# DESCRIBING FUNCTION
# FOR SUCCESSIVE NON-LINEARITIES

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### Abstract

Composite describing function for two consecutive non-linearities in a control system are evaluated by two methods. The first method combines the effects of both the non-linearities into a single non-linear characteristic and the describing function is obtained by integration. The second method makes use of equal area concept for the output waveform. Two examples one with backlash followed by deadzone and the other backlash followed by deadzone and limited proportionality are illustrated. The results obtained by both the methods are compared with those obtained by conventional methods. The simplicity ·in arriving at the formula for the composite describing function by both the methods is of special importance. Describing function for successive non-linearities of similar characteristics are also discussed.

### Introduction

The describing function method of representing non-linear characteristics though an approximation is consistent with the practical performance of many control systems. The evaluation of the describing function necessiates the non-linear characteristic to be defined by mathematical equations with limits. Thus, it is found difficult to arrive at describing functions for non-linearities which cannot be approximated by mathematical equations. The author has developed in this direction two new methods[1,2] of evaluating easily the describing function for any type of non-linearity besides other methods.[8,9]

The present paper is mainly an extension of these methods[1,2] for evaluating the composite decribing function of successive non-linearities not separated by linear blocks. Lakshmi Bai[5] developed a method of transformation. Gronner[3] used the conventional method to evaluate the composite describing function for the case of backlash followed by deadzone. Goldfarb[4] introduced the concept of equivalent admittance.

Two examples are considered in this paper to illustrate the use of the author's two methods[1,2]: (i) Backlash followed by deadzone and (ii) Backlash followed by deadzone and limited dynamic range. The results obtained seem to compare favourably with those obtained by Gronner[3] and Goldfarb[4]. The ease with which these results are obtained by the author's methods[1,2] is to be noted, indicating possible extension of these methods to other types of non-linearities in series. The effect of each nonlinearity on the combined describing function for the two examples is also brought out.

*Method No. 1.*

Let a nonlinear charactertistic be represented by the equation

$$y = f(x) \qquad [1]$$

where $x$ is the input and $y$ the output of the nonlinear element.

The output waveform $F(\theta)$ from the nonlinearity for an input sinusoid $A \sin \theta$ is obtained by putting

$$x = A \sin \theta \qquad [2]$$

in equation [1], giving

$$F(\theta) = f(A \sin \theta) \qquad [3]$$

The fundamental Fourier components of $F(\theta)$ are given by

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} f(A \sin \theta) \cos \theta \, d\theta \qquad [4]$$

$$b_1 = \frac{1}{\pi} \int_0^{2\pi} f(A \sin \theta) \sin \theta \, d\theta \qquad [5]$$

As the output waveform is obtained from given nonlinear characteristic, the author gets the Fourier fundamental components from the nonlinear characteristic itself, without going to the output waveform. This simplifies the work compared to finding the equation for each portion of the output waveform and integrating. Putting equation [2] in equation [5] gives

$$b_1 = \frac{1}{\pi A} \left[ \int_0^A \frac{f(x) \, x \, dx}{\sqrt{(A^2 - x^2)}} + \int_A^0 \frac{f(x) \, x \, dx}{\sqrt{(A^2 - x^2)}} + \int_0^{-A} \frac{f(x) \, x \, dx}{\sqrt{(A^2 - x^2)}} + \int_{-A}^0 \frac{f(x) \, x \, dx}{\sqrt{(A^2 - x^2)}} \right] \qquad [6]$$

Upto this is derived by many authors[6, 7] and the equation [6] is either interpreted graphically or cursors used to evaluate the describing function. The author's method proceeds a step further as follows.

Integrating equation [6] by parts leads to

$$b_1 = \frac{1}{\pi A} \left[ - \mid f(x) \sqrt{(A^2 - x^2)} \int_0^A + \int_0^A f'(x) \sqrt{(A^2 - x^2)} dx \right.$$

$$- \mid f(x) \sqrt{(A^2 - x^2)} \int_A^0 + \int_A^0 f'(x) \sqrt{(A^2 - x^2)} \, dx$$

$$- \mid f(x) \sqrt{(A^2 - x^2)} \int_0^{-A} + \int_0^{-A} f'(x) \sqrt{(A^2 - x^2)} \, dx$$

B. L. DEEKSHATULU

$$- \left| f(x) \sqrt{(A^2 - x^2)} \Big|_{-A}^{0} + \int\limits_{-A}^{0} f'(x) \sqrt{(A^2 - x^2)} \, dx \right| \qquad [7]$$

Two cases may arise: (a) If the nonlinearity is made up of straigt lines as for backlash, deadband etc., then $f'(x)$ is a constant and can be taken out of the integration sign and $b_1$ is obtained from equation [7] by putting proper limits as is done in this paper. (b) If the nonlinear characteristic is an arbitrary curve, undefined by equations, a simple graphical construction[2] is developed which is applicable to all nonlinearities but is not discussed here.

Putting $x = A \sin \theta$ in equation [4] gives

$$a_1 = \frac{1}{\pi A} \int\limits_{\theta = 0}^{\theta = 2\pi} f(x) \, dx \qquad [8]$$

$$= \frac{1}{\pi A} \left[ \int\limits_{0}^{A} f(x) \, dx + \int\limits_{A}^{0} f(x) \, dx + \int\limits_{0}^{-A} f(x) \, dx + \int\limits_{-A}^{0} f(x) \, dx \right] \qquad [9]$$



$\sin \theta_0 = a/2A ; \quad \theta_0 < \pi/2$
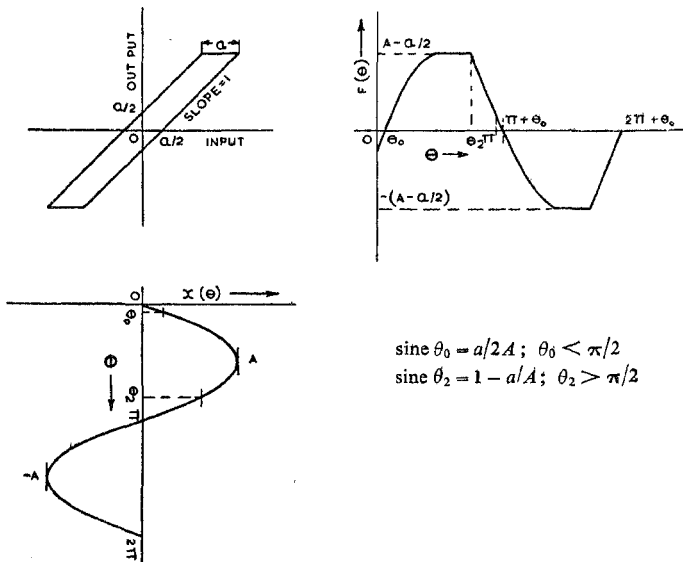$\sin \theta_2 = 1 - a/A ; \quad \theta_2 > \pi/2$

Fig. I
Backlash and its output waveform

Thus $a_1$ exists only when the nonlinearity avoids the origin and is represented by the area enclosed by the nonlinear characteristic around the origin as seen for backlash. The above procedure is shown to be applicable also for two nonlinearities in series.

### *Example I*—BACKLASH FOLLOWED BY DEADBAND

A common phenomena of loose gear trains or mechanical linkages is the backlash illustrated with its output waveform for a sinusoidal input in Fig. 1.



$$\text{sine } \theta_3 = (d + a/2)/A \ ; \ \theta_3 < \pi/2$$
$$\text{sine } \theta_2 = 1 - a/A \ ; \ \theta_2 > \pi/2$$
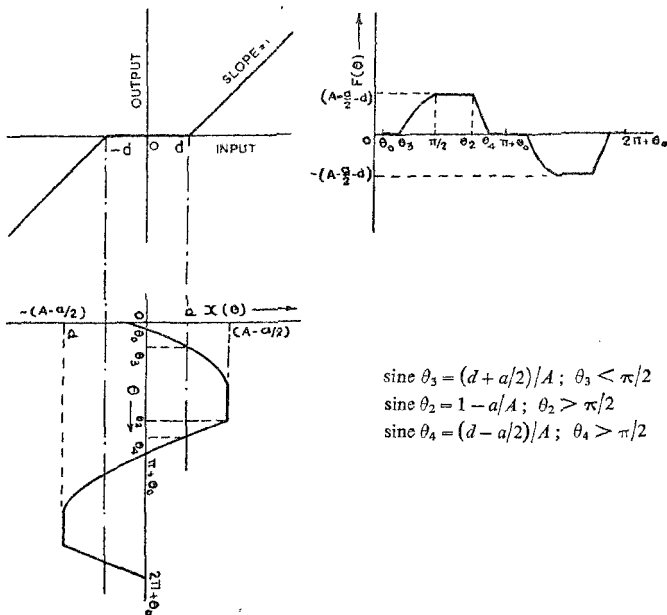$$\text{sine } \theta_4 = (d - a/2)/A \ ; \ \theta_4 > \pi/2$$

FIG. II
Net output for Backlash followed by deadband

A combination of backlash and deadband may appear in a control system with backlash on the the output gear and deadzone in the amplifier. In such a case the output waveform of Fig. I will be the input to the deadband and the net output is shown in Fig. II. Instead of treating the two nonlinearities

separately to evaluate the composite describing function, the given nonlinear characteristics can be modified to suit the combination.

It is seen from Fig. I and II that the net output from the combination is as if the output waveform in Fig. I has been cut flat at $\pm d$ corresponding to the deadband limit.   The same effect is got if instead of cutting in the output waveform of Fig. I, the given nonlinear characteristic is chopped off upto AD and EH (refer to Fig. III) corresponding to $\pm d$.   In Fig. III the deadband
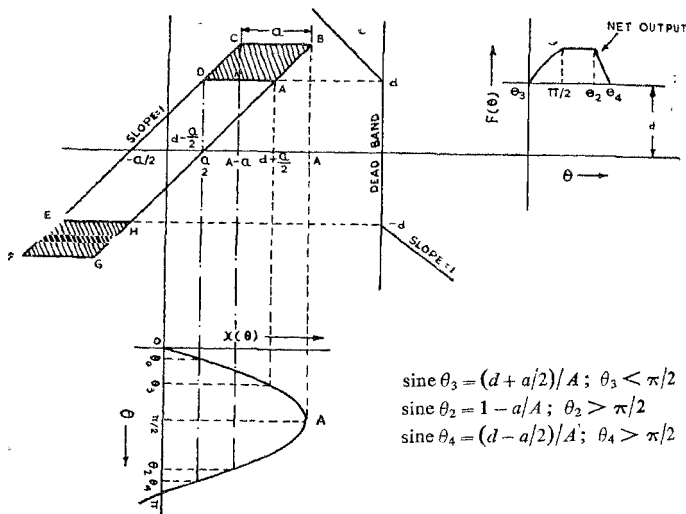


$$\text{sine } \theta_3 = (d + a/2)/A \; ; \; \theta_3 < \pi/2$$
$$\text{sine } \theta_2 = 1 - a/A \; ; \; \theta_2 > \pi/2$$
$$\text{sine } \theta_4 = (d - a/2)/A \; ; \; \theta_4 > \pi/2$$

FIG. III
Effective nonlinear characteristic for Backlash followed by dead zone

characteristic has been titled by $90°$ because the input to it is the output from the backlash element.   Although the given backlash characteristic is GBCF, because of the combination with deadzone the effective nonlinearity reduces to the portions ABCD and EFGH (shown hatched in Fig. III).   The actual output is obtained by subtracting $d$ from the output of the backlash portions ABCD and EFGH.   Now with this modified nonlinearity the describing function is evaluated easily.   Equation [5] is modified to give

$$b_1 = \frac{2}{\pi} \int_0^\pi [f(A \sin \theta) - d] \sin \theta \, d\theta \qquad [10]$$

Putting the equations with proper limits for the straight line portions of the nonlinear characteristic equation [10] becomes.

$$b_1 = (2/\pi)\left\{ - \left| (x - a/2)\sqrt{(A^2 - x^2)} \right|_{d+a/2}^{A} + \int_{d+a/2}^{A} \sqrt{(A^2 - x^2)}\,dx \right.$$
$$- \left| (A - a/2)\sqrt{(A^2 - x^2)} \right|_{A}^{A-a} + 0 - \left| (x + a/2)\sqrt{(A^2 - x^2)} \right|_{A-a}^{d-a/2}$$
$$\left. + \int_{A-a}^{d-a/2} \sqrt{(A^2 - x^2)}\,dx \right\} + \frac{2d}{\pi}\left[ \left| \cos\theta \right|_{\theta_3}^{\pi/2} + \left| \cos\theta \right|_{\pi/2}^{\theta_2} + \left| \cos\theta \right|_{\theta_1}^{\theta_4} \right] \qquad [11]$$

$$b_1/A = (2/\pi)\{(d/a)\sqrt{[1 - (d + \tfrac{1}{2}a)^2/A]} - (d/a)\sqrt{[1 - (d - \tfrac{1}{2}a)^2/A]}$$
$$+ (d/a)(\cos\theta_4 - \cos\theta_3)\}$$
$$+ \frac{1}{\pi}\left\{ \left| \sin^{-1}(x/A) + (x/A)\sqrt{[1 - (x/A)^2]} \right|_{d+a/2}^{A} \right.$$
$$\left. + \left| \sin^{-1}(x/A) + (x/A)\sqrt{[1 - (x/A)^2]} \right|_{A-a}^{d-a/2} \right\} \qquad [12]$$

The first four terms cancel. Putting the different values of $\sin\theta$ and $\cos\theta$ in the above equation we get finally

$$b_1/A = (1/\pi)[\pi/2 + \theta_4 - \theta_3 - \theta_2 + \sin\theta_4\cos\theta_4 - \cos\theta_3\sin\theta_3 - \sin\theta_2\cos\theta_2] \quad [13]$$

The coefficient $a_1$ is given by

$$a_1 = \frac{1}{\pi}\int_{0}^{2\pi} [f(A\sin\theta) - d]\cos\theta\,d\theta \qquad [14]$$

or

$$\frac{a_1}{A} = \frac{2}{\pi A^2}(\text{area } ABCD) = \frac{2}{\pi} \times \frac{a}{A}\left\{1 - \frac{d + a/2}{A}\right\} \qquad [15]$$

The composite describing function

$$G_1 = \sqrt{[(a_1/A)^2 + (b_1/A)^2]} \angle \tan^{-1}(a_1/b_1) \qquad [16]$$

Table I gives the composite describing function for typical values and are compared with those obtained by Gronner[3].

It will be interesting to consider two cases for this combination of non-linearities : (i) with $d = 0.5$ and $A = 1.0$, $a$ is varied between 0 and 1.0 and the describing function is shown in Fig. IV. The case $a = 0$ represents a pure deadband and as $a$ increases the composite describing function decreases in magnitude and increases in phase lag, (ii) With $a = 0.5$ and $A = 1.0$, $d$ can be varied from 0 to 0.75. From Fig. V, it is seen that the phase angle is almost constant upto $d = 0.25$ and then increases.

TABLE I

Backlash followed by deadzone

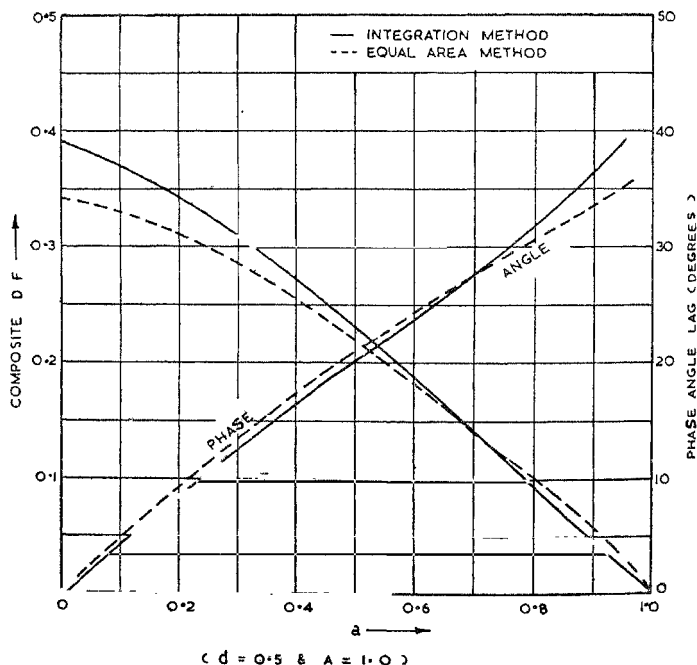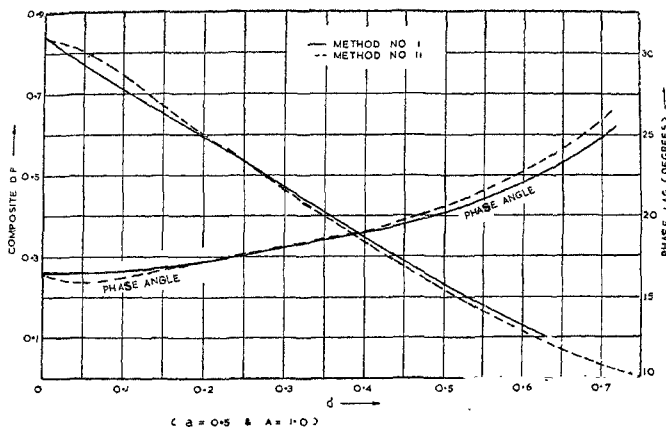| No. | d/a | a/A | D. F. evaluated by integration method | | D. F. evaluated by Gronner (exact analysis) | |
|---|---|---|---|---|---|---|
| | | | $G_1$ | $G_1$ radians | $G_1$ | $G_1$ radians |
| 1 | 1.4 | 0.31 | 0.382 | − 0.212 | 0.382 | − 0.213 |
| 2 | 1.0 | 0.2 | 0.701 | − 0.127 | 0.701 | − 0.127 |
| 3 | 0.6 | 0.5 | 0.463 | − 0.313 | 0.463 | − 0.313 |
| 4 | 0.5 | 0.2 | 0.827 | − 0.123 | 0.827 | − 0.123 |
| 5 | 0.3 | 0.5 | 0.648 | − 0.298 | 0 649 | − 0.298 |



FIG. IV

FIG. V

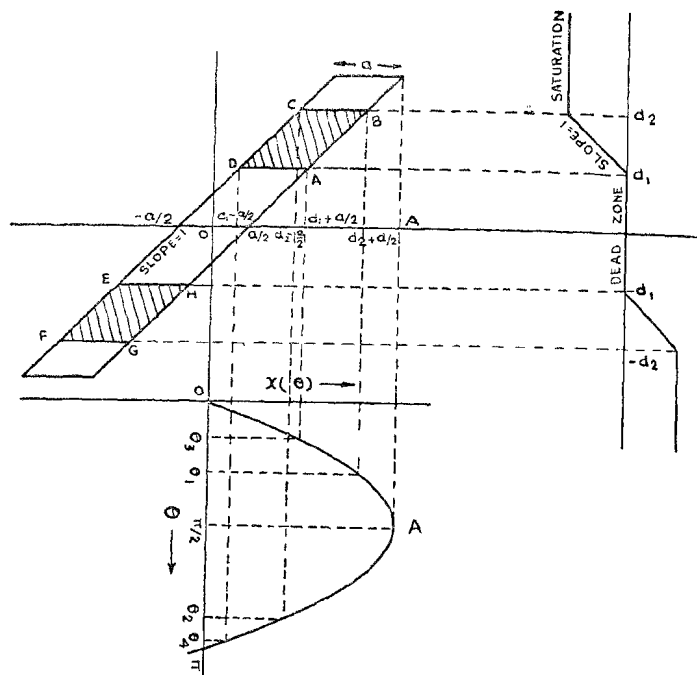*Example II*—BACKLASH FOLLOWED BY DEADBAND AND SATURATION :

A case of this type occurs when one nonlinear element with both viscous and dry friction is cascaded with another of limited dynamic range. As in example I, the second nonlinearity is rotated by $90^\circ$ to give the dynamic limit AD and BC corresponding to the proportionality limits $d_1$ and $d_2$ and shown in Fig. VI. Hence the combined effective nonlinearity is defined by ABCD and EFGH. The various limits and angles are defined in the figure.

As it is again a case of symmetrical nonlinearity we have

$$\frac{b_1}{A} = \frac{2}{\pi A} \int_0^\pi \left[ f(A \sin \theta) - d_1 \right] \sin \theta \, d\theta \qquad [17]$$

Putting the proper limits of integration from Fig. VI, we have :

$$\frac{b_1}{A} = \frac{2}{\pi A} \left[ - \left| (x - a/2) \sqrt{(A^2 - x^2)} \right|_{d_1+a/2}^{d_2+a/2} + \int_{d_1+a/2}^{d_2+a/2} \sqrt{(A^2 - x^2)} \, dx \right.$$

$$- \left| d_2 \sqrt{(A^2 - x^2)} \right|_{d_2+a/2}^{d_2-a/2} - \left| (x + a/2) \sqrt{(A^2 - x^2)} \right|_{d_2-a/2}^{d_1-a/2}$$

$$\left. + \int_{d_2-a/2}^{d_1-a/2} \sqrt{(A^2 - x^2)} \, dx \right] + \frac{2}{\pi A} d_1 \left[ \left| \cos \theta \right|_{\theta_2}^{\theta_1} + \left| \cos \theta \right|_{\theta_1}^{\theta_2} + \left| \cos \theta \right|_{\theta_2}^{\theta_4} \right] \qquad [18]$$

$$d_2 \leqslant A - a/2$$

$$\text{sine } \theta_3 = (d_1 + a/2)/A \,; \quad \theta_3 < \pi/2 \qquad \text{sine } \theta_1 = (d_2 + a/2)/A \,; \quad \theta_2 \leqslant \pi/2$$

$$\text{sine } \theta_2 = (d_2 - a/2)/A \,; \quad \theta_2 > \pi/2 \qquad \text{sine } \theta_4 = (d_1 - a/2)/A \,; \quad \theta_4 > \pi/2$$

### FIG. VI

Effective nonlinear characteristic for Backlash followed by dead zone & saturation

$$= \frac{2}{\pi A^2} \left[ \int_{d_1 + a/2}^{d_2 + a/2} \sqrt{(A^2 - x^2)} \, dx + \int_{d_2 - a/2}^{d_1 - a/2} \sqrt{(A^2 - x^2)} \, dx \right] \qquad [19]$$

Thus the conventional method of obtaining $b_1$ by integrating the equations of the output waveform lends itself to evaluation of a simple standard integral within proper limits by this method.

$$b_1/A = (1/\pi)[\theta_1 + \theta_4 - \theta_3 - \theta_2 + \sin\theta_1\cos\theta_1 + \sin\theta_4\cos\theta_4$$
$$- \sin\theta_3\cos\theta_3 - \sin\theta_2\cos\theta_2] \qquad [20]$$

The coefficient $a_1$ is given by

$$a_1/A = [2/\pi A^2](\text{area } ABCD) = (2/\pi)(a/A)(d_2 - d_1)/A \qquad [21]$$

The composite describing function $G_2$ is therefore

$$G_2 = \sqrt{[(a_1/A)^2 + (b_1/A)^2]} \angle \tan^{-1}(a_1/b_1) \qquad [22]$$

Table II gives the composite describing function for some typical values and are compared with those obtained by Goldfarb[4].

TABLE II
Backlash followed by deadband with limited dynamic range

| No. | $A/a$ | $\dfrac{d_1 + a/2}{d/2}$ | $\dfrac{d_2 + a/2}{d_1 + a/2}$ | D. F. evaluated by Integration method | | D. F. evaluated by Goldfarb (Conventional method) | |
|-----|-------|------|------|--------|--------|--------|--------|
| | | | | $G_2$ | $G_2$ degrees | $G_2$ | $G_2$ degrees |
| 1 | 1 | 1 | 2 | 0.59 | − 32.5 | 0.59 | − 32.0 |
| 2 | 1.5 | 2 | 2 | 0.333 | − 25.5 | 0.33 | − 25.0 |
| 3 | 2.5 | 3 | 2 | 0.388 | − 15.0 | 0.39 | − 16.0 |
| 4 | 3 | 2 | 2 | 0.40 | − 10.2 | 0.4 | − 10.0 |
| 5 | 4 | 3 | 2 | 0.42 | − 8.5 | 0.425 | − 9.0 |

It will be interesting to consider the effect of variation of the dynamic range $d_2/d_1$ on the composite describing function. A typical case with $A/a = 2$ is analyzed for values of $d_1 = 0 \cdot 5a/2$, $a/2$, $1 \cdot 5a/2$ and $a$. The ratio of $d_2/d_1$ is limited with each value of $d_1$. The magnitude and phase of the composite describing function are shown in Fig. VII. The deadzone limit $d_1$ of the second nonlinearity seem to effect the composite describing function to a great extent.

*Method No. II*—THE EQUAL AREA CONCEPT

In this method the output waveform from a nonlinearity is represented by a sinusoid of fundamental frequency whose total area in one period is the same as the total area in one period of the output waveform.

Let the Foureir analysis of $F(\theta)$ in Fig. I give

$$F(\theta) = (an\cos n\theta + bn\sin n\theta); \quad n = 1, 3, 5\cdots \qquad [23]$$

The total area in one loop of the output waveform from $\theta_0$ to $\pi + \theta_0$ is same as from 0 to $\pi$. Integrating $F(\theta)$ within limits from $\theta_0$ to $\pi + \theta_0$ we have
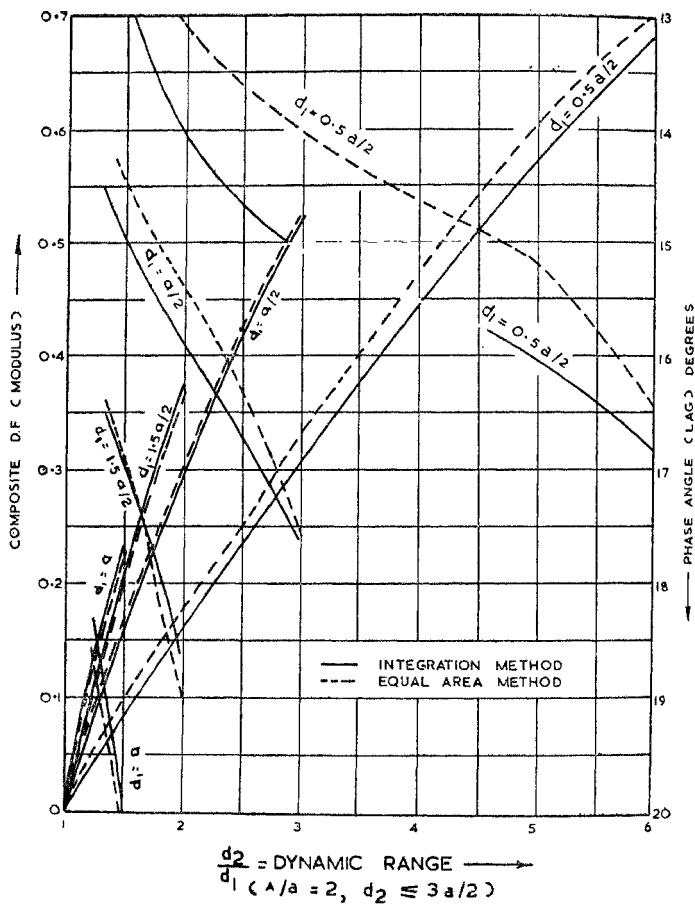
Fig. VII

$$\int\limits_{\theta_0}^{\pi+\theta_0} F(\theta)\,d(\theta) = \int\limits_0^\pi F(\theta)\,d(\theta) = \int\limits_0^\pi a_1\cos\theta\,d\theta + \int\limits_0^\pi a_3\cos 3\theta\,d\theta + \cdots$$

$$+ \int\limits_0^\pi b_1\sin\theta\,d\theta + \int\limits_0^\pi b_3\sin 3\theta\,d\theta + \cdots \quad [24]$$

It is known that the area in one loop of a sinusoid of amplitude $A$ is $2A$ units of area. Representing $F(\theta)$ by an equivalent sinusoid of amplitude $K$ and of fundamental frequency and whose area in its one loop is equal to the area in one loop of $F(\theta)$ (that is the area from $\theta_0$ to $\pi + \theta_0$ or the total area from 0 to $\pi$), we have equation [24] rewritten as

$$\int\limits_0^\pi K\sin\theta\,d\theta = 2K = \int\limits_{\theta_0}^{\pi+\theta_0} F(\theta)\,d\theta = \int\limits_0^\pi F(\theta)\,d\theta = 2(b_1 + b_3/3 + b_5/5 + \cdots) \quad [25]$$

or
$$K = b_1 + b_3/3 + b_5/5 + \cdots \quad [26]$$

The describing function for the nonlinearity

$$= \frac{2K}{2A} = \frac{\text{area in one loop of the equivalent sinusoid}}{\text{area in one loop of the input sinusoid}}$$

$$= K/A = b_1/A + b_3/3A + b_5/5A + \cdots \quad [27]$$

Thus from equation [26] the amplitude of the equivalent sinusoid obtained from the consideration of equal area of the output waveform represents the fundamental amplitude plus (or minus depending on the algebraic signs of the harmonics) a fraction of the harmonic amplitudes and therefore $K/A$ represents the describing function when the harmonics are small as usually the case with many nonlinear elements.

The coefficient $a_1$ is evaluated in terms of area given by

$$\frac{a_1}{A} = \frac{2}{\pi A^2}(\text{area enclosed by the nonlinear characteristic}) \quad [28]$$

The equal area method is a very helpful and a rapid graphical technique, and the results obtained are sufficiently accurate. The method is applicable to all nonlinearities.

### EXAMPLE I—BACKLASH FOLLOWED BY DEAD ZONE

The output waveform is symmetrical for this combination as seen in Fig. II and hence area in one loop of the output waveform need be considered. Referring to Fig. II and VIII the area in the net output $PQRS$ can be represented by a sinusoid of fundamental frequency and amplitude $K_1$ given by; area in one loop of the equivalent sinusoid $= 2K_1$ where
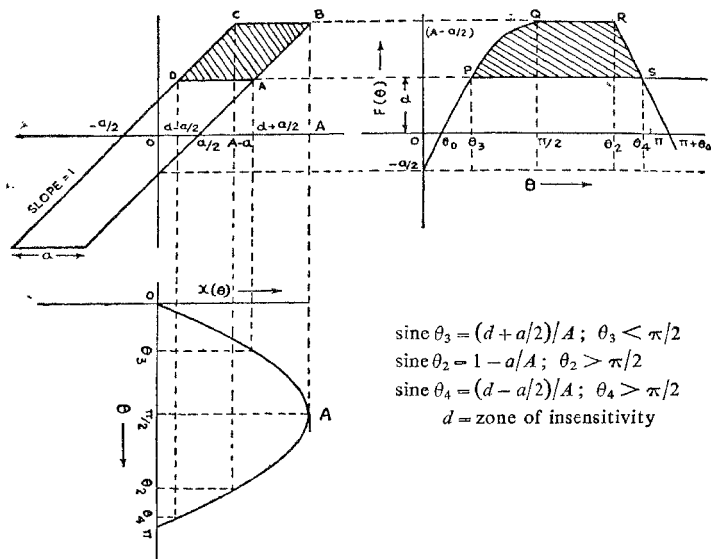
FIG. VIII
Equal area method of evaluating the composite describing function for
Backlash followed by dead zone

$$\sin e\ \theta_3 = (d + a/2)/A \ ; \ \theta_3 < \pi/2$$
$$\sin e\ \theta_2 = 1 - a/A \ ; \ \theta_2 > \pi/2$$
$$\sin e\ \theta_4 = (d - a/2)/A \ ; \ \theta_4 > \pi/2$$
$$d = \text{zone of insensitivity}$$

$$2K_1 = \text{area } PQRS = \int\limits_{\theta_3}^{\pi/2} A \sin \theta \ d\theta - (d + a/2)(\pi/2 - \theta_3)$$

$$+ (\theta_2 - \pi/2)\left\{A - (d + a/2)\right\} + \int\limits_{\theta_3}^{\theta_4} (A \sin \theta + a) d\theta - (d + a/2)(\theta_4 - \theta_2) \quad [29]$$

The fundamental component $b_1/A$ of the conventional describing function is then approximately given by

$$\frac{b_1}{A} = \frac{2K_1}{2A} = \frac{\cos \theta_3}{2} - \frac{d + a/2}{2A}(\theta_4 - \theta_3) + \frac{\theta_2 - \pi/2}{2}$$
$$- \frac{\cos \theta_4}{2} + \frac{\cos \theta_2}{2} + (a/2A)(\theta_4 - \theta_2) \quad [30]$$

$$\frac{a_1}{A} = \frac{2}{\pi A^2}(\text{area } ABCD) = \frac{2}{\pi}(a/A)\left(1 - \frac{d + a/2}{A}\right) \qquad [31]$$

The composite describing function from equation [30] and [31] is

$$G_1 = \sqrt{[(a_1/A)^2 + (b_1/A)^2]} \angle \tan^{-1}(a_1/b_1) \qquad [32]$$

Table III gives the values of the composite describing function and are compared for accuracy with those obtained by Gronner[3].

TABLE III

Backlash followed by dead zone

| No. | $d/a$ | $a/A$ | D. F. evaluated by equal area concept | | D. F. evaluated by Gronner (exact analysis) | |
|-----|-------|-------|-------|-------|-------|-------|
| | | | $G_1$ | $G_1$ radian | $G_1$ | $G_1$ radians |
| 1 | 0.3 | 0.5 | 0.66 | − 0.291 | 0.649 | − 0.298 |
| 2 | 0.5 | 0.2 | 0.816 | − 0.124 | 0.827 | − 0.123 |
| 3 | 0.6 | 0.5 | 0.445 | − 0.328 | 0.463 | − 0.313 |
| 4 | 1.0 | 0.2 | 0.678 | − 0.129 | 0.701 | − 0.127 |
| 5 | 1.4 | 0.31 | 0.354 | − 0.228 | 0.382 | − 0.213 |

From Table III, the maximum error in the values obtained by equal area method is about 8 per cent. Refer also to Figures IV and V.

EXAMPLE II—BACKLASH FOLLOWED BY DEAD ZONE AND SATURATION

Referring to Figure IX, the area in one loop of the output waveform *PQRS* is represented by an equivalent sinusoid of amplitude $K_2$ given by

$$\text{area } PQRS = 2K_2 = \int_{\theta_3}^{\theta_1} A \sin\theta \, d\theta - (a/2 + d_1)(\theta_1 - \theta_3)$$

$$+ (\theta_2 - \theta_1)(d_2 - d_1) + \int_{\theta_2}^{\theta_4}(A \sin\theta + a) \, d\theta - (d_1 + a/2)(\theta_4 - \theta_2) \quad [33]$$

$$= A \cos\theta_3 - A \cos\theta_1 - A \cos\theta_4 + A \cos\theta_2 - (d_1 - a/2)(\theta_4 - \theta_2)$$

$$+ (\theta_2 - \theta_1)(d_2 - d_1) - (d_1 + a/2)(\theta - \theta_3) \quad [34]$$

or $\dfrac{b_1}{A} = \dfrac{2K_2}{2A} = \dfrac{\cos\theta_3}{2} - \dfrac{\cos\theta_1}{2} - \dfrac{\cos\theta_4}{2} + \dfrac{\cos\theta_2}{2}$

$$-\dfrac{\sin\theta_4}{2}(\theta_4 - \theta_2) + (\theta_2 - \theta_1)\dfrac{(d_2 - d_1)}{2A} - \dfrac{\sin\theta_3}{2}(\theta_1 - \theta_3) \qquad [35]$$

$$\dfrac{a_1}{A} = (2/\pi) \times (a/A) \times \dfrac{d_2 - d_1}{A} \qquad [36]$$



$d_2 \leq A - a/2$

$\text{sine }\theta_3 = (d_1 + a/2)/A \; ; \; \theta_3 < \pi/2$

$\text{sine }\theta = (d_2 + a/2)/A \; ; \; \theta < \pi/2$

$\text{sine }\theta_2 = (d_2 - a/2)/A \; ; \; \theta_2 > \pi/2$

$\text{sine }\theta_4 = (d_1 - a/2)/A \; ; \; \theta_4 > \pi/2$

$d_1 = $ dead zone limit
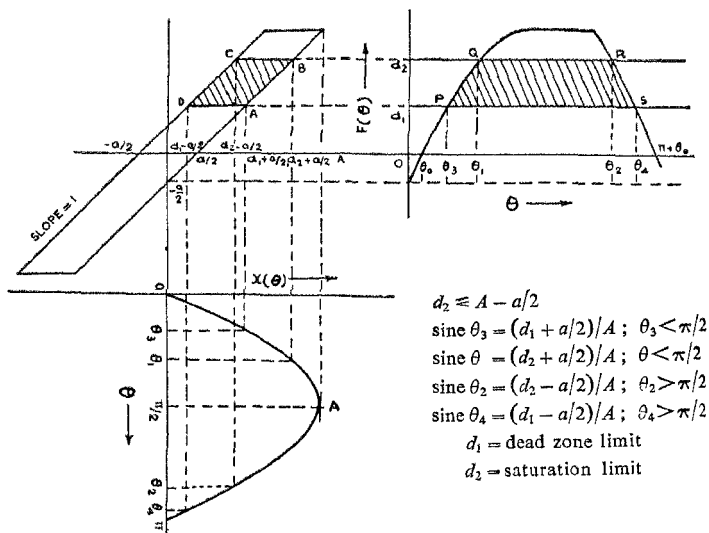
$d_2 = $ saturation limit

FIG. IX

*Equal area method of evaluating composite describing function for Backlash followed by dead zone & saturation*

The describing functions evaluated with equations [35] and [36] are shown for comparison in Table IV.

Referring to Figure VII it can be stated that the equal area method also gives sufficiently accurate results compared to those obtained by conventional methods.

TABLE IV
Backlash followed by dead zone and limited dynamic range

| No. | $A/a$ | $\dfrac{d_1+a/2}{a/2}$ | $\dfrac{d_2+a/2}{d_1+a/2}$ | D. F. evaluated by equal area concept | | D. F. evaluated by Gold-farb(conventional method) | |
|---|---|---|---|---|---|---|---|
| | | | | $G_2$ | $G_2$ degrees | $G_1$ | $G_2$ degrees |
| 1 | 1 | 1 | 2 | 0.593 | − 32.8 | 0.59 | − 32 |
| 2 | 1.5 | 2 | 2 | 0.354 | − 23.5 | 0.33 | − 25 |
| 3 | 2.5 | 3 | 2 | 0.37 | − 15.8 | 0.39 | − 16 |
| 4 | 3 | 2 | 2 | 0.415 | − 9.5 | 0.4 | − 10 |
| 5 | 4 | 3 | 2 | 0.42 | − 8.2 | 0.425 | − 9 |

*Nonlinearitis of similar characteristics cascaded together* :—Uptill now cases have been investigated for nonlinearities of dissimilar characteristics. Few cases of composite nonlinearities of similar characteristics are also studied and the following conclusions are drawn. Such cases may arise in a servo system with the saturation of the amplifier followed by saturation of the motor or dead band in the amplifier followed by dead band in the motor.

(i) Two nonlinearities with limited range of unit proportionality output followed by saturation are considered. The composite describing function in such a ease can be got by considering only that single nonlinearity which has the minimum saturated output.

(ii) Consider two nonlinearities with dead zone limits $d_1$ and $d_2$ followed by unit proportionality outputs. The net output from such a combination for an input sinusoid $A \sin \theta$ exists only if $A > d_1 + d_2$ and the composite describing function can be evaluated by considering a single nonlinearity with a zone of insensitivity $d_1 + d_2$.

## CONCLUSIONS

(1) The first method known as the 'integration method' is used to evaluate the composite describing function by considering the portion of the nonlinear characteristic due to the combination. This metod is accurate but is appplicable only if the nonlinear characteristic is made up of straight line portions as for backlash, dead zone, etc.

(2) If the two nonlinear characteristics are arbitrary curves, then the effective single nonlinear characteristic is to be obtained from the ordinates and abscisse of the two elements (because the output from one forms the input to the other nonlinearity). The graphical technique[2] can then be applied to the single nonlinear characteristic to evaluate the composite describing function.

(3) The equal area method is a rapid graphical technique. It is sufficiently accurate and may be extended to any complex combination of nonlinearities in series.

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