COMPOSITE DESCRIBING FUNCTION OF SUCCESSIVE NON-LINEAR COMPONENTS IN FEED BACK CONTROL SYSTEMS

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Abstract

Recent work by the author on a transformation method for the study of non-linear components is utilized to evaluate the composite describing function of successive non-linear components in feed-back control systems. Characteristic equations for different values of the non-linearizing parameters are given for commonly encountered non-linear components, in control system practice. Composite describing functions are evaluated in case of a backlash followed by a dead-zone, and also in case of a rate-limiting device followed by a variable gain amplifier. In the latter case, the composite describing function is dependent on the input amplitude and frequency.

Introduction

In a recent paper published by the author[1] in the Transactions, AIEE, the transformation method is discussed in great detail. The work reported here is an extension of the above study, for the evaluation of composite describing functions of two consecutive non-linear components in closed-loop feedback control systems. The describing function technique[2-4] has become a useful tool for the sinusoidal analysis of non-linear closed-loop systems. However, the underlying assumptions are such that it is useful only when a single non-linearity is present in the closed-loop system. An extension of this technique is possible when two non-linear components are separated by linear filtering networks so that higher harmonics in the output of the first non-linear component are sufficiently attenuated before appearing as input to the second non-linear component.

If two or more non-linear components are present in a system without being separated by linear filtering networks, the only approach possible is to treat all the non-linear elements enblock, and to evaluate the composite describing functions. Gronner[5] has adopted this approach and derived analytically the composite describing function of a backlash followed by a dead-zone. Mikhail and Fett[6] have given a graphical method when two successive non-linear components are in series, with one or both of them having describing functions dependent on amplitude and frequency of the sinusoidal input. The present paper uses the characteristic equations of the non-linear components to evaluate the composite describing function.

In the case of a dead-zone, saturation and backlash, Fourier coefficients[7] and the respective characteristic equations are tabulated in Tables I to IV. Example 1, deals with a backlash followed by dead-zone and Example 2 deals with a rate-limiting device followed by variable gain amplifier. In case of Example 1, results are compared with those evaluated by Gronner, with the aid of a computer.

Basic Theory

An important point is to be observed while using the characteristic equations given in Tables I to V. In all these Tables, though the values of the non-linearizing parameter and Fourier coefficients are given in the normalized form, the characteristic equations are true only for the unity input amplitude.

However, these characteristic equations can be utilized for studying the frequency response of the non-linear element to non-unity input amplitudes, by a proper change of variable.

Let
$$f(x) = ax + bx^3 + cx^5, \qquad -1 \leq x \leq +1 \tag{1}$$

be the characteristic equation of a certain non-linear element. Then for an input

$$x = A \sin wt, \qquad \text{for } |A| < 1 \tag{2}$$

the modified equation to be used is given by

$$f\left(A \cdot \frac{x}{A}\right) = f(Ay) = F(y)$$
$$= aA \,.\, y + bA^3y^3 + cA^5y^5 \tag{3}$$

Now, treating $x/A = y$, as a new variable in the recurrence relations[8], the Fourier coefficient in $F(y)$ to an input $\sin wt$ are exactly the same as those in $f(x)$ for an input $x = A \sin wt$.

The case when $|A| > 1$, can be dealt only when the amplitude A is a scalar and is independent of the input frequency. In such a case, an equivalent scalar gain $K > 1$, is taken out to be represented as a scalar amplifier following the non-linear element, thus making the input amplitude $|A/K| < 1$.

The values of the Fourier coefficients in Table I are evaluated from, Fig. I, as per equations:

$$b_1/A = (2/\pi)\left(\tfrac{1}{2}\pi - B - \tfrac{1}{2}\sin 2B\right) \tag{4}$$

and
$$\frac{b_n}{A} = \frac{-2}{n\pi}\left[\frac{\sin(n-1)B}{(n-1)} + \frac{\sin(n+1)B}{(n+1)}\right] \text{ for } n = 3, 5, \ldots \tag{5}$$

where,
$$B = \sin^{-1}(d/A); \tag{6}$$

and input
$$X = A \sin wt. \tag{7}$$

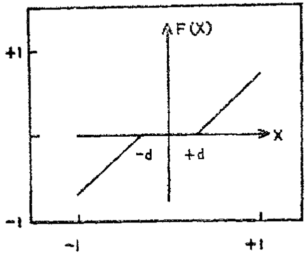

FIG. I A

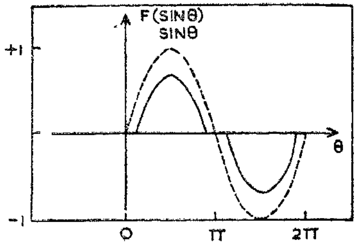

FIG. I B

TABLE I

Fourier coefficients and characteristic equations for dead-space

d/A	b_1/A	b_3/A	b_5/A	Characteristic equation for $A=1$
0.9	0.06	−0.06	+0.0216	$-0.012x-0.192x^3+0.3456x^5$
0.8	0.107	−0.0714	+0.029	$+0.042x-0.295x^3+0.464x^5$
0.7	0.188	−0.1045	+0.0212	$-0.0195x-0.006x^3+0.3392x^5$
0.6	0.285	−0.13	−0.0096	$-0.155x+0.72x^3-0.16x^5$
0.5	0.[illegible]9	−0.138	−0.0276	$-0.162x+1.104x^3-0.4416x^5$
0.4	0.505	−0.13	−0.0452	$-0.111x+1.424x^3-0.723x^5$
0.3	0.624	−0.11	−0.0505	$+0.044x+1.45x^3-0.808x^5$
0.2	0.75	−0.08	−0.0426	$0.297x+1.172x^3-0.6816x^5$
0.1	0.874	−0.0415	−0.0243	$0.628x+0.652x^3-0.388x^5$

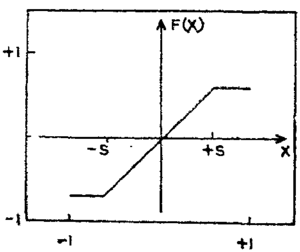

FIG. II A

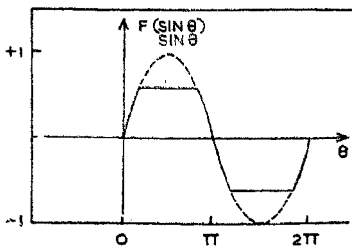

FIG. II B

TABLE II

Fourier coefficients and characteristic equations for saturation

S/A	b_1/A	b_3/A	b_5/A	Characteristic equation for $A=1$
1	1	0	0	x
0.9	0.965	0.06	− 0.0216	$1.3x+0.19x^3-0.347x^5$
0.8	0.875	0.0714	− 0.029	$x+0.295x^3-0.464x^5$
0.7	0.812	0.1045	− 0.0212	$1.02x+0.006x^3-0.339x^5$
0.6	0.72	0.13	0.0096	$1.16x-0.72x^3+0.16x^5$
0.5	0.61	0.138	0.0276	$1.162x-1.104x^3+0.4416x^5$
0.4	0 495	0.13	0.0452	$1.111x-1.424x^3+0.7232x^5$
0.3	0.377	0.11	0.0505	$0.9595x-1.45x^3+0.808x^5$
0.2	0.253	0.08	0.0426	$0.706x-1.172x^3+0.6816x^5$
0.1	0.127	0.0415	0.0243	$0.373x-0.652x^3+0.388x^5$

The Fourier coefficients for the saturating element as given in Table II are evaluated from Fig. II, as per equations

$$b_1/A=(2/\pi)\ (B+\tfrac{1}{2}\sin 2\,B) \qquad [8]$$

$$\frac{b_n}{A}=\frac{2}{n\pi}\left[\frac{\sin(n-1)\,B}{(n-1)\,B}+\frac{\sin(n+1)\,B}{(n+1)}\right]\text{for } n=3,5,\ldots \qquad [9]$$

wherein $$B=\sin^{-1}(S/A) \qquad [10]$$

and input $$X=A\sin wt; \qquad [11]$$

The Fourier coefficients for the backlash element as given in Table III, are evaluated from Fig. III, as per equations

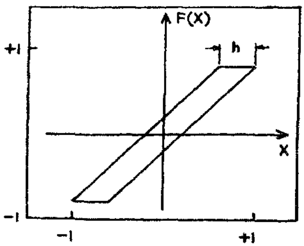

FIG. III A

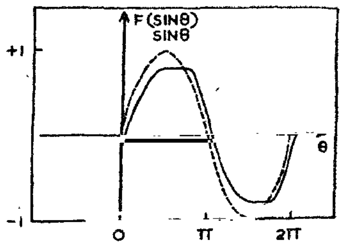

FIG. III B

TABLE III

Fourier co-efficients for Backlash

h/A	1.2	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
B	$-11°32'$	$0°$	$+5°45'$	$+11°32'$	$+17°29'$	$+23°35'$	$+30°$	$+36°52$	$+44°26'$	$+53°8'$	$+64°10'$
a_1/A	-0.306	-0.318	-0.315	-0.305	-0.29	-0.268	-0.239	-0.231	-0.162	-0.115	-0.059
b_1/A	$+0.373$	$+0.495$	$+0.5636$	$+0.627$	$+0.688$	$+0.747$	$+0.808$	$+0.856$	$+0.907$	$+0.950$	$+0.980$
d_1/A	$+0.483$	0.593	$+0.645$	$+0.697$	$+0.745$	$+0.795$	$+0.84$	$+0.880$	$+0.92$	$+0.955$	$+0.983$
ϕ_1	$-39°20'$	$-32°30'$	$-29°15'$	$-26°$	$-22°50'$	$-19°42'$	$-16°30'$	$-15°16$	$-10°6'$	$-6°54'$	$-3°30'$
a_3/A	$+0.094$	$+0.106$	$+0.103$	$+0.0934$	$+0.079$	$+0.0606$	$+0.039$	$+0.019$	$+0.0015$	-0.0107	-0.0124
b_3/A	-0.04	0	$+0.021$	$+0.04$	$+0.0554$	$+0.0656$	$+0.069$	$+0.065$	$+0.0545$	$+0.0368$	$+0.0159$
d_3/A	-0.104	$+0.106$	$+0.105$	$+0.104$	$+0.096$	$+0.089$	$+0.078$	$+0.068$	$+0.0546$	$+0.0384$	$+0.0202$
ϕ_3	$-66°54'$	$+90°$	$+78°30'$	$+66°54'$	$+55°1'$	$+42°46'$	$+30°$	$+16°17'$	$+1°35'$	$-16°10'$	$-38°0'$
a_5/A	-0.00945	-0.021	-0.0181	-0.00945	-0.0028	-0.00178	$+0.0238$	$+0.0268$	$+0.022$	-0.0107	-0.0006
b_5/A	-0.024	0	$+0.0122$	$+0.024$	$+0.0252$	$+0.0225$	$+0.0137$	$+0.0016$	-0.0097	-0.0155	-0.011
d_5/A	-0.0258	$+0.021$	$+0.0218$	$+0.0258$	$+0.0254$	$+0.0227$	$+0.0275$	$+0.027$	-0.024	-0.0188	-0.011
ϕ_5	$\pi+21°18'$	$90°$	$-55°57$	$-21°18$	$-6°17'$	$-4°30'$	$+60°6'$	$+86°36$	$\pi-66°2$	$\pi+35°$	$\pi+3°6'$

$$a_1/A = (h/A\pi)\ (h/A - 2), \quad [12]$$

$$b_1/A = \tfrac{1}{2} + (B/\pi) + (1/\pi)\ (1 - h/A) \cos B, \quad [13]$$

$$a_3/A = (1/6\pi)\ [\cos 2B + \tfrac{1}{2}\ (1 + \cos 4B)], \quad [14]$$

$$b_3/A = (1/6\pi)\ [\tfrac{1}{2} \sin 4B + \sin 2B] \quad [15]$$

$$a_5/A = (1/10\pi)\ [\tfrac{1}{6} - \tfrac{1}{2} \cos 4B - \tfrac{1}{3} \cos 6B], \quad [16]$$

$$b_5/A = (1/10\pi)\ [\tfrac{1}{3} \sin 6B + \tfrac{1}{2} \sin 4B], \quad [17]$$

wherein, $B = \sin^{-1}\ (1 - h/A),$ [18]

and input, $X = A \sin wt.$

TABLE IV

Characteristic Equations for Backlash

h/A	Characteristic equations determined from Fourier coefficients for $A=1$
1.2	$0.133x + 0.64x^3 - 0.384x^5 \pm [\sqrt{(1 - x^2)}]\ [-0.2214 - 0.263x^2 - 0.1512x^4]$
1.0	$0.495x \pm [\sqrt{(1 - x^2)}]\ [-0.233 - 0.172x^2 - 0.336x^2]$
0.9	$0.6876x - 0.328x^3 + 0.1952x^5 \pm [\sqrt{(1 - x^2)}]\ [-0.2301 - 0.1948x^2 - 0.2896x^4]$
0.8	$0.867x - 0.64x^3 + 0.385x^5 \pm [\sqrt{(1 - x^2)}]\ [-0.221 - 0.26x^2 - 0.151x^4]$
0.7	$0.361x - 0.7256x^3 + 0.4032x^5 \pm [\sqrt{(1 - x^2)}]\ [-0.2138 - 0.2844x^2 - 0.0448x^4]$
0.6	$1.0563x - 0.7124x^3 + 0.36x^5 \pm [\sqrt{(1 - x^2)}]\ [-0.209 - 0.264x^2 - 0.0286x^4]$
0.5	$1.0835x - 0.55x^3 + 0.2192x^5 \pm [\sqrt{(1 - x^2)}]\ [-0.1762 - 0.4416x^2 + 0.3808x^4]$
0.4	$1.059x - 0.292x^3 + 0.0256x^5 \pm [\sqrt{(1 - x^2)}]\ [-0.1852 - 0.3976x^2 + 0.4288x^4]$
0.3	$1.022x - 0.024x^3 - 0.1552x^5 \pm [\sqrt{(1 - x^2)}]\ [-0.1385 - 0.27x^2 + 0.352x^4]$
0.2	$0.9829x + 0.1628x^3 - 0.258x^5 \pm [\sqrt{(1 - x^2)}]\ [-0.1364 + 0.1712x^2 - 0.1712x^4]$
0.1	$0.9727x + 0.1564x^3 - 0.176x^5 \pm [\sqrt{(1 - x^2)}]\ [-0.072 + 0.0568x^2 - 0.0096x^4]$

Table IV gives the characteristic equations for backlash. Table V gives the Fourier coefficients, for variable gain elements, derived directly from their characteristic equations of type $y = x^n$, $n = 1, 3, 5 \ldots$ by making use of the transformation method. Table VI gives the composite describing function for

Table V

Fourier coefficients and characteristic equations for variable gain elements

n	Ch. equation for $A=1$	b_1/A	b_3/A	b_5/A	b_7/A
1	x	1	0	0	0
3	x^3	0.75	-0.25	0	0
5	x^5	0.625	-0.312	$+0.0625$	0
7	x^7	0.548	-0.328	$+0.109$	-0.156

Table VI

Composite describing function for backlash followed by dead-zone

$B=d/h$	d	h	Composite describing function: Evaluated by tranformation method		Analytically derived and computed	
			Magnitude	Phase radians	Magnitude	Phase radians
0.1	0.1	1	0.485	-0.550	0.459	-0.584
0.5	0.2	0.4	0.605	-0.236	0.628	-0.245
1.4	0.7	0.5	0.059	-0.2168	0.066	Not available
3.0	0.6	0.2	0.263	-0.14	0.240	-0.159

backlash followed by dead-zone. The values evaluated by using the transformation method are compared with those analytically derived and computed by Gronner[9]. An attempt is made to cover a wide range for $B=d/h$, the parameter which characterizes the composite non-linearity. The following illustrative example shows the details of the calculation.

Example 1

$$\text{Let } h=0.2, \quad d=0.6, \quad \therefore \; B=d/h=3$$

with reference to Fig. IV, $U=\sin wt$ [19]

From Table III, for $h=0.2$, the output Y is approximated, considering only the first and third harmonics, by

$$Y=0.95 \sin (wt-6^\circ 54')+0.0384 \sin (3wt-16^\circ 10') \qquad [20]$$

Equation [20] can be re-written as,

$$Y=0.95 \sin \gamma+0.0384 \sin (3\gamma+4^\circ 32') \qquad [21]$$

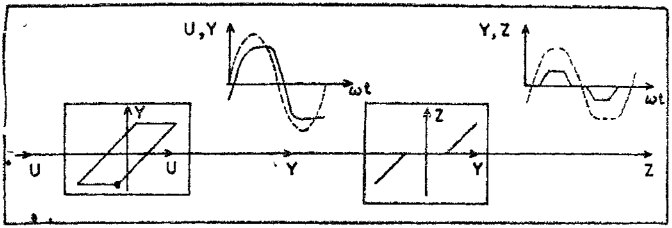

FIG. IV

where $$\gamma = wt - 6^\circ 54' \qquad [22]$$

Now, let $$\alpha = \sin\gamma \qquad [23]$$

Equation [21] can be written as

$$Y = 0.95 \sin\gamma + 0.0384 \cos 4^\circ 32' \sin 3\gamma + 0.0384 \sin 4^\circ 32' \cos 3\gamma \qquad [24]$$

Using equation [23], [24] can be written as

$$Y = 1.0652\ \alpha - 0.1536\ \alpha^3 + 0.03\,[\sqrt{(1-\alpha^2)}]\,(1 - 4\,\alpha^2) \qquad [25]$$

Since the input amplitude corresponding to the fundamental in equation [25] is greater than 1, the input is considered as $Y/1.0652$ and following the non-linear element a hypothetical scalar amplifier of gain 1.0652 is introduced.

Now, from Table I, the characteristic equation for a dead-space element with $d = 0.6$, is given by

$$f(x) = -0.155\,x + 0.72\,x^3 - 0.16\,x^5 \qquad [26]$$

To evaluate the composite describing function for a backlash element $(h = 0.2)$, followed by dead-zone element $(d = 0.6)$, we need evaluate the fundamental Fourier co-efficient of the overall output wave form.

Let modified equation [25] be identified as

$$Y/1.0652 = \alpha - 0.144\ \alpha^3 + 0.0028\,[\sqrt{(1-\alpha^2)}]\,(1 - 4\,\alpha^2)$$
$$= a\,\alpha + b\,\alpha^3 + c\,[\sqrt{(1-\alpha^2)}]\,(1 - 4\alpha^2), \qquad [27]$$

Let the dead-zone characteristic equation [26] be identified as

$$f(y) = Ly + My^3 + Ny^5, \qquad [28]$$

Now, the response of the characteristic [28] to an output as per equation [27], can be represented by

$$La\alpha + (Ma^3 + Lb)\,\alpha^3 + (Na^5 + 3\,Ma^2b)\,\alpha^5 + Lc\,[\sqrt{(1-\alpha^2)}]\,(1 - 4\alpha^2), \qquad [29]$$

Equation [29] is obtained after ignoring terms in c^2, and higher powers of c in the substitution of [27] into equation [28]. Bringing the equation into a standard form like [29] is desirable, so that repeated calculation can be avoided, and equation [29] used with suitable values for a, b, c, and L, M, N, depending on the values of h and d. for $h = 0.2$, and $d = 0.6$, [29] gives

$$-0.155\,\alpha + 0.7423\,\alpha^3 - 0.264\,\alpha^5 - 0.0004\,[\sqrt{(1-\alpha^2)}]\,(1-4\alpha^2) \qquad [30]$$

From eqation [23] where α stands for $\sin\gamma$, if B_n denotes the n^{th} Fourier sine coefficient of the composite wave form, equation [30] gives

$$B_1 = 0.258, \quad B_3 = -0.1056, \quad B_5 = -0.016, \quad A_1 = -0.0004.$$

Taking into account, the hypothetical scalar amplifier introduced, the composite describing function is given by

$$1.0652\ \sqrt{(A_1^2 + B_1^2)}\ \angle\left(-6^\circ 54' + \tan\frac{-0.0004}{0.258}\right)$$

$$= 0.263\ \angle(-0.14 \text{ radians}) \qquad [31]$$

Table VI *shows differences upto 10%. Besides the error in slide rule* calculation, this error is mainly due to the fact, that fifth and higher harmonics are neglected in the input to the second non-linear element.

Example 2

A rate-limiting device followed by a variable gain amplifier, as per Fig. VII. (Refer to Fig. V and VI for individual elements.)

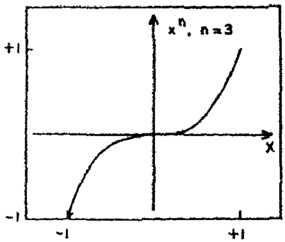

Fig. V A

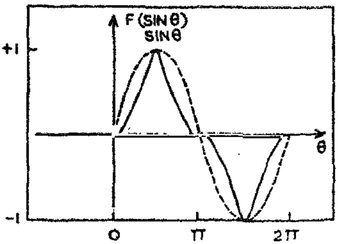

Fig. V B

To an input $X = a \sin wt$, the rate-limiting device has a triangular wave form for its response as shown in Fig. V. This wave form is given by

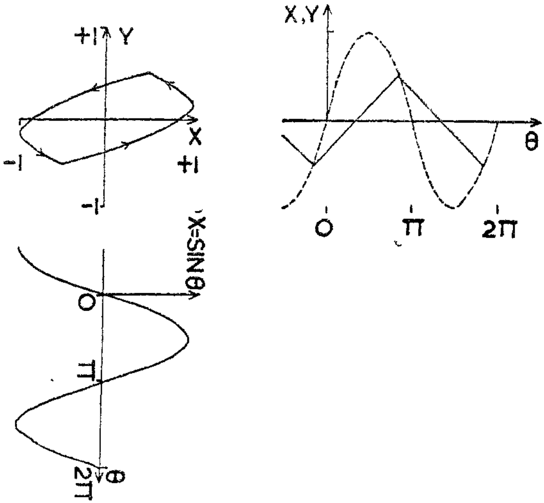

FIG. VI

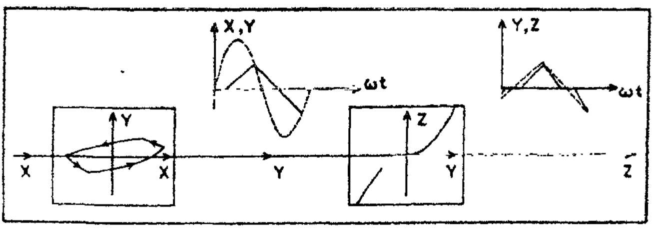

FIG. VII

$$Y = C_1 \sin(wt + \phi) - C_3 \sin(3\,wt + 3\,\phi) + C_5 \sin(5\,wt + 5\,\phi) \;\ldots \quad [32]$$

wherein,

$$\cos\phi = \frac{\pi V}{2\,\omega a} \quad [33]$$

$$C_1 = \frac{8}{\pi^2}\ \frac{V\pi}{2\omega a},\quad C_3 = \frac{C_1}{9},\quad C_5 = \frac{C_1}{25} \qquad [34]$$

In equation [33], ω is the frequency and a is the amplitude of the input sinusoid, and V is the maximum value of the output velocity, determined by the rate limiting device.

Now, let the input to the variable gain amplifier be approximated by

$$Y \simeq C_1 \sin(wt + \phi) - (C_1/9) \sin(3wt + 3\phi) \qquad [35]$$

Assuming $|C_1| < 1$, equation [35] can be written as

$$Y \simeq C_1 \sin\gamma - (C_1/9) \sin 3\gamma, \qquad [36]$$

where $\gamma = wt + \phi$

Let

$$Z = y^3 \qquad [37]$$

be the characteristic equation of the variable gain amplifier, then from equations [36] and [37],

$$Z = [C_1 \sin\gamma - (C_1/9) \sin 3\gamma]^3 \qquad [38]$$

Let $\alpha = \sin\gamma$ [39]

From [39], equation [38] can be written as

$$Z = \left[C_1\alpha - \frac{C_1}{3}\alpha + \frac{4C_1}{9}\alpha^3\right]^3 \qquad [40]$$

Expanding the right hand side of equation [40], and collecting coefficients, one gets

$$Z = 0 + \frac{8C_1^3}{27}\alpha^3 + \frac{16C_1^3}{27}\alpha^5 \qquad [41]$$

Now, using the transformation, and the recurrence relation in $\sin n\gamma$, one gets the composite harmonic components as

$$B_5 = +\frac{C_1^3}{27};\quad B_3 = -\frac{7C_1^3}{27};\ \text{and}\ B_1 = \frac{16C_1^3}{27}; \qquad [42]$$

Substituting for C_1 from equation [34], the composite describing function is given by

$$\frac{16 \times 64\, V^3}{27\,\pi^3\,\omega^3\,a^3} \qquad [43]$$

and phase

$$\phi = \cos^{-1}(\pi V/2a\,\omega) \qquad [44]$$

It is to be observed that both magnitude and phase of the composite describing function are functions of the amplitude, and frequency of the input sinusoid to the first of the non-linear elements.

The main advantage of this method depends on the experimental evaluation of the composite describing function, wherein not even the higher harmonics are ignored in the input to the second non-linear element. The composite wave form of two successive non-linear elements in series, to an input sin wt, is analysed by the aid of a harmonic analyzer. The phase of each of these harmonics can be fixed up as plus or minus, by a knowledge of the wave form as recorded on a cathode ray oscilloscope. Now, knowing the magnitude and phase of the harmonic components in the composite wave form, a composite characteristic equation can be fixed up for the two non-linear elements in series. This composite characteristic equation, in turn, can be used for studying multiple sinusoidal inputs. In fact the principle can be extended to any number of non-linear components in series.

Conclusions

(i) The purpose of this paper is mainly to illustrate the principle and bring out the practical application of the author's transformation method.

(ii) A detail reference table of composite describing functions is not attempted, because of the lack of a digital computer.

(iii) Characteristic equations for different values of non-linearizing parameter are given in case of dead-space, saturation and backlash.

(iv) The principle used here is aptly suited for the study of 'The improved describing function, accounting for circulating harmonics in the closed-loop'. This aspect will be discussed in a subsequent paper.

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