balance, according to the principle of virtual work, the work done by these two systems of forces in any suitable small (*i.e.*) infinitesimal displacement form the equilibrium position. Since the work done against a force in a small displacement represents the change in potential energy associated with that force, balancing of the internal and external forces means that the gain in potential energy of one is equal to the loss in potential energy of the other, thus requiring that the change in the total potential energy, (*i.e.* composed of the internal and external forces) be zero. Now, the strain energy in a deformed medium is the internally stored up energy, brought about as the result of the work done against the action of internal forces resisting such a deformation. Hence the strain energy is the potential energy of the internal forces and the criteria of equilibrium require this internal potential energy to be related to the potential of the external forces. If δW be the work done by the external forces acting over a unit initial volume of the deformable medium in a small displacement, then $-\delta W$ is the associated change in the external potential, and the equilibrium conditions require that the change in the internal and external potentials taken together is zero, (*i.e.*) that $\delta V = \int(\delta\phi - \delta W) = 0$, or that $\delta W = \delta\phi$, where ϕ is the strain energy per unit initial volume and $\delta\phi$ is the change in the same, computed up to the first order of the infinitesimals specifying the change. Analytical expressions can be worked out for δW. The criterion that $\delta V = 0$ in every virtual displacement, leads first to the equations of equilibrium in the form $(\partial T_{ik}/\partial x_k) + \rho F_i = 0$ connecting the stress matrix T with the body force F, ρ being the density. On using this relation, the expression for δW, is given in the infinitesimal theory by

$$\delta W = T_1\delta_1 + T_2\delta_2 + T_3\delta_3 + 2T_4\delta_4 + 2T_5\delta_5 + 2T_6\delta_6 \qquad [1]$$

where T and δ are the stress and infinitesimal strain matrices represented by

$$T = \begin{vmatrix} T_1 & T_6 & T_5 \\ T_6 & T_2 & T_4 \\ T_5 & T_4 & T_3 \end{vmatrix}; \qquad \delta = \begin{vmatrix} \delta_1 & \delta_6 & \delta_5 \\ \delta_6 & \delta_2 & \delta_4 \\ \delta_5 & \delta_4 & \delta_3 \end{vmatrix} \qquad [2]$$

The relation $\delta W = \delta\phi$ then leads to the standard relation between stress and strain energy in the classical theory $T = \partial\phi/\partial\eta$ assuming that ϕ is written as a function of the 9 independent components of the strain matrix, disregarding the symmetry relation $\eta_{ik} = \eta_{ki}$. In the finite theory, where the distinction between an initial and final state has to be maintained, the expression for virtual work is more complicated (Murnaghan 1951). Let J be the matrix specifying the deformation from an initial state (state 'a') to a final state (state 'x'), ρ_a and ρ_x being the corresponding densities, so that a small volume element dV_a in the initial state, which becomes dV_x in the final state is connected by the relation

$$\frac{dV_a}{dV_x} = \frac{\rho_x}{\rho_a} = \frac{1}{\text{Determinant } J} \qquad [3]$$

The elastic medium is in equilibrium in the final 'x' state, which is characterized by a specified amount of finite strain η (measured from conditions in the initial state) and a corresponding magnitude of finite stress T (which is always referred to the conditions in the final state). We now impose a 'δ' displacement on the system in the 'x' state, specified say by a 'δ' matrix of the type given in [2]. Then Murnaghan's analysis shows that the work of the external forces in the 'δ' displacemement is still given by [1], with the additional feature however that [1] now refers to a unit volume of the 'x' state and not of the initial 'a' state. It will be noted here that δW in [1] is the same thing as the trace (*i.e.*) sum of the diagonal elements of the product martix $T.\delta$. If an unsymmetrical 'δ' displacement is chosen, we form the symmetric matrix $D = [\delta + \delta^*]/2$ where δ^* is the transpose of δ, and δW is then given by the trace of the matrix $T.D$. Now, the strains η and consequently the variations $\delta\eta$ in the same are however referred to the initial conditions while the δ or D refer to the 'x' state. Murnaghan shows that these two quantities are connected by the relation $\delta\eta = J^*.D.J$, which reduces to $J^*.\delta.J.$ in our present symmetric case. On using this relation, the virtual work of all the external forces acting on a unit volume of the 'x' state is obtained as the trace of the product martix $J^{-1}T(J^*)^{-1}\delta\eta$, while the corresponding first order increase in strain energy is given by $\delta\phi$ per unit initial volume or, from [3] by $(\rho_x/\rho_a)\delta\phi$ per unit volume of the 'x' state. In order therefore that a specified volume V_x of the deformable medium remain in equilibrium, we must have

$$\int_{V_x} J^{-1}T(J^*)^{-1}\delta\eta = \int_{V_x} \frac{\rho_x}{\rho_a}\delta\phi \qquad [4]$$

However Murnaghan points out that every portion of the deformable medium is in equilibrium. Thus V_x is arbitrary and the integrands in [4] must be equal. We thus obtain the fundamental stress energy (or equivalently the stress-strain) relationship valid in the realm of non-linear elasticity as

$$T = \frac{\rho_x}{\rho_a} J \frac{\partial\phi}{\partial\eta} J^* \qquad [5]$$

STABILITY CRITERION AND EFFECTIVE ELASTIC ENERGY

The fundamental contribution of Murnaghan presented above reveals incidentally two important aspects. Firstly, the principle of virtual work, or in other words, the requirement of the stationary value of the total potential energy can be applied over any arbitrary portion of the deformable medium, to derive the criteria of *equilibrium of that portion.* It hence follows that the supplementary criterion regarding the *stability* of such an equilibrium state will be determined by application of the criterion of the *minimum* of potential energy *viz.* $\delta^2 V > 0$ over any arbitrary portion of the deformable medium, this approach having the same validity as the applicability of the stress strain law [5] at any point with in the medium. As the portion over which V is computed is thus arbitrary, it follows that the criterion that any selected unit volume element of the deformed state is in stable equilibrium is given by $\delta^2 v > 0$ where v is the total potential energy in that *unit volume.* The variation in v is however composed of the variation in the strain energy *minus* the work done by the external forces. To quantitatively obtain its expression, we note that the second important point in Murnaghan's work is that when the strain energy is referred to a unit volume of the finitely deformed state (maintained in equilibrium by external forces), and the variation of this strain energy consequent on a 'δ' displacement from this equilibrium state is expanded as a power series in the δ's, then the terms linear in the δ's in this expansion, balance the work of the external forces, the equilibrium stresses just developing to such a magnitude so as to ensure this. This necessitates that the development of v in terms of δ_r can have no linear term, while the second order terms $\delta^2 v$ can only arise from terms of the second degree in the expansion of $(\rho_x/\rho_a)\phi$, there being no contribution to the same from the work of the external forces. Thus

$$\delta^2 v = \frac{\rho_x}{\rho_a}\delta^2\phi \qquad [6]$$

In the earlier papers of Bhagavantam and Chelam referred to, terms of the second degree in the δ_r, contained in the expansion of $(\rho_x/\rho_a)\phi$, have been denoted by ϕ_e and called the effective elastic energy. Thus $\phi_e = (\rho_x/\rho_a)\,\delta^2\phi$ and we hence get

$$\delta^2 v = \phi_e \qquad [7]$$

Thus we deduce the criterion that the effective elastic energy should remain positive definite for elastic stability to be maintained. If in any portion of the elastic medium, ϕ_e assumes negative values (which can be the case when any of its coefficients in the normal form is negative) then elastic instability and release of kinetic energy appear in that portion of the elastic medium. This establishes the important role of the effective elastic energy in relation to elastic instability.

The criterion regarding the positive definiteness of ϕ_e at any point within an elastic medium subjected to large initial stresses is generally simpler to apply than the conventional criterion regarding the minimization of the integral of the total potential energy, although the two are equivalent, in the ultimate analysis. In the earlier paper by Bhagavantam and Chelam, the detailed method of deriving the expressions for the coefficients occurring in the quadratic form ϕ_e was outlined, assuming that ϕ_e was developed as a function of the 6 infinitesimal strain components $\delta_1, \delta_2 \ldots \delta_6$ in the form

$$\phi_e = \tfrac{1}{2} \Sigma c'_{11} \delta_1^2 + \Sigma c'_{12} \delta_1 \delta_2 + 2 \Sigma c'_{44} \delta_4^2 + 2 \Sigma c'_{45} \delta_4 \delta_5 + 2 \Sigma c'_{14} \delta_1 \delta_4$$

Each c'_{rs} was developed as a function of the elements of the Jacobian Matrix specifying the initial finite displacement. It thus becomes easier to determine the magnitude of the initial finite strain at which the c'_{rs} or rather an appropriate combination of them in a normal form of ϕ_e, becomes negative. This procedure thus provides a convenient approach to determine the onset of elastic instability.

A study of the stability of the system after it has reached some postulated state of finite strain, has also the advantage as poined out by Prager (1947) that the problem of elastic instability is clearly separated from the more involved investigations into the type of initial forces which led to the development of such a state of finite strain.

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