ON THE NATURE OF THE BOUNDARY LAYER
NEAR THE LEADING EDGE OF A FLAT PLATE
WITH UNIFORM SUCTION

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ABSTRACT

In the presence of uniform suction, we find a solution of Navier-Stokes equations near the leading edge of a flat plate. We find analytical expressions for the first two terms of the stream function by a process of successive approximation. In the particular case of no suction, we get the expression for the stream function, obtained by Carrier and Lin[5].

INTRODUCTION

It is well known that Prandtl Boundary Layer Equations do not hold at the leading edge of a flat plate. Several attempts by the previous workers have been made to extend the boundary layer solution near the leading edge of the flat plate. Alden[1] solved this problem by taking second and higher order approximations, but he found that the higher order approximations become progressively singular. Lighthill[2] gave the technique by which the solution of an approximated non-linear equation can be extended in the neighbourhood of the singularity merely by straining the argument of the solution. Kuo[3] used this technique to improve the Blasius[4] solution so as to extend its validity upto near the leading edge. This improved Blasius solution has no singularity except at the leading edge and bring out the effect of the leading edge. It is found to satisfy Stokes equation of slow viscous motion in the immediate vicinity of the leading edge.

Carrier and Lin[5] solved the full Navier Stokes equations in the neighbourhood of the leading edge and found analytical expressions for the first two terms of the series expansion for ψ. In what follows, an attempt has been made to modify their solution to suit the requirements of uniform suction near the leading edge. The method essentially consists of solving the equation in terms of the stream function ψ, obtained by eliminating the pressure terms by cross-multiplication from the Navier-Stokes equations for the two-dimensional incompressible viscous flow. This equation is solved by successive approximations. Analytical expressions for the first two terms of the series expansion for ψ satisfying the boundary conditions for uniform suction have been obtained. Further, explicit expression for the skin-friction on the plate has also been found. It is to be noted that although, velocity components parallel and

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normal to the plate directions are of the same order of magnitude, yet, the contribution to the wall shearing stresses, arising due to the presence of the normal velocity component is zero even in the case of uniform suction. Secondly the solution holds within a certain radius of convergence of the order of ν/U_0 where ν is the kinematic viscosity and U_0 is the velocity of the incoming fluid.

This paper is the extension of the previous paper[6] where linear law of suction was assumed. It is shown here that by the proper choice of the stream function, it is possible to extend the results for any power law of the suction and in fact, results can be extended to any arbitrary law of suction velocity distribution, expressed by a series in terms of the ascending powers of the distance on the plate from the leading edge.

The necessity for this investigation arises, as in most cases of interest, boundary layer separates just near the leading edge. Therefore, by applying suction near the leading edge and joining the solution thus obtained, with the solution, given on the basis of boundary layer theory, it is possible to shift the point of separation, far downstream. Experiments on Sailplanes with suction, applied near the leading edge have been performed by Raspet[7] and his co-workers.

2. EQUATIONS OF MOTION

The Navier-Stokes equations for the two-dimensional, incompressible, viscous flow on the plate in cartesian co-ordinates are

$$u_1 \frac{\partial u_1}{\partial x_1} + v_1 \frac{\partial u_1}{\partial y_1} = -\frac{1}{\rho_1} \frac{\partial p_1}{\partial x_1} + \nu \left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial y_1^2} \right),$$ [2.1]

$$u_1 \frac{\partial v_1}{\partial x_1} + v_1 \frac{\partial v_1}{\partial y_1} = -\frac{1}{\rho_1} \frac{\partial p_1}{\partial y_1} + \nu \left(\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial y_1^2} \right),$$ [2.2]

and the equation of continuity is,

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial y_1} = 0.$$ [2.3]

Here, x_1, y_1 are the distances, measured along and perpendicular to the plate, u_1, v_1 are the components of the velocity in these directions. ρ_1, p_1, ν are respectively the density, pressure and kinematic viscosity of the fluid.

These equations are rendered dimensionless by the following substitutions:

$$p = \frac{p_1}{\rho_1 U_0^2}, \quad x = \frac{x_1 U_0}{\nu}, \quad y = \frac{y_1 U_0}{\nu}; \quad u = \frac{u_1}{U_0}, \quad v = \frac{v_1}{U_0},$$ [2.4]

where U_0 is the free-stream velocity.

The equations [2.1] − [2.3] assume the forms

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$ [2.5]

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \qquad [2.6]$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad [2.7]$$

The equation of continuity is automatically satisfied by assuming the existence of the stream function ψ such that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \qquad [2.8]$$

Eliminating the pressure terms from [2.5] and [2.6] by cross-multiplication and using [2.8], the above equations [2.5], [2.6] reduce to the single following equation :

$$\frac{\partial (\psi, \ \nabla^2 \psi)}{\partial (y, x)} = \nabla^4 \psi, \qquad [2.9]$$

where

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}. \qquad [2.10]$$

The equation [2.9] expressed in terms of the polar co-ordinates can be written in the form,

$$\frac{\partial^4 \psi}{\partial r^4} + \frac{2}{r} \frac{\partial^3 \psi}{\partial r^3} - \frac{1}{r^2} \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^3} \frac{\partial \psi}{\partial r} + \frac{2}{r^2} \left(\frac{\partial^4 \psi}{\partial r^2 \partial \theta^2} - \frac{1}{r} \frac{\partial^3 \psi}{\partial r \partial \theta^2} + \frac{2}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right) + \frac{1}{r^4} \frac{\partial^4 \psi}{\partial \theta^4}$$

$$= \frac{1}{r} \left\{ \frac{\partial \psi}{\partial \theta} \frac{\partial}{\partial r} \left(\nabla^2 \psi \right) - \frac{\partial \psi}{\partial r} \frac{\partial}{\partial \theta} \left(\nabla^2 \psi \right) \right\}, \qquad [2.11]$$

where

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2}. \qquad [2.12]$$

This equation [2.11] is to be integrated under the boundary conditions :

$$\left. \begin{array}{ll} \left(\dfrac{\partial \psi}{\partial r} \right)_{\theta = 0} = V, & \left(\dfrac{\partial \psi}{\partial r} \right)_{\theta = 2\pi} = -V, \\[3mm] \left(\dfrac{1}{r} \dfrac{\partial \psi}{\partial \theta} \right)_{\theta = 0} = 0, & \left(\dfrac{1}{r} \dfrac{\partial \psi}{\partial \theta} \right)_{\theta = 2\pi} = 0, \end{array} \right\} \qquad [2.13]$$

where V is the velocity of uniform suction. For the sake of symmetry, suction is assumed to be applied on both sides of the plate.

Further conditions to be satisfied by the stream function can be derived from symmetry considerations :

$$\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)_{\substack{r=R\\\theta=\alpha}} = \left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)_{\substack{r=R\\\theta=2\pi-\alpha}}, \qquad [2.14\,a]$$

$$\left(\frac{\partial\psi}{\partial r}\right)_{\substack{r=R\\\theta=\alpha}} = -\left(\frac{\partial\psi}{\partial r}\right)_{\substack{r=R\\\theta=2\pi-\alpha}}. \qquad [2.14b]$$

For the sake of convenience, the equation $[2.11]$ is written in the form

$$L\left(\psi\right) = L^{*}\left(\psi\right), \qquad [2.15]$$

when L denotes the biharmonic operator and L^{*} denotes the non-linear operator on the right.

We assume a solution of $[2.15]$ in the form :

$$\psi = \psi_0 + \psi_1 + \psi_2 + \psi_3 + \cdots \qquad [2.16]$$

where $\psi_0, \psi_1, \psi_2, \cdots, \psi_n$ are defined by

$$L\left(\psi_0\right) = 0, \qquad [2.17a]$$

$$L\left(\psi_1\right) = L^{*}(\psi_0), \qquad [2.17b]$$

$$L\left(\psi_2\right) = L^{*}(\psi_0 + \psi_1) - L^{*}(\psi_0), \qquad [2.17c]$$

. . . . . .

$$L\left(\psi_n\right) = L^{*}(\psi_0 + \psi_1 + \cdots + \psi_{n-1}) - L^{*}(\psi_0 + \psi_1 + \cdots + \psi_{n-2}). \qquad [2.17d]$$

We shall choose ψ_0 such that it satisfies the boundary conditions $[2.13]$ completely, *i.e.*

$$\left.\begin{array}{ll}\left(\dfrac{\partial\psi_0}{\partial r}\right)_{\theta=0} = V, & \left(\dfrac{\partial\psi_0}{\partial r}\right)_{\theta=2\pi} = -V, \\[2ex] \left(\dfrac{1}{r}\dfrac{\partial\psi_0}{\partial\theta}\right)_{\theta=0} = 0, & \left(\dfrac{1}{r}\dfrac{\partial\psi_0}{\partial\theta}\right)_{\theta=2\pi} = 0,\end{array}\right\} \qquad [2.18]$$

so that the remaining components of ψ, *viz.* $\psi_1, \psi_2 \cdots$ satisfy the following conditions

$$\frac{\partial\psi_i}{\partial r} = \frac{1}{r}\frac{\partial\psi_i}{\partial\theta} = 0 \quad \text{at} \quad \theta = 0, \ 2\pi \ \text{for} \ i > 0. \qquad [2.19]$$

The symmetric conditions [2.14] is to be satisfied separately by each component of ψ.

It can be seen easily that a solution of [2.17a] with the boundary conditions [2.18] and the symmetry conditions [2.14] can be written in the following form :

$$\psi_0 = r f(\theta), \qquad [2.20]$$

where

$$f^{iv}(\theta) + 2 f''(\theta) + f(\theta) = 0, \qquad [2.21]$$

and

$$f(0) = V, \quad f(2\pi) = -V, \quad f'(0) = 0, \quad f'(2\pi) = 0. \qquad [2.22]$$

The required solution of [2.17a] is

$$\psi_0 = (Vr/\pi)[(\pi - \theta)\cos\theta + \sin\theta]. \qquad [2.23]$$

It is seen that the solution of the problem, without suction cannot be obtained as the particular case from [2.23] by putting $V = 0$. The equation [2.17a] is a linear equation, therefore, we add to its solution [2.23] with suction, the solution without suction which corresponds to the following boundary conditions, besides the symmetry conditions [2.14],

$$\left.\begin{array}{ll}
\left(\dfrac{\partial\psi}{\partial r}\right)_{\theta=0} = 0, & \left(\dfrac{\partial\psi}{\partial r}\right)_{\theta=2\pi} = 0, \\[3mm]
\left(\dfrac{\partial\psi}{\partial\theta}\right)_{\theta=0} = 0, & \left(\dfrac{\partial\psi}{\partial\theta}\right)_{\theta=2\pi} = 0.
\end{array}\right\} \qquad [2.24]$$

This is found to be of the form

$$\psi_0 = 2A\, r^{3/2}\left(\cos\tfrac{1}{2}\theta - \cos\tfrac{3}{2}\theta\right), \qquad [2.25]$$

where A is an arbitrary constant to be evaluated later on.

Hence, ψ_0 can be taken as

$$\psi_0 = 2A\, r^{3/2}\left(\cos\tfrac{1}{2}\theta - \cos\tfrac{3}{2}\theta\right) + (Vr/\pi)\left[(\pi - \theta)\cos\theta + \sin\theta\right], \qquad [2.26]$$

Substituting this value of ψ_0 in [2.17b], we get

$$L(\psi_1) = (2A^2/r)(2\sin\theta - 3\sin 2\theta)$$
$$+ \frac{AV}{r^{3/2}\pi}\left[\; 2(\pi - \theta)\sin\frac{3\theta}{2} + 6\cos\frac{5\theta}{2} - \left(\cos\frac{\theta}{2} + 5\cos\frac{3\theta}{2}\right)\right]$$
$$- \frac{V^2}{r^2\pi^2}\left[2(\pi - \theta)\cos 2\theta + \sin 2\theta\;\right]. \qquad [2.27]$$

We find the solution of [2.27] in the form

$$\psi_1 = A^2 r^3 \left[f_1(\theta) + \phi_1(\theta) \log r \right]$$
$$+ \frac{AV}{\pi} r^{5/2} \left[x_1(\theta) + x_2(\theta) \log r \right] + \frac{V^2}{\pi^2} r^2 f_2(\theta). \qquad [2.28]$$

Substituting [2.28] in [2.27], we get the following system of equations, determining the θ-functions;

$$\phi_1^{iv} + 10\phi_1'' + 9\phi_1 = 0, \qquad [22.9a]$$

$$f_1^{iv} + 10f_1'' + 9f_1 = -24\phi_1 - 8\phi_1'' + 4\sin\theta - 6\sin 2\theta, \qquad [2.29b]$$

$$x_2^{iv} + \tfrac{1\,3}{2} x_2'' + \tfrac{2\,5}{1\,6} x_2 = 0, \qquad [2.29c]$$

$$x_1^{iv} + \tfrac{1\,3}{2} x_1'' + \tfrac{2\,5}{1\,6} x_1$$
$$= -\tfrac{1\,5}{2} x_2 - 6x_2'' + 2(\pi - \theta)\sin\tfrac{3}{2}\theta + 6\cos\tfrac{5}{2}\theta - (\cos\tfrac{1}{2}\theta + 5\cos\tfrac{3}{2}\theta), \quad [2.29d]$$

$$f_2^{iv} + 4f_2'' = -2(\pi - \theta)\cos 2\theta - \sin 2\theta. \qquad [2.29c]$$

These equations are to be integrated with the boundary conditions,

$$\left. \begin{array}{l} f_1(\theta) = \phi_1(\theta) = f_2(\theta) = x_1(\theta) = x_2(\theta) = 0 \text{ at } \theta = 0, 2\pi, \\ f_1'(\theta) = \phi_1'(\theta) = f_2'(\theta) = x_1'(\theta) = x_2'(\theta) = 0 \text{ at } \theta = 0, 2\pi, \end{array} \right\} \qquad [2.30]$$

and the symmetry considerations [2.14].

The equations [2.29] admit of direct solutions, and we get

$$\psi_1 = \frac{V^2 r^2}{32\pi^2} \left[7(\theta - \pi)(1 - \cos 2\theta) - 2\theta(\theta - 2\pi)\sin 2\theta \right]$$
$$+ \frac{AVr^{5/2}}{\pi} \left[\frac{\pi - \theta}{240} \left(65\sin\tfrac{1}{2}\theta - 60\sin\tfrac{3}{2}\theta + 23\sin\tfrac{5}{2}\theta \right) \right.$$
$$+ \tfrac{1\,3}{1\,6} \left(\cos\tfrac{3}{2}\theta - \cos\tfrac{5}{2}\theta \right) + \tfrac{5}{4\,8} \left(\cos\tfrac{1}{2}\theta - \cos\tfrac{5}{2}\theta \right) \log Br \left. \right]$$
$$+ A^2 r^3 \left[\frac{\pi - \theta}{8} \left(\cos 3\theta - \cos\theta \right) + \tfrac{2}{3} \left(\sin 2\theta - 2\sin\theta \right) \right.$$
$$+ \tfrac{1}{8} \left(3\sin\theta - \sin 3\theta \right) \log Cr \left. \right], \qquad [2.31]$$

where B, C are the additional constants, yet undetermined.

3. DETERMINATION OF THE CONSTANT A

In the case of no suction *i.e.* $V = 0$, the stream function ψ is given by

$$\psi = 2A\,r^{3/2}\left(\cos\tfrac{1}{2}\theta - \cos\tfrac{3}{2}\theta\right)$$
$$+ A^2\,r^3\left[\tfrac{1}{8}(\pi - \theta)(\cos 3\theta - \cos\theta) + \tfrac{2}{5}\left(\sin 2\theta - 2\sin\theta\right)\right.$$
$$\left. + \tfrac{1}{8}\left(3\sin\theta - \sin 3\theta\right)\log C r\right]. \tag{3.1}$$

For small values of θ, the velocity u is given by

$$u \simeq 4A\,r^{\frac{1}{2}}\theta + \tfrac{8}{3}A\,r^{\frac{1}{2}}\theta^3 + A^2 r^3\left[-\pi\theta + \tfrac{3}{10}\left(1 + 5\log C r\right)\theta^2\right.$$
$$\left. + \tfrac{2}{3}\pi\,\theta^3 - \left(\tfrac{14}{15} + \tfrac{1}{2}\log C r\right)\theta^4\right] + \cdots \tag{3.2}$$

The velocity profile, obtained by Blasius for the flow, downstream is given by

$$u = \alpha\eta - \frac{\alpha^2\eta^4}{2\,(4!)} + - \cdots \tag{3.3}$$

where

$$\eta = \frac{y}{\sqrt{x}}.$$

For small values of r and θ

$$u \simeq \alpha r^{\frac{1}{2}}\theta - \frac{\alpha^2 r^2\theta^2}{48} + \cdots \tag{3.4}$$

For sufficiently small values of r and θ, the leading term of the expansion describes the flow. Hence, comparing the coefficients of $r^{1/2}\theta$ in the above expansions [3.2] and [3.4], we get

$$A = \frac{\alpha}{4} = \frac{0.332}{4} = 0.083.$$

4. DISCUSSION OF SKIN-FRICTION

The complete expression for ψ is

$$\psi = (Vr/\pi)\left[(\pi - \theta)\cos\theta + \sin\theta\right] + 0.166\,r^{3/2}\left(\cos\tfrac{1}{2}\theta - \cos\tfrac{3}{2}\theta\right)$$
$$+ \frac{V^2 r^2}{32\,\pi^2}\left[7\left(\theta - \pi\right)\left(1 - \cos 2\theta\right) - 2\theta\left(\theta - 2\pi\right)\sin 2\theta\right]$$
$$+ \frac{0.083}{\pi}\,Vr^{5/2}\left[\tfrac{1}{240}(\pi - \theta)\left(65\sin\tfrac{1}{2}\theta - 60\sin\tfrac{3}{2}\theta + 23\sin\tfrac{5}{2}\theta\right)\right.$$
$$+ \tfrac{1}{16}\left(\cos\tfrac{3}{2}\theta - \cos\tfrac{5}{2}\theta\right) + \tfrac{5}{48}\left(\cos\tfrac{1}{2}\theta - \cos\tfrac{5}{2}\theta\right)\log Br\Big]$$
$$+ (0.083)^2\,r^3\left\{\left[(\pi - \theta)/8\right]\left(\cos 3\theta - \cos\theta\right) + \tfrac{2}{5}\left(\sin 2\theta - 2\sin\theta\right)\right.$$
$$\left. + \tfrac{1}{8}\left(3\sin\theta - \sin 3\theta\right)\log C r\right\} + \cdots \tag{4.1}$$

The skin-friction

$$\tau_0 = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)_{y=0} , \qquad [4.2]$$

$$= \mu \left[\frac{1}{r} \left(\frac{\partial \psi}{\partial r} \right)_{\theta=0} + \frac{1}{r^2} \left(\frac{\partial^2 \psi}{\partial \theta^2} \right)_{\theta=0} - \left(\frac{\partial^2 \psi}{\partial r^2} \right)_{\theta=0} \right] ,$$

$$= \mu \left[\frac{0 \cdot 332}{r^{\frac{1}{2}}} + \frac{0 \cdot 083}{\pi} \, V r^{\frac{1}{2}} \left(\frac{13}{4} + \frac{5}{8} \, \log \, Br \right) - (0 \cdot 083)^2 r \pi - \frac{5}{8} \, \frac{V^2}{\pi} \right] . \quad [4.3]$$

It will be seen from here that although near the leading edge, u and v are of the same order of magnitude, yet the contribution to the wall shearing stresses, arising out of the additional term *viz.* $\mu \left(\partial v / \partial x \right)$ in [4.2] is zero even in the case when uniform suction is applied. This means that the law of resistance is given by the usual formula

$$\tau_0 = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0} . \qquad [4.4]$$

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