ON THE STABILITY OF A RIGIDLY ROTATING LIQUID COLUMN WITH AXIAL MAGNETIC FIELD

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Abstract

We have discussed the stability of a rigidly rotating infinite liquid column of infinite conductivity in the presence of a uniform axial magnetic field. We have derived the dispersion relation in the general case taking the cylinder to be self-gravitating and accounting for the surface tension forces on the free surface of the liquid. We have shown that previous results obtained by Rayleigh,[5] Hocking[4] and Simon[6] are particular cases of the present dispersion relation. We have discussed the present dispersion relation for large wave-length disturbances neglecting surface tension forces in the cases of (*i*) non-gravitating cylinder with the velocity of rotation small on the surface compared with the Alfven wave velocity (*ii*) non-gravitating cylinder with velocity of rotation large on the surface compared with the Alfven wave velocity and (*iii*) gravitating cylinder. We find that in case (*i*) there is stability but in the other two cases disturbances are unstable.

Introduction

The problem of gravitational and magnetogravitational stability of an infinite, homogeneous and rotating fluid medium has been studied recently by a number of authors [Chandrasekhar[1], Pacholczyk and Stodolkiewicz[2]]. The corresponding problem for a rotating fluid column of finite radius does not seem to have attracted the same attention. Recently Hocking and Michael[3] and Hocking[4] have considered the effect of rotation on the stability of a non-gravitating infinite liquid column without magnetic field. They find that rotation always tends to reduce the stability.

In the present paper we have discussed the stability of an infinitely conducting, infinite, rigidly rotating liquid column with a uniform axial magnetic field taking it to be continuous across the surface of the cylinder. In deducing the dispersion relation we have considered the cylinder to be self-gravitating and have taken account of the surface tension forces on the free surface of the cylinder. In the axisymmetric case we can deduce the stability criterion given by Rayleigh[5] for a nonrotating liquid jet on putting $\Omega = 0$, $G = 0$ and $B = 0$ in the present dispersion relation, where Ω denotes the angular velocity of rotation, G the gravitational constant and B the magnitude of the axial magnetic field. We can get the dispersion relation given by Hocking[4] when we put $G = 0$ and $B = 0$ in the axisymmetric case of our present dispersion relation, while the case considered by Simon[6], *viz.*, hydromagnetic oscillations of an infinitely conducting gravitating liquid column at rest in equilibrium with an axial uniform magnetic field can be obtained by putting $\Omega = 0$.

In the present paper we have discussed the dispersion relation for the axisymmetric disturbances with large wave length in the following three cases: (i) $G = 0$, $V_A >> \Omega r_0$, (ii) $G = 0$, $V_A << \Omega r_0$, (iii) $G \neq 0$, where V_A is the Alfven wave velocity in the cylinder and r_0 is the radius of the undisturbed cylinder.

We find that in the case (i) the cylinder is stable against large wavelength disturbances, however it is unstable in cases (ii) and (iii).

Basic Equations

In terms of the usual symbols p, ρ, $\mathbf{v}$, $\mathbf{E}$, $\mathbf{B}$, $\mathbf{j}$, ϵ, μ_0 and k_0 for pressure, density, velocity, electric field, magnetic induction, volume current density, electric charge density, magnetic permeability and dielectric constants respectively we have in the M.K.S. system of units the following equations for the conducting liquid when the displacement current is neglected.

$$\rho(d\mathbf{v}/dt) = \mathbf{j} \times \mathbf{B} + \epsilon \mathbf{E} - \nabla p + \rho \nabla \phi_i , \tag{1}$$

$$\nabla \cdot \mathbf{v} = 0 , \tag{2}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 , \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} , \tag{3, 4}$$

$$\nabla \cdot \mathbf{B} = 0 , \qquad \nabla \times \mathbf{E} = -(\partial \mathbf{B}/\partial t) , \tag{5, 6}$$

$$\nabla \cdot \mathbf{E} = \epsilon/k_0 , \qquad \nabla^2 \phi_i = -4\pi G \rho , \tag{7, 8}$$

where ϕ_i is the gravitational potential inside the conducting liquid.

Vacuum Equations

The equations holding inside the vacuum are Maxwell's equations (4) to (7) with $\mathbf{j} = 0$ and $\epsilon = 0$ and the equation for gravitational potential

$$\nabla^2 \phi_0 = 0 , \tag{9}$$

where ϕ_0 is the gravitational potential in the vacuum outside the liquid.

Following Kruskal and Schwarzchild[7] we can write down the surface conditions in the following form even when surface tension is present. (*c.f.* Chakraborty and Bhatnagar[8])

$$u = \mathbf{n} \cdot \mathbf{v} , \tag{10}$$

$$\mathbf{n} \times [\mathbf{B}] = \mu_0 \mathbf{j}^* , \qquad \mathbf{n} \cdot [\mathbf{B}] = 0 , \tag{11, 12}$$

$$\mathbf{n} \times [\mathbf{E}] = u[\mathbf{B}] , \qquad \mathbf{n} \cdot [\mathbf{E}] = \epsilon^*/k_0 , \tag{13, 14}$$

$$\mathbf{j}^* \times \bar{\mathbf{B}} + \epsilon^* \bar{\mathrm{E}} - \mathbf{n}[p] + \mathbf{n}\, T(k_a + k_b) = \mathbf{0}\,, \tag{15}$$

where $k_a + k_b$ is the sum of the principal curvatures of the surface of liquid at the point under considerations. Here $\mathbf{n}$ is the unit normal to the surface and is directed into the conducting liquid, u is the normal component of the velocity of the surface, p and $\mathbf{u}$ are the pressure and the velocity of the liquid on the surface, $\mathbf{j}^*$ and ϵ^* are the surface current density and surface charge density and T is the surface tension. The brackets denote the jump in the enclosed quantity upon crossing the surface from the vacuum into the liquid, and a bar above a quantity denotes the arithmatic mean of the values of that quantity just on each side of the boundary. The boundary conditions satisfied by the gravitational potential on the liquid surface are

(i) the gravitational potential should be continuous,

(ii) $\nabla \phi$ should be continuous.

Following Chandrasekhar[9] we shall neglect the term $\epsilon \mathbf{E}$ in (1) as its contribution can be shown to be negligible in comparison with the Lorentz for $\mathbf{j} \times \mathbf{B}$ or the inertia force in our present problem. We shall similarly neglect $\epsilon^* \bar{\mathrm{E}}$ in comparison with $\mathbf{j}^* \times \bar{\mathbf{B}}$ in (15).

STEADY STATE

In the steady state the liquid cylinder is rotating rigidly about its axis with uniform angular velocity Ω. The magnetic permeability μ_0 of the liquid is taken to be the same as in the vacuum and the magnetic induction $\mathbf{B} = (0, 0, B)$ in the cylindrical polar co-ordinates (r, θ, z), z-axis coinciding with the axis of the cylinder, is taken uniform in the liquid as well outside. From the equations (1) to (7) we have for the liquid

$$\mathbf{v} = (0,\ \Omega r,\ 0), \quad \mathbf{j} = 0, \quad \mathbf{E} = (-\Omega r B,\ 0,\ 0), \quad \epsilon = -2 k_0 \Omega B,$$
$$p = \pi G \rho^2 (r_0^2 - r^2) + (\rho \Omega^2/2)(r^2 - r_0^2) + p_{is},$$

where p_{is} is the pressure of the liquid on the surface and r_0 is the radius of the cylinder. The gravitational potential is given by

$$\phi_i = -\pi G \rho\, (r^2 - r_0^2)$$
$$\phi_0 = -2\pi G \rho r_0^2 \log (r/r_0) \tag{16}$$

From the surface equations (10) to (15), we have, assuming continuity of $\mathbf{B}$ and the normal component of $\mathbf{E}$,

$$\mathbf{j}^* = 0, \quad \epsilon^* = 0, \quad T/r_0 - p_{is} = 0. \tag{17}$$

By definition $\mathbf{n}$ is given by $\mathbf{n} = (-1, 0, 0)$ in the cylindrical polar co-ordinates.

Small Perturbations and the Solutions of the Linearized Equations

We consider the follownig perturbations from the steady configuration.

$$q = q_0 + \tilde{q}\,, \quad \tilde{q} = \hat{q}(r)\,\exp.(\,i m\theta + i k z + \omega t\,)\,, \tag{18}$$

where q_0 denotes the steady state value of any physical quantity q. Since we are interested in the stability problem we shall neglect the powers and products of the small quantities $\hat{q}(r)$.

We linearize the equations and finally obtain the following equations that hold inside the liquid.

$$\nabla\cdot\tilde{\mathbf{v}} = 0\,, \tag{19}$$

$$\tilde{\mathbf{E}} + \tilde{\mathbf{v}}\times\mathbf{B} + \mathbf{v}\times\tilde{\mathbf{B}} = 0\,, \qquad \nabla\times\tilde{\mathbf{B}} = \mu_0\,\tilde{\mathbf{j}}\,, \tag{20, 21}$$

$$\nabla\cdot\tilde{\mathbf{B}} = 0\,, \qquad \nabla\times\tilde{\mathbf{E}} = -\,\partial\tilde{\mathbf{B}}/\partial t\,, \tag{22, 23}$$

$$\nabla\cdot\tilde{\mathbf{E}} = \tilde{\mathbf{E}}/k_0\,, \qquad \nabla^2\phi_i = 0\,, \tag{24, 25}$$

$$\rho\,[\,(\,\omega + \Omega\, i\, m\,)\,\tilde{\mathbf{v}} + 2\,\Omega\,\tilde{\mathbf{v}}_r\mathbf{e}_\theta - 2\,\Omega\,\tilde{\mathbf{v}}_\theta\,\mathbf{e}_r\,] = -\,\mathrm{grad}\,\tilde{P} + \tilde{\mathbf{j}}\times\mathbf{B}\,, \tag{26}$$

where $\tilde{P} = \tilde{p} - \rho\,\tilde{\phi_i}$ and $\mathbf{e}_r$, $\mathbf{e}_\theta$ are the unit vectors in the directions of $r-$ and $\theta-$increasing respectively and $\tilde{\mathbf{v}} = (\tilde{v}_r\,,\ v_\theta\,,\ \tilde{v}_2)$. In view of (18) we can express all the perturbed quantities of interest in terms of $\tilde{\mathbf{v}}$ with the help of equations (20) to (24). Using (19) and (26) we have

$$r^2\,(d^2\,\hat{v}_z/dr^2) + r\,(d\,\hat{v}_z/dr) - [m^2 + (k^2r^2/C^2)(C^2 + 4\,\Omega^2\rho^2)]\,\hat{v}_z = 0 \tag{27}$$

where

$$A = \omega + i\,m\,\Omega, \quad C = \rho A + k^2B^2/(\mu_0\,A) \tag{28}$$

The solution of (27) regular on the axis is given by

$$\hat{v}_z = D\,I_m\,\{kr\,[\,(C^2 + 4\,\Omega^2\rho^2)/C^2]^{1/2}\,\}\,, \tag{29}$$

where I_m is the modified Bessel function of the first kind and order m and D is an arbitrary constant. All the perturbed quantities can now be expressed in terms of $\hat{v}_z$ and its derivatives.

The perturbed equations in the vacuum are the Maxwell's equations (21) to (24) with $\tilde{\mathbf{j}} = 0$ and $\epsilon = 0$; together with the equation for perturbations in the gravitational potential

$$\nabla^2\phi = 0 \tag{30}$$

Solving (25) and (30) under the appropriate boundary conditions we have

$$\hat{\phi}_i = 4\,\pi\,G\,r_0\,\rho\,k_m\,(kr_0)\,I_m(kr)\,\delta r$$

$$\hat{\phi}_0 = 4 \pi G r_0 \rho I_m (k r_0) K_m (k r) \delta r, \tag{31}$$

where K_m is the modified Bessel function of the second kind and m th order and δr is given by the equation for the disturbed liquid surface, *viz.*,

$$r = r_0 + \delta r \exp. (i k z + i m \theta + \omega t). \tag{32}$$

By solving the Maxwell equations in the vacuum we get

$$\hat{\mathbf{B}} = E_0 [k K_m'(kr), (i m/r) K_m (k r), i k K_m (k r)], \tag{33}$$

where E_0 is a constant.

BOUNDARY CONDITIONS AND THE DISCUSSION OF THE DISPERSION RELATION

Substituting the solutions for the perturbed quantities in the surface equations (10) to (15) we obtain the following dispersion relation

$$B^2 k K_m' (k r_0)/[\mu_0 K_m (k r_0)] + 4 \pi G r_0 \rho^2 K_m (k r_0) I_m (k r_0) - 2 \pi G \rho^2 r_0 + \rho \Omega^2 r_0$$
$$- (T/r_0^2)(m^2 - 1 + r_0^2 k^2) = N/D$$

where $$N = - i A \{C^2 + 4 \Omega^2 \rho^2\} I_m \{k r_0 [(C^2 + 4 \Omega^2 \rho^2)/C^2]^{1/2}\}$$

and $$D = (2 m \Omega \rho / r_0) I_m \{k r_0 [(C^2 + 4 \Omega^2 \rho^2)/C^2]^{1/2}\}$$
$$- ikC[(C^2 + 4 \Omega^2 \rho^2)/C^2]^{1/2} I_m' \{k r_0 [(C^2 + 4 \Omega^2 \rho^2)/C^2]^{1/2}\} \tag{34}$$

PARTICULAR CASES

(i) *Rayleigh's Result.* When the cylinder is non-gravitating, non-rotating and the magnetic field is absent, (34) in the axisymmetric case reduces to

$$(T/r_0^2)(1 - r_0^2 k^2) = [\rho \omega^2 I_0(k r_0)]/[k I_0' (k r_0)]. \tag{35}$$

From (35) we can see that stable disturbances are possible only when

$$1 < r_0^2 k^2.$$

(ii) *Hocking's Result.* If the cylinder is non-gravitating, disturbances axi-symmetric, and the magnetic field is absent, (34) reduces to

$$\alpha J_0 (\alpha) = J_1 (\alpha) \{ T(1 - r_0^2 k^2)/(r_0^3 \rho \Omega^2) + 1\} (\Omega^2 r_0^2 k^2/\omega^2), \tag{36}$$

where $\alpha^2 = - k^2 r_0^2 [1 + 4\Omega^2/\omega^2]$.

(iii) *Simon's Result.* If the cylinder is non-rotating and the surface tension is absent the dispersion relation (34) reduces to

$$\{[B^2/(\mu_0 \rho)]k^2 K_m(k r_0)/[K_m'(k r_0)] + 4\pi G \rho k r_0 K_m(k r_0) I_m(k r_0) - 2\pi G \rho r_0 k\} \times \{I_m'(k r_0)/[I_m(k r_0)]\} = \omega^2 + k^2 B^2/(\mu_0 \rho). \tag{37}$$

Dispersion relation in the present case. The dispersion relation (34) in the axisymmetric case and in the absence of surface tension reduces to

$$R K_0(R) I_0(R) - \frac{R}{2} + \frac{\Omega^2 R}{4\pi G \rho} - \frac{V_A^2 R^2 K_0(R)}{4\pi G r_0^2 \rho K_1(R)} = \frac{[4\Omega^2 r_0^2 \pm \sqrt{\{[2V_A^2(W^2-1)R^2 - 4\Omega^2 r_0^2]^2 - 4R^4 V_A^4 (W^2-1)^2\}}]\, W I_0(RW)}{8\pi G \rho r_0^2 (W^2-1) I_1(RW)}, \tag{38}$$

where we have introduced the quantities $k r_0 = R$, $V_A = B/\sqrt{(\mu_0 \rho)}$, (Alfven wave velocity), $V_w = \omega/k$ and

$$W^2 = 1 + \frac{4\Omega^2 r_0^2}{R^2 V_w^2 (1 + V_A^2/V_w^2)^2}. \tag{39}$$

We shall discuss the dispersion relation for disturbances of large wavelength, *i.e.*, R is small.

Case (i).

$V_A >> \Omega r_0$ and the material is non-gravitating. We shall neglect $-[V_A^2 R^2 K_0(R)]/[r_0^2 K_1(R)]$ in comparison with $\Omega^2 R$ as R is small. Putting $W = iW_1$, (38) can be written as

$$2R\Omega^2 r_0^2 J_1(RW_1)(1 + W_1^2) = -W_1 J_0(RW_1)\{4\Omega^2 r_0^2 \pm \sqrt{[16\Omega^4 r_0^4 + 16\Omega^2 r_0^2 V_A^2 R^2 (1 + W_1^2)]}\}. \tag{40}$$

In order to discuss the nature of the roots W_1 of (40) we shall follow a method adopted by Hocking.[4] We consider the ratio

$$\left| \frac{W_1 J_0(RW_1)\{4\Omega^2 r_0^2 \pm \sqrt{[16\Omega^4 r_0^4 + 16\Omega^2 r_0^2 V_A^2 R^2 (1 + W_1^2)]}\}}{2R\Omega^2 r_0^2 (1 + W_1^2) J_1(RW_1)} \right|, \tag{41}$$

when $|W_1| = \alpha_n$, where $W_1 = \alpha_n$ is the n th non-zero zero of $J_1(RW_1)$ and n is large. Now $|J_1(RW_1)|$ and $|J_0(RW_1)|$ are of the same order in the circumference of the circle $|W_1| = \alpha_n$ except near $W_1 = \pm\alpha_n$, where $|J_1(RW_1)| < |J_0(RW_1)|$. We can easily see that on the circumference of $|W_1| = \alpha_n$, the ratio (41) is of the order of the large quantity $V_A/(\Omega r_0)$ except near the points $W_1 = \pm \alpha_n$, where the ratio tends to an indefinitely large quantity as either of these points are approached along the circumference. Hence by Rouche's Theorem

$$W_1 J_0(RW_1)\{4\Omega^2 r_0^2 \pm \sqrt{[16\Omega^4 r_0^4 + 16\Omega^2 r_0^2 V_A^2 R^2(1+W_1^2)]}\} = 0 \qquad (42)$$

and the equation (40) have the same number of roots in $|W_1| < \alpha_n$, $W_1 = 0$ is a root of (42) as well as of (40) and from (39) V_w^2 is negative indicating stability. If we take the negative sign in the expression within the curly bracket in (42) and in right hand side in (40), $W_1^2 = -1$ is root of the disperion relation. Remembering that $W = i W_1$, (39) shows that the corresponding V_w is zero. We shall now locate the roots of (40) corresponding to the $2n$ zeros $\pm\beta_1, \pm\beta_2, \cdots$, $\pm\beta_n$ of $J_0(RW_1)$ in $|W_1| < \alpha_n$. We write the dispersion relation (40) as

$$\frac{2R\Omega^2 r_0^2 J_1(RW_1)}{-W_1 J_0(RW_1)\{4\Omega^2 r_0^2 \pm \sqrt{[16\Omega^4 r_0^4 + 16\Omega^2 r_0^2 V_A^2 R^2(1+W_1^2)]}\}} = \frac{1}{1+W_1^2} \qquad (43)$$

We can easily see that the graph of the left hand side of (43) drawn against W_1 (real) has n vertical asymptotes at n zeros $\beta_1, \beta_2, \cdots, \beta_n$ of $J_0(RW_1)$ where β's are positive. We can show that the graph for the right hand side of (43) intersects the previous graph n times as W_1 varies from $W_1 = 0$ to $W_1 = \alpha_n$. Thus we have located n roots (all real and positive) of the dispersion relation. As (40) is even in W_1 equal and opposite real roots occur in pair. Hence the $2n$ roots (all real) that correspond to the roots $\pm\beta_1, \pm\beta_2 \cdots, \pm\beta_n$ of (42) are located. Remembering that $W = i W_1$, we can varify from (39) that any real root W_1 of the dispersion relation (40) implies negative V_w^2 indicating stability. Thus in this case disturbances with large wavelength are stable.

Case (ii)

In this case we take the matter to be non-gravitating and $\Omega r_0 >> V_A$. We consider the value of the ratio (41) on the circle $|W_1| = \beta_n$, where β_n is the n th zero $J_0(RW_1)$. By a similar reasoning as in the case (i) we can show that this ratio is of the order of the small quantity $V_A/\Omega r_0$ at all points on the above circle near $W_1 = \pm\beta_n$, where this ratio tends to zero as these points are approached along the circumference. Hence by Rouche's theorem

$$2R\Omega^2 r_0^2(1+W_1^2) J_1(RW_1) = 0 \qquad (44)$$

and the dispersion relation (40) have the same number of roots in $|W_1| < \beta_n$. By considering the equation (43) in the same way as above we can show that corresponding to the $2(n-1)$ roots $\pm\alpha_1, \pm\alpha_2, \ldots, \pm\alpha_{n-1}$ of (44), where *α's are non-zero zeros of $J_1(RW_1)$*, we have $2(n-1)$ real roots in $|W_1| < \beta_n$. These are the only real roots. Also $W_1 = 0$ is a root of (40) as well as of (44). It remains now to locate the roots of (40) that correspond to the roots

$W_1^2 = -1$ of (44). When we take the negative sign in the expression within the curly bracket on the righthand side of (40), we find that $W_1^2 = -1$ are also roots of the dispersion relation (40), when we take the the positive sign instead of the negative, $W_1^2 = -1$ are not roots of (40). All the real root of (40) have already been located. Hence its remaining roots in this case must be given by either complex or imaginary W_1. If W_1 is imaginary we can put $W = i W_1$ and write the dispersion relation (40) in the form

$$2\Omega^2 r_0^2 R (W_1^2 - 1) I_1(RW) = \{4\Omega^2 r_0^2 + \sqrt{[16\Omega^4 r_0^4 - 16\Omega^2 r_0^2 V_A^2 (W^2 - 1) R^2]}\} W I_0(RW). \qquad (45)$$

It is clear that when $W^2 < 1$, no solution is possible. When $W^2 > 1$, in order that the right hand side of (45) may be real we must have

$$\Omega^2 r_0^2 > V_A^2 R^2 (W^2 - 1). \qquad (46)$$

But from (39) in this case V_w^2 is positive indicating instability. If no real W is a root of (45), the only alternative is that the root is complex. From (39) it is clear that V_w^2 is not real. In view of (28) in the axisymmetric case we easily see that (34) is even in ω, and hence a non real V_w^2 will imply that one of the corresponding ω has positive real part. This again will indicate instability.

Case (iii)

We take the cylinder to be self-gravitating. As R is small the dispersion relation can be written as

$$8\pi G \rho r_0^2 R \log(R/2)(1 + W_1^2) J_1(R W_1) = W_1 J_0(R W_1)\{4\Omega^2 r_0^2 \pm \sqrt{[16\Omega^4 r_0^4 + 16\Omega^2 r_0^2 V_A^2 R^2 (1 + W_1^2)]}\}. \qquad (47)$$

We consider the ratio

$$\left| \frac{8\pi G\rho r_0^2 R \log(R/2)(1 + W_1^2) J_1(R W_1)}{W_1 J_0(R W_1)\{4\Omega^2 r_0^2 \pm \sqrt{[16\Omega^4 r_0^4 + 16\Omega^2 r_0^2 V_A^2 R^2 (1 + W_1^2)]}\}} \right|$$

on the circle $|W_1| = \beta_n$ where β_n is the nth zero of $J_0(RW_1)$ and n is large.

We find that this ratio is large of the order of

$$\frac{2\pi G \rho r_0 |\log(R/2)|}{\Omega V_A}$$

at all points except near the points $W_1 = \pm\beta_n$, where this ratio tends to indefinitely large quantity as these points are approached. By proceeding exactly as in case (ii) we find that for this case also there is instability.

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