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HYDROMAGNETIC WAVES IN A COMPRESSIBLE FLUID

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ABSTRACT

The equations satisfied by vorticity and current density are derived for a compressible infinitely conducting fluid in the presence of a magnetic field. It has been shown that the components of vorticity and current density along the direction of the magnetic field will be propagated with Alfven velocity only if the charge separation takes place in the medium. The propagation of the disturbance caused by homogeneous and inhomogeneous condensations at a point in the medium are considered. Finally, a proof of Walen's problem, namely an annular disturbance is propagated along the magnetic lines of force with Alfven velocity without distortion has been given.

INTRODUCTION

The problem of propagation of waves in compressible fluids in the presence of uniform magnetic field was initiated by Herlofson[1] and Van de Hulst[2]. Recently Carstoiu[3] has discussed this problem from the consideration of the propagation of vorticity and current density fields associated with the wave of infinitesimally small amplitude. In section 1, we establish the equations determining density fluctuation, vorticity, and current density fields and solve them in section 2 for the case of a plane harmonic wave. We find, that in the absence of free charge the components of vorticity and current in the direction of primitive magnetic field are zero unless we take into account the separation of charge, in which case they are propagated with Alfven velocity. It is of importance to compare this statement with the corresponding statement of Carstoiu.

In section 3 and section 4, we have studied the propagation of disturbance produced by sudden density fluctuation at some point in the medium and in section 5, we have obtained explicit solution for the Walen problem[4] dealing with the propagation of a ring of disturbance along the magnetic field when the electrical conductivity is infinite. In this case, compressibility has no effect, as expected. In passing, we may mention that Walen has given the solution of this problem when conductivity is finite; but it does not appear possible to deduce the present solution from that of Walen.

1. BASIC EQUATIONS OF THE PROBLEM

Equation of continuity:

$$\partial \rho/\partial t + \operatorname{div}(\rho \mathbf{v}) = 0, \quad [1.1]$$

Equation momentum:

$$\rho[\partial \mathbf{v}/\partial t + (\mathbf{v}\cdot\nabla)\mathbf{v}] = -\operatorname{grad} p + \mu \mathbf{J}\times\mathbf{H}, \quad [1\cdot2]$$

(neglecting external forces),

Maxwell's Equations:

$$\operatorname{curl} \mathbf{H} = 4\pi\, \mathbf{J}, \quad [1.3]$$

(neglecting the displacement current),

$$\operatorname{div} \mathbf{H} = 0, \quad [1.3a]$$

$$\operatorname{curl} \mathbf{E} = -\mu\, \partial \mathbf{H}/\partial t, \quad [1.4]$$

$$\operatorname{div} \mathbf{E} = 0, \text{ (no free charge)}, \quad [1.5a]$$

$$= (4\pi/K)\epsilon, \quad [1.5b]$$

where ϵ is the free charge density and K the dielectric constant.

Current equation: In the case of infinite electrical conductivity,

$$\mathbf{E} = -\mu \mathbf{v}\times\mathbf{H}. \quad [1.6]$$

We assume that the undisturbed fluid is uniform and is embedded in a uniform magnetic field $\mathbf{H}_0\,(0, 0, H_0)$. Let ρ_0, p_0 be the density and pressure of the fluid in the undisturbed state. Further let the density, pressure, magnetic field, electric field and current in the disturbed state be noted by $\rho_0+\rho$, p_0+p, $\mathbf{H}_0+\mathbf{h}$, $\mathbf{E}$ and $\mathbf{J}$, respectively. In order to study the small disturbances we shall linearize the equations by neglecting the powers and products of the small quantities $|\mathbf{h}|$, ρ, p, $|\mathbf{E}|$ and $|\mathbf{J}|$. The linearized set of equations is

$$\partial \rho/\partial t + \rho_0 \operatorname{div} \mathbf{v} = 0, \quad [1\cdot7]$$

$$\rho_0\, \partial \mathbf{v}/\partial t = -\operatorname{grad} \phi + (\mu H_0/4\pi)\, \partial \mathbf{h}/\partial z, \quad [1.8]$$

$$\phi = p + (\mu/4\pi)\, \mathbf{H}_0\cdot\mathbf{h}, \quad [1.9]$$

$$\text{curl } \mathbf{h} = 4\pi \mathbf{J}, \quad \text{div } \mathbf{h} = 0, \qquad [1.10], [1.11]$$

$$\text{curl } \mathbf{E} = -\mu\, \partial \mathbf{h}/\partial t, \quad \text{div } \mathbf{E} = 0, \text{(no free charge)}, \qquad [1.12], [1.13]$$

$$\mathbf{E} = -\mu \mathbf{v} \times \mathbf{H}_0. \qquad [1.14]$$

Eliminating **E** between (1.12) ahd (1.14) we have the equation determining the induced magnetic field :

$$\partial \mathbf{h}/\partial t = H_0 \partial \mathbf{v}/\partial z - \mathbf{H}_0 \text{ div } \mathbf{v}. \qquad [1.15]$$

Performing curl operations on (1·8) and (1·15), we have equations for vorticity $\vec{w}$ and current **J** :

$$\partial \vec{w}/\partial t = (\mu H_0/\rho_0)\, \partial \mathbf{J}/\partial z, \qquad [1.16]$$

and,
$$\partial \mathbf{J}/\partial t = (H_0/4\pi) \partial \vec{w}/\partial z - \mathbf{H}_0/4\pi\rho_0 \times \text{grad}\, (\partial \rho/\partial t), \qquad (1.17)$$

on using (1.7).

From (1.13) and (1.14) we can show that in the absence of free charge

$$w_z = 0. \qquad [1.18]$$

Since the second term on the right hand side of (1.17) is a vector in (x, y) plane and $w_z = 0$,

$$J_z = 0. \qquad [1.19]$$

However, if we allow the separation of charges due to the propagation of disturbance we use (1.5*b*), namely

$$\text{div } \mathbf{E} = (4\pi/K)\epsilon. \qquad [1.20]$$

Then from (1.14) and (1.20), we get

$$w_z = (-4\pi/K)\epsilon/H_0. \qquad [1.21]$$

Taking the z components of (1.16) and (1.17), we get

$$\frac{\partial w_z}{\partial t} = \frac{\mu H_0}{\rho_0} \cdot \frac{\partial J_z}{\partial z}, \qquad \frac{\partial J_z}{\partial t} = \frac{H_0}{4\pi} \cdot \frac{\partial w_z}{\partial z}, \qquad [1.22]$$

So that,
$$\frac{\partial^2 w_z}{\partial t^2} = a^2 \frac{\partial^2 w_z}{\partial z^2}, \qquad \frac{\partial^2 J_z}{\partial t^2} = a^2 \frac{\partial^2 J_z}{\partial z^2} \qquad [1\cdot23,\ 1\cdot24]$$

where, $a = [(\mu H_0^2)/(4\pi\rho_0)]^{1/2}$, the Alfven wave volocity.

Thus we see that, *if there is no free charge in the medium the components of vorticity and current in the direction of the primitive magnetic field vanishes ; it is only when we allow the separation of charges due to the propagation of disturbance, that the vorticity and current components parallel to the primitive* *magnetic field are propagated with the Alfven wave velocity* $\pm a$ *in the direction of the field.* It will be interesting to compare the above statement with that of Carstoiu[3].

The following equations determine the components of vorticity and current perpendicular to $\mathbf{H}_0$:

$$\frac{\partial^2 w_x}{\partial t^2} - a^2 \frac{\partial^2 w_x}{\partial z^2} = \frac{a^2}{\rho_0} \frac{\partial^3 \rho}{\partial t \partial y \partial z}, \qquad [1.25]$$

$$\frac{\partial^2 w_y}{\partial t^2} - a^2 \frac{\partial^2 w_y}{\partial z^2} = -\frac{a^2}{\rho_0} \frac{\partial^3 \rho}{\partial t \partial x \partial z}, \qquad [1.26]$$

$$\frac{\partial^2 J_x}{\partial t^2} - a^2 \frac{\partial^2 J_x}{\partial z^2} = \frac{H_0}{4\pi \rho_0} \frac{\partial^3 \rho}{\partial t^2 \partial y}, \qquad [1.27]$$

$$\frac{\partial^2 J_y}{\partial t^2} - a^2 \frac{\partial^2 J_y}{\partial z^2} = -\frac{H_0}{4\pi \rho_0} \frac{\partial^3 \rho}{\partial t^2 \partial x}. \qquad [1.28]$$

We note that whether we allow for the separation of charge or not, the equations determining the components of vorticity and current perpendicular to the primitive magnetic field are the same.

Performing the divergence operation on (1.8) and using (1.9)-(1.11), we get

$$\frac{\partial^2 \rho}{\partial t^2} = \Delta p + \mu H_0 \left(\frac{\partial J_x}{\partial y} - \frac{\partial J_y}{\partial x} \right) \qquad [1.29]$$

If we assume now that the disturbance is propagated adiabatically, we have

$$p = c^2 \rho, \qquad [1.30]$$

where $c^2 = \gamma p_0/\rho_0$ is the velocity of sound in the undisturbed medium and in the absence of magnetic field.

Eliminating p, J_x and J_y between (1.27)-(1.30), we get the equation determining the propagation of density disturbance, namely,

$$\left(\frac{\partial^2}{\partial t^2} - a^2 \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial t^2} - c^2 \Delta \right) \rho = a^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \frac{\partial^2 \rho}{\partial t^2}, \qquad [1.31]$$

where Δ is the Laplacian operator.

2. PROPAGATION OF PLANE WAVES

We now assume that the perturbed quantities are proportional to exp. $[i(wt + \mathbf{k} \cdot \mathbf{r})]$. From (1.31), we get two modes of propagation with wave-velocities.

$$V_w^2 = \tfrac{1}{2}\left[(a^2 + c^2) \pm \{(a^2 + c^2)^2 - 4a^2c^2n^2\}^{1/2}\right], \qquad [2.1]$$

where $(1, m, n)$ are the direction cosines of $\mathbf{k}$. The wave-velocities depend on the inclination of the direction of propagation to the primitive magnetic field. The wave-velocity of the fast wave $V_{wf}(>c)$, is maximum when the propagation is perpendicular to $\mathbf{H}_0$, while the wave-velocity of the slow wave, $V_{ws}(<c)$, is maximum when the propagation is along $\mathbf{H}_0$. If $a=c$, the two wave-velocities for the propagation in the direction of $\mathbf{H}_0$ are equal. We can easily show that $V_{wf}>a$ and that $V_{ws}<a$ always. When a and c are unequal, there is no propagation with velocity lying between a and c.

From (1.25) – (1.28), we get

$$w_x = \left(\frac{i\rho}{\rho_0}\right)a^2 \frac{V_w\, mnk}{V_w^2 - a^2n^2}, \quad w_y = -\left(\frac{i\rho}{\rho_0}\right)a^2 \frac{V_w\, lnk}{V_w^2 - a^2n^2}, \qquad [2.2,\ 2.3]$$

and

$$J_x = \left(\frac{i\rho}{\rho_0}\right)\frac{H_0}{4\pi} \frac{V_w^2\, mk}{V_w^2 - a^2n^2}, \quad J_y = -\frac{i\rho}{\rho_0}\cdot\frac{H_0}{4\pi}\cdot\frac{V_w^2\, lk}{V_w^2 - a^2n^2}. \qquad [2.4,\ 2.5]$$

We notice that the vorticity vector

$$\vec{w} = \left(\frac{i\rho}{\rho_0}\right)\frac{a^2}{H_0} \frac{V_w\, n}{V_w^2 - a^2n^2}\, \mathbf{k}\times\mathbf{H}_0 \qquad [2.6]$$

is perpendicular to the primitive magnetic field and to the direction of wave propagation. Similarly the current vector

$$\mathbf{J} = \left(\frac{i\rho}{\rho_0}\right)\frac{1}{4\pi} \frac{V_w^2}{V_w^2 - a^2n^2}\, \mathbf{k}\times\mathbf{H}_0 \qquad [2.7]$$

is also normal to the plane containing the primitive magnetic field and the direction of propagation. The expressions for $\vec{w}$ as well as $\mathbf{J}$ contain a factor $(i\rho/\rho_0)$ showing that the propagation of these vectors is always out of phase by $\pi/2$ from the propagation of density disturbance.

To complete the solution we note below the expressions for velocity, induced magnetic field and electric field :

$$\mathbf{v} = -\frac{\rho}{\rho_0}V_w\left[\frac{\mathbf{k}}{k} + \frac{a^2\rho_0}{H_0}\frac{n}{V_w^2 - a^2n^2}\mathbf{k}\times(\mathbf{k}\times\mathbf{H}_0)\right], \qquad [2.8]$$

$$\mathbf{h} = -\frac{\rho}{\rho_0}\frac{V_w^2}{V_w^2 - a^2n^2}\frac{\mathbf{k}\times(\mathbf{k}\times\mathbf{H}_0)}{\mathbf{k}^2} \qquad [2.9]$$

and

$$\mathbf{E} = \frac{\mu\rho}{\rho_0}\frac{V_w}{k}\left[1 + \frac{a^2\rho_0\, nk^2}{V_w^2 - a^2n^2}\right](\mathbf{k}\times\mathbf{H}_0). \qquad [2.10]$$

We may note that the present investigation is more general than that of Carstoiu[3] in as much as his discussion is based on the relations [(20) – (22) of his paper]

$$J_x = \pm (H_0/2\pi a)\, w_x + F(y, z, t),$$
$$J_y = \pm (H_0/2\pi a)\, w_y + G(x, z, t),$$

which are only the particular cases of the more general solution

$$\mathbf{J} = \pm (H_0/2\pi a)\, \vec{w} + \text{curl } \mathbf{A},$$

where $(\text{curl } \mathbf{A})_z = 0$ if we allow the separation of charge. If there is no separation of charge, then $w_z = J_z = 0$ and there is no interlocking between his equations (17) and (18). This particular choice of F and G results in reducing the order of density equation by two.

To conclude this section we note that the dispersion relation (2·1), which we have obtained through the equation determining the density fluctuation is a particular case of the general dispersion relation obtained by Van de Hulst[2] directly dealing with the scalar equations determining the physical quantities. We have obtained the equations in the present form in order to study in the following sections, the propagation of a small disturbance produced by a sudden density fluctuation and similar initial value problems.

3. PROPAGATION OF SMALL DISTURBANCES ARISING DUE TO A SUDDEN DENSITY FLUCTUATION IN THE MEDIUM

Let us assume that a density fluctuation is produced at $t = 0$ at the origin $(0, 0, 0)$ which can be represented by

$$\rho = \tau\, \delta(\mathbf{r}) \text{ at } t = 0, \qquad [3.1]$$

where $\delta(\mathbf{r}) = \delta(x)\, \delta(y)\, \delta(z)$, the three dimensional Dirac delta function.

We take the Fourier transform of (1.31) under the conditions that ρ and its derivatives up to third order with respect to space corodinates tend to zero as r rends to infinity ; then,

$$[\partial^4/\partial t^4 + (a^2 + c^2) k^2\, \partial^2/\partial t^2 + a^2 c^2 \zeta^2 k^2]\bar{\rho} = 0, \qquad [3.2]$$

where $$\bar{\rho} = \frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{+\infty} \rho(x, y, z, t) \exp. i(\xi x + \eta y + \zeta z)\, dx\, dy\, dz,$$

and $$k^2 = \xi^2 + \eta^2 + \zeta^2.$$

In what follows, the bar over any perturbed quantity would be taken to mean its Fourier Transform.

From (3.1), at $t = 0$

$$(\bar{\rho})_0 = \tau/(2\pi)^{3/2}, \qquad [3.3]$$

where the suffix 0 denotes the value of the quantity at time $t = 0$. In addition. we assume that

$$\frac{\partial \bar{\rho}}{\partial t} = \frac{\partial^2 \bar{\rho}}{\partial t} = \frac{\partial^3 \bar{\rho}}{\partial t^3} = 0 \quad \text{at } t = 0. \tag{3.4}$$

The solution of (3.2) satisfying the above initial conditions is

$$\bar{\rho} = \frac{\tau}{(2\pi)^{3/2}} \cdot \frac{1}{(m_1^2 - m_2^2)} [m_1^2 \cos m_2 t - m_2^2 \cos m_1 t], \tag{3.5}$$

where $$m_{1,2} = \tfrac{1}{2}k\,[\sqrt{(a^2 + 2ac\cos\theta + c^2)} \pm \sqrt{(a^2 - 2ac\cos\theta + c^2)}], \tag{3.6}$$

and $$\cos\theta = \zeta/k. \tag{3.7}$$

We shall now determine the equations and the initial conditions satisfied by the Fourier transforms of other physical quantities. From (1.7), (1.8) and (1.11), we get

$$\partial\bar{\rho}/\partial t = i\rho_0 \mathbf{k}\cdot\bar{\mathbf{v}}, \tag{3.8}$$

$$\rho_0(\partial\bar{\mathbf{v}}/\partial t) = ic^2\bar{\rho}\mathbf{k} + \mu\bar{\mathbf{J}} \times \mathbf{H}_0, \tag{3.9}$$

$$\mathbf{k}\cdot\bar{\mathbf{h}} = 0, \tag{3.10}$$

and $$\bar{\mathbf{J}} = (-i/4\pi)\,\mathbf{k} \times \bar{\mathbf{h}}. \tag{3.11}$$

From the above equations, we can show that

$$\bar{h}_x = -\frac{\xi\zeta}{\xi^2 + \eta^2}\bar{h}_z, \quad \bar{h}_y = -\frac{\eta\zeta}{\xi^2 + \zeta^2}\bar{h}_z, \tag{3.12}$$

$$\bar{J}_x = -\frac{i}{4\pi}\frac{\eta k^2}{\xi^2 + \eta^2}\bar{h}_z, \quad \bar{J}_y = -\frac{i}{4\pi}\frac{\xi k^2}{\xi^2 + \eta^2}\bar{h}_z, \tag{3.13}$$

$$\frac{\partial^2 \bar{\rho}}{\partial t^2} = -c^2k^2\bar{\rho} - \frac{\mu H_0}{4\pi}k^2\bar{h}_z, \tag{3.14}$$

and $$\frac{\partial^3 \bar{\rho}}{\partial t^3} = -c^2k^2\frac{\partial\bar{\rho}}{\partial t} - \frac{\mu H_0}{4\pi}k^2\frac{\partial \bar{h}_z}{\partial t}. \tag{3.15}$$

In view of (3.3), (3.4), (3.14) and (3.15), we obtain

$$(\bar{h}_z)_0 = -\frac{4\pi c^2}{\mu H_0}(\bar{\rho})_0 \quad \text{and} \quad \left(\frac{\partial \bar{h}_z}{\partial t}\right)_0 = 0. \tag{3.16}$$

The initial values of **h** and **J** can easily be written from (3.12), (3.13) and first condition in (3.16). Using the second condition of (3.16) we find that

$$\left(\frac{\partial\bar{\mathbf{h}}}{\partial t}\right)_0 = \left(\frac{\partial\bar{\mathbf{J}}}{\partial t}\right)_0 = 0. \tag{3.17}$$

We also note that

$$\bar{v} = \frac{i}{H\zeta}\left[\frac{\partial \bar{h}}{\partial t} - \frac{H_0}{\rho_0}\frac{\partial \bar{\rho}}{\partial t}\right], \quad [3.18]$$

and

$$\bar{E} = -\mu\,\bar{v} \times H_0 \quad [3.19]$$

Hence

$$(\bar{v})_0 = (\bar{E})_0 = 0.$$

But from (3.9), (3.13) and (3.16) we have

$$\left(\frac{\partial \bar{v}}{\partial t}\right)_0 = \frac{i c^2}{\rho_0}(\bar{\rho})_0\,\mathbf{k} + \frac{i\mu}{4\pi\rho_0}\,\mathbf{H}_0 \times \{\mathbf{k} \times (\bar{h})_0\}, \quad [3.20]$$

giving non-zero initial values of the time derivative of $\bar{v}$.

We consider below the two special cases : (i) $c >> a$ and (ii) $c << a$, as the inversion in these cases in less complicated than in the general case, *Case* (*i*). When $c >> a$, we neglect quantities of the order of $(a/c)^2$. We have,

$$m_1 \simeq ck, \quad m_2 \simeq a\zeta,$$

so that,

$$\bar{\rho} \simeq \frac{\tau}{(2\pi)^{3/2}} \cos a\,\zeta\,t. \quad (3.22)$$

On inverting this we have

$$\rho = \tfrac{1}{2}\tau\,\delta(x)\,\delta(y)\,[\delta(z+at) + \delta(z-at)]. \quad [3.23]$$

Thus the density disturbance travels with Alfven wave velocity in the direction parallel and anti-parallel to the primitive magnetic field and at time t the density fluctuation exists only at the points $(0, 0, \pm\, at)$.

We may easily deduce from (1.25) and (1.26) that v_x, v_y, v_z are determined by

$$\left(\frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial z^2}\right)\Delta\mathbf{v} = -\frac{1}{\rho_0}\nabla\left(\frac{\partial^3 \rho}{\partial t^3}\right) + \frac{a^2}{\rho_0}\left(\frac{\partial^2}{\partial t\,\partial z}\Delta\rho\right)\hat{z}, \quad [3.24]$$

where $\hat{z}$ is the unit vector along z direction.

The Fourier transform of (3.24), under the assumption that the velocity and its first order derivatives with respect to space coardinates tend to zero as r tends to infinity, yields

$$\left(\frac{\partial^2}{\partial t^2} + a^2\zeta^2\right)\bar{v} = -\frac{i\,\mathbf{k}}{\rho_0 k^2}\frac{\partial^3 \bar{\rho}}{\partial t^3} - \left(\frac{i a^2 \zeta}{\rho_0}\frac{\partial \bar{\rho}}{\partial t}\right)\hat{\zeta}, \quad (3.25)$$

where $\hat{\zeta}$ is the unit vector along ζ direction in wave vector space. Using (3.22) and integrating (3.25) under the initial conditions determined already, we have

$$\bar{\mathbf{v}} = \frac{-i\tau}{2\rho_0(2\pi)^{3/2}} \frac{a\zeta\mathbf{k}}{k^2}\sin a\zeta t + \frac{i\tau t}{2\rho_0(2\pi)^{3/2}} \frac{a^2\zeta^2\mathbf{k}}{k^2}\cos a\zeta t$$
$$+\hat{\zeta}\left[\frac{ia\tau}{2\rho_0(2\pi)^{3/2}} \frac{\sin a\zeta t}{k^2} - \frac{i\tau t}{2\rho_0(2\pi)^{3/2}} \frac{a^2\zeta}{k^2}\cos a\zeta t\right] \quad (3.26)$$

The inversion then gives

$$\mathbf{v} = -\frac{\tau}{16\pi\rho_0}\nabla\frac{\partial}{\partial t}\left(\frac{1}{R}+\frac{1}{R'}\right) + \frac{\tau t}{16\pi\rho_0}\nabla\frac{\partial^2}{\partial t^2}\left(\frac{1}{R}+\frac{1}{R'}\right)$$
$$+\hat{z}\left[\frac{a\tau}{16\pi\rho_0}\left(\frac{1}{R'}-\frac{1}{R}\right) + \frac{a^2\tau t}{16\pi\rho_0}\frac{\partial}{\partial z}\left(\frac{1}{R}+\frac{1}{R'}\right)\right].$$

where R and R' are the distances of (x, y, z) from $(0, 0, at)$ and $(0, 0, -at)$ respectively.

Proceeding similarly, we can see that

$$\bar{\vec{w}} = \left[\frac{\tau c^2}{a\rho_0(2\pi)^{3/2}} \frac{k^2}{\xi^2+\eta^2}\sin a\zeta t - \frac{a^2\tau t}{2\rho_0(2\pi)^{3/2}}\zeta\cos a\zeta t\right]\mathbf{k}\times\hat{\zeta}, \quad [3.27]$$

to our approximation. Their inverses are zero at all points except at $(0, 0, \pm at)$ where they are highly singular, the nature of the singularity being given by

$$\vec{w} = -\frac{a^2\tau t}{4\pi\rho_0}(\hat{z}\times\nabla)\frac{\partial}{\partial z}\left[\delta(\mathbf{R})+\delta(\mathbf{R}')\right]$$

The transforms of current densities are

$$\bar{\mathbf{J}} = \left[\frac{i\tau c^2}{\mu H_0(2\pi)^{3/2}} \frac{k^2}{\xi^2+\eta^2}\cos a\zeta t + \frac{i\tau t H_0}{8\pi\rho_0(2\pi)^{3/2}} a\zeta\sin a\zeta t\right]\mathbf{k}\times\hat{\zeta} \quad [3.28]$$

so that
$$\mathbf{J} = +\frac{\tau t H_0}{16\pi\rho_0}(\hat{z}\times\nabla)\frac{\partial}{\partial t}\left[\delta(\mathbf{R})+\delta(\mathbf{R}')\right].$$

Also to this approximation $h_x = h_y = 0$ except at $(0, 0, \pm at)$ where they are undefined, and

$$h_z = -(2\pi c^2/\mu H_0)\,\delta(x)\,\delta(y)\left[\delta(z+at)+\delta(z-at)\right] \quad [3.29]$$

Thus the magnetic disturbance also travels with Alfven velocity along the primitive magnetic field. From the knowledge of the velocity field, using (1.14) the electric field can immediately be obtained.

Case (ii): When $c \lesssim < a$, neglecting quantities of order $(c/a)^2$ we get as before

$$m_1 \simeq ak, \quad m_2 \simeq c\zeta, \tag{3.30}$$

and
$$\bar{\rho} = [\tau/(2\pi)^{3/2}] \cos c\zeta t, \tag{3.31}$$

so that
$$\rho = \tfrac{1}{2}\tau\,\delta(x)\,\delta(y)\,[\delta(z+ct) + \delta(z-ct)]. \tag{3.32}$$

Here the density disturbance travels with velocity of sound in the medium along the primitive magnetic field and at time t, the density fluctuation exists only at $(0, 0, \pm ct)$.

Same procedure as above yields,

$$\bar{v}_x = \frac{c^2\tau}{a^2\rho_0(2\pi)^{3/2}} \frac{a\zeta \sin a\zeta t}{\xi^2+\eta^2}; \quad \bar{v}_y = \frac{-c^2\tau}{a^2\rho_0(2\pi)^{3/2}} \frac{a\zeta \sin a\zeta t}{\xi^2+\eta^2},$$

and
$$\bar{v}_z = \{ic\tau/[\rho_0(2\pi)^{3/2}]\} \sin c\zeta t,$$

so that
$$v_x = v_y = 0; \; v_z = (c\tau/2\rho_0)\,\delta(x)\,\delta(y)\,[\delta(z-ct) - \delta(z+ct)] \tag{3.33}$$

Also the nature of the singularity of vorticity and current are given by

$$\vec{w} = \frac{c\rho_0}{\mu H_0}\mathbf{J} = \frac{c\tau}{2\rho_0}(\hat{z} \times \nabla)\,[\delta(\mathbf{R}') - \delta(\mathbf{R})]. \tag{3.34}$$

The magnetic field is given by

$$\mathbf{h} = \frac{c^2\tau}{2\mu H_0}\nabla\frac{\partial}{\partial z}\left(\frac{1}{R} + \frac{1}{R'}\right) + \frac{2\pi c^2\tau}{\mu H_0}\,[\delta(\mathbf{R}) + \delta(\mathbf{R}')]\,\hat{z}. \tag{3.35}$$

In this case, the electric field vanishes entirely to our approximation.

4. In the previous section we have discussed the fluctuation in density created by sudden spherical condensation or expansion at a point in the medium. Here we shall study the fluctuations produced by means of inhomogeneous condensation or expansion, so that, the element of fluid at the origin has imposed velocities also. But we can show that any imposed transverse velocity involves charge separation and hence we must take into consideration (1.5*b*), (1.23) and (1.24). Rest of the basic equations remain the same. We again adopt the procedure of Fourier Transforms and take the following initial conditions for the Fourier transforms of the perturbed quantities:

$$(\bar{\mathbf{v}})_0 = \frac{\mathbf{V}}{(2\pi)^{3/2}}, \quad \left(\frac{\partial\bar{\mathbf{w}}}{\partial t}\right)_0 = 0, \tag{4.1, 4.2}$$

and
$$(\bar{\rho})_0 = \frac{\tau}{(2\pi)^{3/2}}. \tag{4.3}$$

Initial conditions on other quantities can be obtained from (3.8) – (3.20). For simplicity, we shall restrict the discussion to the limiting case $c >> a$. The limiting case $c << a$ follows exactly on the same lines.

Particular Case :

$c >> a$: Under the initial conditions mentioned above,

$$\bar{\rho} = \frac{\tau}{(2\pi)^{3/2}} \cos ckt + \frac{i\rho_0}{(2\pi)^{3/2}} \mathbf{k}\cdot\mathbf{V}\frac{\sin ckt}{ck} \cdot \qquad [4.4]$$

The first term arises due to spherical part of the condensation and the second due to the imposed initial velocity. Inversion of (4.4) gives

$$\rho = \frac{\tau}{(2\pi)^3}\iiint\limits_{-\infty}^{\infty} \cos ckt \exp.[-i(\xi x + \eta y + \zeta z)]\, d\xi\, d\eta\, d\zeta$$

$$+\frac{i\rho_0}{(2\pi)^3}\iiint\limits_{-\infty}^{\infty} \mathbf{k}\cdot\mathbf{v}[(\sin ckt)/ck] \exp.[-i(\xi x + \eta y + \zeta z)]\, d\xi\, d\eta\, d\zeta.$$

$$= \left[\frac{\rho_0 \mathbf{V}\cdot\nabla}{c^2(2\pi)^3} - \frac{\tau}{c^2(2\pi)^3}\frac{\partial^2}{\partial t^2}\right] \times$$

$$\times \iiint\limits_{-\infty}^{\infty} [(\cos ckt)/k^2] \exp.[-i(\xi x + \eta y + \zeta z)] d\xi\, d\eta\, d\zeta \qquad [4.5]$$

Consider $$I = \iiint\limits_{-\infty}^{\infty} [(\cos ckt)/k^2] \exp.[-i(\xi x + \eta y + \zeta z)]\, d\xi\, d\eta\, d\zeta \qquad [4.6]$$

$$= \int_0^{\pi/2}\int_0^{\infty} [\cos(ckt + kR\cos\beta\cos\theta) + \cos(ckt - kR\cos\beta\cos\theta)] \times$$

$$\times J_0(kR\sin\beta\sin\theta)\sin\theta\, d\theta\, dk,$$

$$= I_1 + I_2 \text{ (say)},$$

on putting

$$\xi = k\sin\theta\cos\phi, \quad \eta = k\sin\theta\sin\phi, \quad \zeta = k\cos\theta,$$

$$x = R\sin\beta\cos\alpha, \quad y = R\sin\beta\sin\alpha, \quad z = R\cos\beta,$$

and integrating with respect to ϕ.

Now $$\int_0^{\infty} \cos ax\, J_0(bx)dx = 1/\sqrt{(b^2 - a^2)} \qquad \text{if } b^2 > a^2,$$

$$= 0 \qquad \text{if } b^2 < a^2.$$

Hence in evaluating these integrals with respect to k, we have to discuss the following three cases :

(*a*) $R < ct$, (*b*) $ct < R < \sqrt{2}\, ct$, (*c*) $\sqrt{2}\, ct < R$.

(*a*) When $R < ct$,

$$I' = \int_0^\infty \cos(c\,t\,k + kR\cos\beta\cos\theta)\, J_0(kR\sin\beta\sin\theta)dk = 0$$

so that

$$I_1 = 0. \tag{4.7}$$

(*b*) When $ct < R < \sqrt{2}\,ct$, the integral can have non-zero value only if $\pi - \theta_0 - \beta < \theta < \pi + \theta_0 - \beta$, where $\cos\theta_0 = ct/R$ so that $0 < \theta_0 < \pi/4$. When this condition is satisfied

$$I' = R^{-1}[\sin^2\beta\sin^2\theta_0 - (\cos\theta + \cos\theta_0\cos\beta)^2]^{-1/2}$$

Now, as β varies from 0 to π, we can verify that

$$\begin{aligned} I_1 &= 0 && \text{if } 0 < \beta < \pi/2 - \theta_0, \\ &= \frac{1}{R}\cos^{-1}\left[\frac{c\,t\,z}{\omega\sqrt{(R^2 - c^2 t^2)}}\right] && \text{if } \pi/2 - \theta_0 < \beta < \pi/2 + \theta_0, \\ &= \frac{\pi}{R} && \text{if } \pi/2 + \theta_0 < \beta < \pi - \theta_0, \\ &= \frac{1}{R}\left[\pi - \cos^{-1}\frac{R^2 + c\,t\,z}{\omega\sqrt{(R^2 - c^2 t^2)}}\right] && \text{if } \pi - \theta_0 < \beta < \pi. \end{aligned} \tag{4.8}$$

(*c*) When $R > \sqrt{(2)}\,ct$, so that $\pi/4 < \theta_0 < \pi/2$, I' is given by the same formulae and we have

$$\begin{aligned} I_1 &= 0 && \text{if } 0 < \beta < \pi/2 - \theta_0, \\ &= \frac{1}{R}\cos^{-1}\left[\frac{c\,t\,z}{\omega\sqrt{(R^2 - c^2 + t^2)}}\right] && \text{if } \pi/2 - \theta_0 < \beta < \pi - \theta_0 \\ &= \frac{1}{R}\left[\cos^{-1}\frac{c\,t\,z}{\omega\sqrt{(R^2 - c^2 t^2)}} - \cos^{-1}\frac{R^2 + c\,t\,z}{\omega\sqrt{(R^2 - c^2 t^2)}}\right] && \text{if } \pi - \theta_0 < \beta < \pi/2 + \theta_0, \\ &= \frac{1}{R}\left[\pi - \cos^{-1}\frac{R^2 + c\,t\,z}{\omega\sqrt{(R^2 - c^2 t^2)}}\right] && \text{if } \tfrac{1}{2}\pi + \theta_0 < \beta < \pi. \end{aligned} \tag{4.9}$$

We shall now evaluate I_2. Let

$$I'' = \int_0^\infty \cos(c\,t\,k - k\,R\cos\beta\cos\theta)\, J_0(k\,R\sin\beta\sin\theta)\,dk. \tag{4.10}$$

(*a*) When $R < ct$, $I'' = 0$, and therefore

$$I_2 = 0. \qquad [4.11]$$

(*b*) When $ct < R < \sqrt{2}\cdot ct$,

$$I'' = \frac{1}{R[\sin^2\theta_0 \sin^2\beta - (\cos\theta - \cos\theta_0\cos\beta)^2]^{1/2}},$$

provided $\beta - \theta_0 < \theta < \beta + \theta_0$. When this condition is satisfied,

$$\begin{aligned} I_2 &= \frac{1}{R}\left[\pi - \cos^{-1}\frac{R^2 - ctz}{\omega\sqrt{(R^2 - c^2t^2)}}\right], && \text{if } 0 < \beta < \theta_0 \\ &= \frac{\pi}{R} && \text{if } \theta_0 < \beta < \pi/2 - \theta_0 \\ &= \frac{1}{R}\left[\cos^{-1}\frac{-ctz}{\omega\sqrt{(R^2 - c^2t^2)}}\right] && \text{if } \pi/2 - \theta_0 < \beta < \pi/2 + {}_0 \\ &= 0 && \text{if } \tfrac{1}{2}\pi + \theta_0 < \beta < \pi \quad [4.12] \end{aligned}$$

(*c*) If $R > \sqrt{2}\cdot ct$,

$$\begin{aligned} I_2 &= \frac{1}{R}\left[\pi - \cos^{-1}\frac{-ctz}{\omega\sqrt{(R^2 - c^2t^2)}}\right] && \text{if } 0 < \beta < \pi/2 - \theta_0 \\ &= \frac{1}{R}\left[\cos^{-1}\frac{-ctz}{\omega\sqrt{(R^2 - c^2t^2)}} - \cos^{-1}\frac{R^2 - ctz}{\omega\sqrt{(R^2 - c^2t^2)}}\right] && \text{if } \pi/2 - \theta_0 < \beta < \theta_0 \\ &= \frac{1}{R}\left[\cos^{-1}\frac{-ctz}{\omega\sqrt{(R^2 - c^2t^2)}}\right] && \text{if } \theta_0 < \beta < \pi/2 + \theta_0 \\ &= 0 && \text{if } \tfrac{1}{2}\pi + \theta_0 < \beta < \pi. \quad [4.13] \end{aligned}$$

where $\omega^2 = x^2 + y^2$.

Thus we see that the whole region is divided broadly into three parts by the spherical wave fronts $R = ct$ and $R\sqrt{2}ct$. Then again, the region outside the wave fronts and the annular region between them are divided into four parts by certain cones. In each of these regions, appropriate values have to be associated. In what follows in this section, we shall limit the discussion by exhibiting the various quantities explicitly in terms of the integral I, whose exhaustive treatment we have just now given. Proceeding as before, we get

$$\begin{aligned} \bar{v}_x = & \left(\frac{V_x}{(2\pi)^{3/2}} - \frac{\xi\,\mathbf{k}\cdot\mathbf{V}}{(2\pi)^{3/2}k^2}\right)\cos a\zeta t \\ & + \frac{i\xi}{\rho_0}\left[\frac{c^2\tau}{(2\pi)^{3/2}}\,\frac{\sin ckt}{ck} - \frac{i\rho_0\,\mathbf{k}.\mathbf{V}}{(2\pi)^{3/2}}\,\frac{\cos ckt}{k^2}\right] \qquad [4.14] \end{aligned}$$

Changing the respective variable cyclically we obtain $\bar{v}_y$ and $\bar{v}_z$, the extra term in the latter being neglected since it is of the order a^2/c^2. Inverting, we have

$$v_x = V_x \delta(x)\,\delta(y)\,[\delta(z+at)+\delta(z-at)] + \frac{\mathbf{V}\cdot\nabla}{4\pi}\frac{\partial}{\partial x}\left(\frac{1}{R}+\frac{1}{R'}\right) + \left[\frac{\tau}{\rho_0(2\pi)^3}\frac{\partial^2}{\partial x\partial t} - \frac{\mathbf{V}\cdot\nabla}{(2\pi)^3}\frac{\partial}{\partial x}\right] I. \quad [4.15]$$

v_y and v_z are obtained cyclically changing the appropriate variables. We note, that, apart from travelling with the Alfven wave velocity $\pm a$ along the primitive magnetic field, due to the effect of interaction between compressibility and magnetic field, the last two distributions in (4.15) are created. The vorticity components are of the order of $(a/c)^2$ except at the singularities $(0, 0, \pm at)$ and their nature at the singularities are of the following forms:

$$\vec{w} = \tfrac{1}{2}(\mathbf{V}\times\nabla)[\delta(\mathbf{R})+\delta(\mathbf{R}')] + (a\tau/2\rho_0)(\hat{z}\times\nabla)[\delta(\mathbf{R}')-\delta(\mathbf{R})]. \quad [4.16]$$

As is expected, we note that the density fluctuation does not affect the vorticity in the direction of the primitive magnetic field. From Maxwell's equations, (1.27), (1.28) and (1.24) we get the following equations for the components of perturbed magntic field :

$$\left(\frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial z^2}\right)\Delta\,\mathbf{h} = -\frac{H_0}{\rho_0}\nabla\left(\frac{\partial^3\rho}{\partial z\,\partial t^2}\right) - \frac{H_0}{\rho_0}\left(\Delta\frac{\partial^2\rho}{\partial t^2}\right)\hat{z}, \quad [4.17]$$

whose transforms are

$$\left(\frac{\partial^2}{\partial t^2} + a^2\zeta^2\right)\bar{\mathbf{h}} = -\frac{H_0}{\rho_0}\frac{\zeta\,\mathbf{k}}{k^2}\frac{\partial^2\bar{\rho}}{\partial t^2} + \frac{H_0}{\rho_0}\frac{\partial^2\bar{\rho}}{\partial t^2}\hat{\zeta}. \quad [4.18]$$

Using the appropriate initial conditions and [4.4], we see that

$$h_x = \frac{H_0\tau}{64\pi^2\rho_0}\frac{\partial^2}{\partial x\,\partial z}(1/R+1/R') - \frac{H_0 V_x}{2a}\delta(x)\delta(y)\{\delta(z-at)-\delta(z+at)\} + \frac{H_0}{64\pi^2 a\rho_0}\mathbf{V}\cdot\nabla\,(1/R'-1/R) - \left[\frac{H_0\tau}{32\pi^4\rho_0}\frac{\partial^2}{\partial x\,\partial z} - \frac{H_0\mathbf{V}\cdot\nabla}{32\pi^4}\frac{\partial^2}{\partial x\partial z}\int_0^t dt\right] I.$$

Changing the variable from x to y, we obtain h_y.

$$h_z = \frac{H_0\,\tau}{64\pi^2\rho_0}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{1}{R}+\frac{1}{R'}\right) + \frac{H_0}{2}\left(V_x\frac{\partial}{\partial x}+V_y\frac{\partial}{\partial y}\right)\int_0^t dt\,[\delta(x)\,\delta(y)\{\delta(z+at)+\delta(z-at)\}], + \frac{H_0}{64\pi^2}\mathbf{V}\cdot\nabla\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\int_0^t dt\left(\frac{1}{R}+\frac{1}{R'}\right)$$

$$-\left[\frac{H_0\tau}{32\pi^4\rho_0}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-\frac{H_0\mathbf{V}\cdot\nabla}{32\pi^4}\left(\frac{\delta^2}{\delta x^2}+\frac{\delta^2}{\delta y^2}\right)\int_0^t dt\right]I. \qquad [4.19]$$

These clearly show that $(0, 0, \pm at)$ are the singular points for the magnetic field. The nature of each term in itself shows the order of the singularity.

Similarly, the singular current density is obtained from,

$$J_x-\frac{H_0\tau}{8\pi\rho_0}\frac{\partial}{\partial y}\left[\delta(x)\delta(y)\{\delta(z+at)+\delta(z-at)\}\right]$$

$$-\frac{H_0}{8\pi a}\left(V_z\frac{\partial}{\partial y}-V_y\frac{\partial}{\partial z}\right)\left[\delta(x)\delta(y)\{\delta(z-at)-(z+at)\}\right]$$

$$+\left[\frac{H_0}{32\pi^3c^2}\mathbf{V}\cdot\nabla\frac{\partial^2}{\partial y\partial t}-\frac{H_0\tau}{32\pi^3\rho_0c^2}\frac{\partial^3}{\partial y\partial t^2}\right]I. \qquad [4.20]$$

Changing x to y, we obtain J_y.

$$J_z-=\frac{H_0}{8\pi a}\left(V_y\frac{\partial}{\partial x}-V_x\frac{\partial}{\partial y}\right)\left[\delta(x)\,\delta(y)\,\{\delta(z-at)-\delta(z+at)\}\right].$$

5. Propagation of Ring Disturbances: (Walen's Problem)

In this section we reconsider the Walen's problem namely the propagation of disturbance in the form of a ring in the plane perpendicular to the initial magnetic field in an incompressible fluid. We note that the disturbance being transverse to the direction of propagation, the compressibilty does not play any part. Thus the present section may be regarded as supplementary to Walen's work in as much as it provides the complete solution of his problem in mathematical form when electrical conductivity is infinite.

Let the initial disturbance be

$$\Omega_z=\Omega\,\delta(z)\,\delta(\omega-\omega_0)\ \text{ at }\ t=0, \qquad [5.1]$$

Ω being angular velocity of the element of fluid. Hence, at $t=0$, we have

$$v_x=-\frac{\Omega y}{\omega_0^2}\,\delta(z)\,\delta(\omega-\omega_0),\quad v_y=\frac{\Omega x}{\omega_0^2}\,\delta(z)\,\delta(\omega-\omega_0),\quad v_z=0, \qquad [5.2]$$

and

$$w_x=w_y=0\ ;\ w_z=\frac{2\Omega}{\omega_0^2}\,\delta(z)\,\delta(\omega-\omega_0). \qquad [5.3]$$

Thus from (5.3), we have

$$(\bar{w}_z)_0=\frac{2\Omega}{\omega_0}\frac{J_0(\omega_0\rho)}{(2\pi)^{1/2}},\quad \rho^2=\xi^2+\eta^2. \qquad [5.4]$$

In addition we assume that

$$\left(\frac{\partial\bar{w}_z}{\partial t}\right)_0=0. \qquad [5.5]$$

Taking Fourier transform of (1.23), we get

$$\left(\frac{\partial^2}{\partial t^2} + a^2 \zeta^2\right) \bar{w}_z = 0, \qquad [5.6]$$

whose solution under the above initial conditions is

$$\bar{w}_z = \frac{2\Omega}{(2\pi)^{1/2}\,\omega_0}\, J_0(\omega_0\rho) \cos a\,\zeta\, t. \qquad [5.7]$$

On inversion, we obtain

$$w_z = \frac{2\Omega}{(2\pi)^2\,\omega_0} \iiint_{-\infty}^{\infty} \cos a\,\zeta\, t\, J_0(\omega_0\rho) \exp.\left[-i(\xi x + \eta y + \zeta z)\right] d\xi\, d\eta\, d\zeta$$

$$= \frac{2\Omega}{(2\pi)^2\omega_0} \int_{-\infty}^{\infty} \cos a\,\zeta\, t \exp.\left[-i\,\zeta\, z\right] d\zeta \iint_{-\infty}^{\infty} J_0(\omega\rho) \exp.\left[-i\,(\xi x + \eta y)\right] d\xi\, d\eta.$$

Since $$\int_{-\infty}^{\infty} \cos a\zeta t \exp.\left[-i\zeta z\right] d\zeta = \pi\left[\delta(z + at) + \delta(z - at)\right] \qquad [5.8]$$

and $$\iint_{-\infty}^{\infty} J_0(\omega_0\,\rho) \exp.\left[-i\,(\xi x + \eta y)\right] d\xi\, d\eta$$

$$= \int_0^{2\pi} \int_0^{\infty} J_0(\omega_0\,\rho) \exp.\left[-i\omega\rho \cos(\phi - \alpha)\right] \rho d\rho d\phi$$

$$= 2\pi \int_0^{\infty} J_0(\omega\rho)\, J_0(\omega_0\,\rho)\, \rho d\rho = 2\pi\,\delta\,(\omega - \omega_0) \qquad [5.9]$$

we get $$w_z = (\Omega/\omega_0)\left[\delta\,(z + at) + \delta\,(z - at)\right] \delta\,(\omega - \omega_0). \qquad [5.10]$$

Thus, the ring of initial disturbance travels with the Alfven wave velocity in the positive and negative directions of the primitive magnetic field, the plane of the ring always keeping parallel to itself.

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