EQUAL AREA METHOD OF EVALUATING THE DESCRIBING FUNCTION

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ABSTRACT

A graphical method of approximation to the describing function is presented and evaluated as the ratio of the total area in one period of the output waveform to the total area in one period of the input sinusoid. The values so obtained are compared for different input amplitudes and for different nonlinearities with those obtained by conventional methods. The error in all these cases is shown to be within 8 per cent emphasizing the usefulness and simplicity of this new method for many types of nonlinearities.

In sinusoidal analysis[1-3] the describing function is determined solely from the peak amplitude of the fundamental in the output waveform and the peak amplitude of the input sinusoid. The harmonics are neglected as they are supposed to be small and filtered out by the system characteristics. In other words, the output from the nonlinearity is treated only as the fundamental, which is a pure sinusoid of the same frequency as the input sinusoid.

EQUAL AREA CONCEPT OF THE EQUIVALENT SINUSOID

According to this method, the periodic output waveform from a nonlinear characteristic (for an input sinewave) is replaced by an equivalent sinewave such that the total area in its one period is equal to the total area in one period of the output waveforms.[5] While this concept is theoretical, the author wants to show by this paper that it gives sufficiently accurate results in the describing function analysis.

Consider the output waveform from a nonlinearity to be consisting of only sine terms in its Fourier Series. One can then confine to the interval 0 to π as shown in Fig. I. At any instant the value of the output waveform is obtained by adding to the amplitude of the fundamental the amplitudes of the harmonics at that particular instant. Hence considering the output to be only a fundamental sinusoid (essential in describing function analysis) involves an error by an amount proportional to the area intercepted between the fundamental and the original output waveform as shown in Fig. I. Call this area Δ. Here, it should be noted that only odd sine harmonics contribute certain net area* to $f(\theta)$, whereas the even harmonics do not (in practice also, even harmonics are rarely present). Hence the cause of the existence of the area Δ is due to the presence of only odd sine terms (refer to appendix).

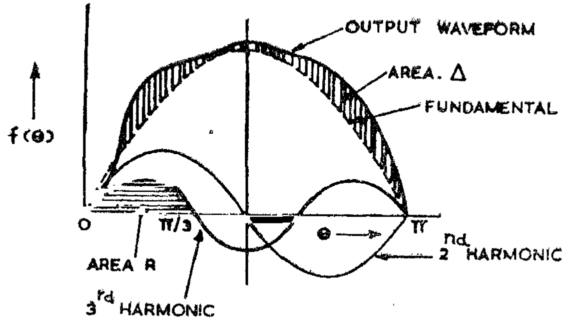

FIG. 1

The describing function according to this method is defined as

$$\text{D.F.} = \frac{\text{Peak amplitude of the equivalent sinusoid}}{\text{Peak amplitude of the input sinusoid}}$$

$$= \frac{\text{Output area in one period}}{\text{Input area in one period}}$$

The phase shift, if any, is determined from the area enclosed by the nonlinear characteristic as shown in example (*b*).

GRAPHICAL IMPORTANCE OF THE METHOD

While the mathematical methods of evaluating the describing function are laborious even for slightly complex nonlinearities like backlash, the equal area method gives the describing function of the nonlinear element quickly. All that is necessary is to plot the output waveform for a given nonlinearity to scale, corresponding to the given input sinusoid (a). If the nonlinearity

*The net area contributed by a third harmonic is given by

$$R = \int_0^{\pi/3} b_3 \sin 3\theta \, d\theta = \frac{2\, b_3}{3}$$

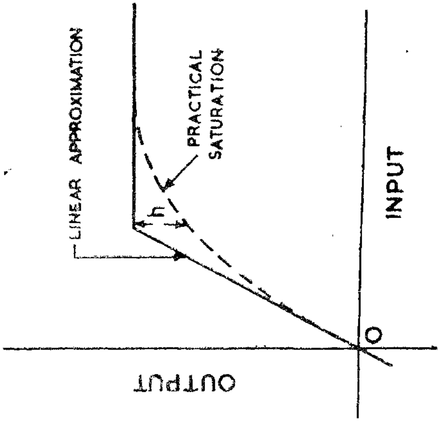

FIG. II (*a*)

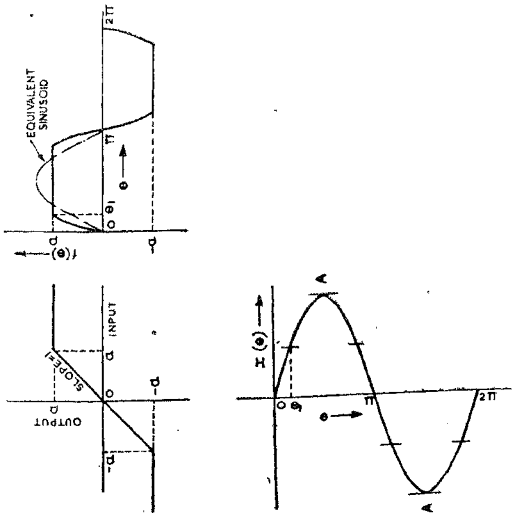

FIG. II
Output Waveform for Saturation

is asymmetrical, the output must be plotted for the full period (0 to 2π) and the arithmetic sum of areas in its loops divided by the arithmetic sum of areas in both the loops of the input sinusoid gives the describing function. (*b*) If the nonlinearity is symmetrical it is still more simple. One need plot the output waveform to scale for one loop ($0-\pi$) or even $0-\pi/2$ and doubled, and the ratios of the areas in the output and input loops give the describing function. Thus, the process of integration in analytical methods lends itself to areas in this simple method. This is the strong point in favour of the equal area method.

EXAMPLES

To indicate the accuracy of this method, in the following examples, formulas are derived for describing functions of nonlinearities by the equal area method (the graphical method gives the same result) and are compared with those obtained by Fourier Analysis. The peak amplitude of the input sinusoid and of the equivalent sinusoid are represented by A and K respectively. The output fundamental amplitude is represented by b_1.

(*a*) *Saturation*:—This is a symmetrical nonlinearity (refer to Fig. II). The area is one loop of the equivalent sinusoid is given by

$$2K = 2A - 2\int_{\theta_1}^{\pi/2} A \sin\theta \, d\theta + 2a(\pi/2 - \theta_1)$$

The describing function by the equal area method is therefore given by

$$K/A = 1 - \cos\theta_1 + \sin\theta_1 (\pi/2 - \theta_1) \quad \text{where} \quad \theta_1 = \sin^{-1}(a/A)$$

The describing function by Fourier method is

$$\frac{b_1}{A} = \frac{2}{\pi}\left(\theta_1 + \frac{\sin 2\theta_1}{2}\right)$$

In Fig. II, $A = 3$ cms., $a = 1.5$ cms. The output area in one loop ($0/\pi$) of $f(\theta)$ is found to be 3.8 sq. cms. radians. As one radian in the figure represents $3/\pi$ cms., $K = 3.8 \times \pi/(2 \times 3) = 1.98$ cms. and so the describing function by equal area method is $1.98/3.0 = 0.66$. The describing function by conventional method gives 0.61. The error is 8.2 per cent. Curves of nonlinear gain by both methods are shown in Fig. III for different input amplitudes. It is seen that the equal area method gives higher values since the coefficients b_3, b_5 etc. in equation (8) are positive.

At this stage, the following points are worth mentioning:

1. Generally because of practical limitations, A/a ratio cannot exceed beyond 3 or 4. And, the error in the amplitude of a limit cycle (if it exists)

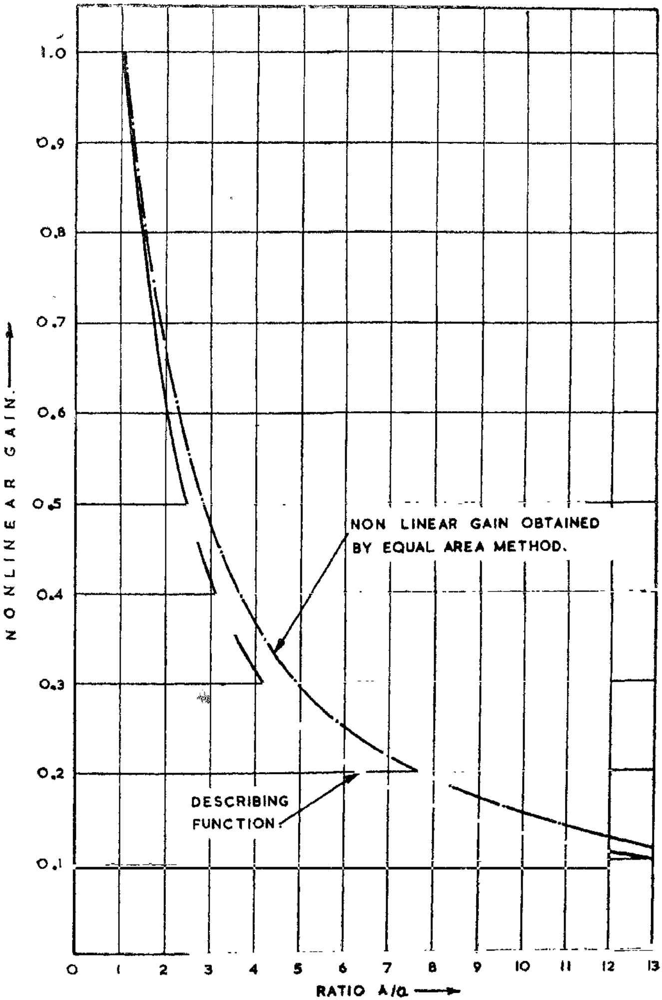

FIG. III
Transfer function curves for saturation

caused by the error in the describing function evaluated by equal area method, is very small.

2. For simple nonlinearities whose output does not cause a phase shift in its describing function, the method does not in any way affect the frequency of limit cycle.

3. The linear approximation to an experimentally obtained saturation curve always lies higher up. Also, there will be no abrupt change in the slope of an actual n.l. characteristic. The values of the describing function for the actual nonlinearity evaluated by the equal area method will equal those obtained by Fourier Analysis with negligible error. This is because the harmonic content in the output waveform of a practical nonlinear characteristic will be much less than when the actual nonlinearity is approximated by linear segments. (The hump h in Figure II a which is a measure of b_3, will be absent in an actual saturation curve).

(*b*) *Dead-zone* :—The applicability of the method to deadband nonlinear characteristic can be easily seen from Figures IV and V.

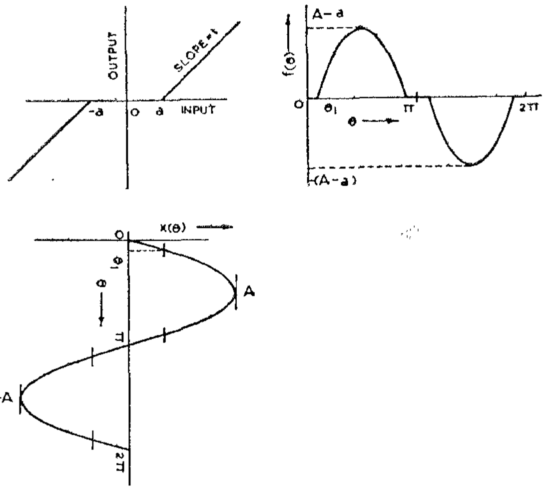

FIG. IV
Output waveform for deadband

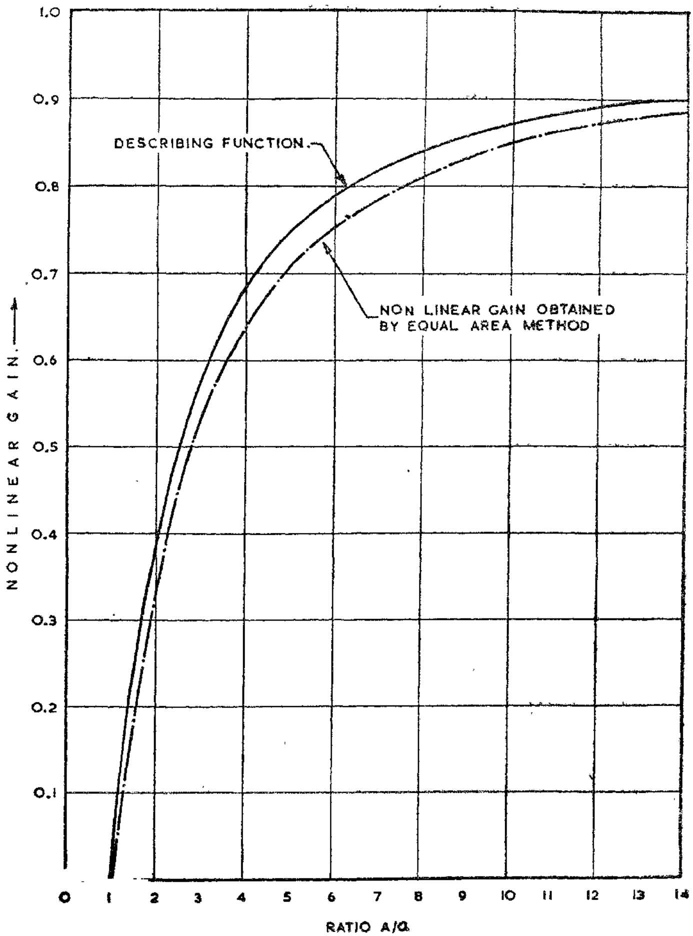

FIG. V
Transfer function curves for deadband

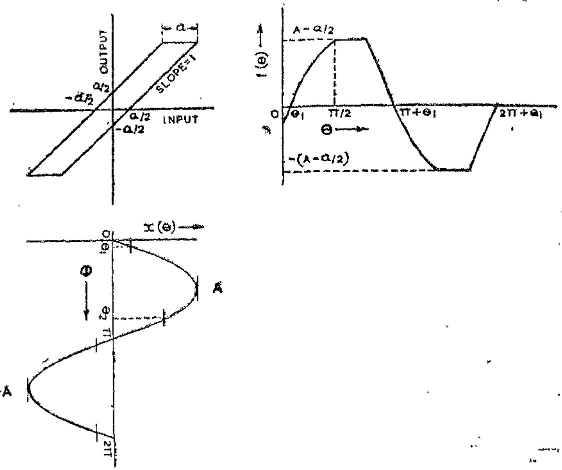

FIG. VI
Output waveform for backlash

(*c*) *Backlash* :—Consider a regular backlash characteristic as shown in Figure VI. Taking care of the units, the phase shift θ in the describing function is given by

$$\theta = \tan^{-1}\frac{a_1}{b_1} = \tan^{-1}\frac{1}{\pi A} \times \frac{\text{Area enclosed by the nonlinear characteristic}}{\text{Amplitude of the equivalent sinusoid } b_1}$$

from which the cosine component a_1 can be calculated. The amplitude of the equivalent sinusoid is obtained by considering the area in the loop from $\theta = \theta_1$ to $\theta = \pi + \theta_1$ since it is again a case of symmetrical nonlinearity.

$$\therefore \quad 2K = \int_{\theta_1}^{\pi/2} A \sin\theta \, d\theta - \frac{a}{2}\left(\frac{\pi}{2} - \theta_1\right) + \left(\theta_2 - \frac{\pi}{2}\right)\left(A - \frac{a}{2}\right)$$

$$+ \int_{\theta_2}^{\pi} (a + A \sin\theta) \, d\theta - \frac{a}{2}\left(\frac{\pi}{2} - \theta_2\right) + \frac{a\theta}{2}\,\theta_1 - \int_{0}^{\theta_1} A \sin\theta \, d\theta$$

Where $\theta_2 = \sin^{-1}\left(1 - \frac{a}{A}\right)$; and $\theta_1 = \sin^{-1}\frac{a}{2A}$

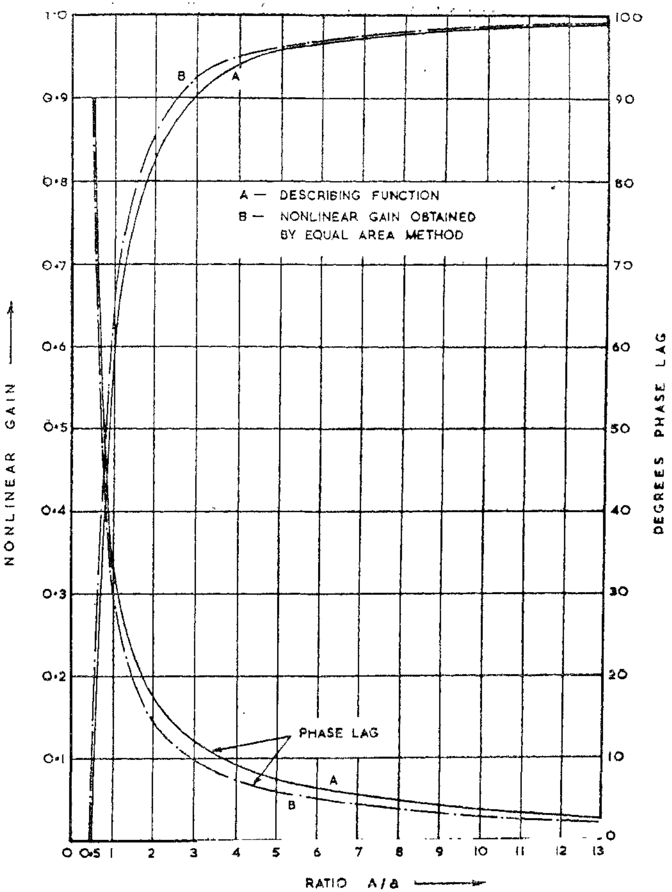

FIG. VII
Transfer function curves for backash

The sine component of the describing function by the equal area method is

$$\frac{K}{A} = \cos\theta_1 + \frac{\cos\theta_2}{2} + \left(\frac{\pi}{2} + \theta_1 - \theta_2\right)\sin\theta_1 + \frac{\theta_2 - \pi/2}{2}$$

which is also obtained by finding the ratios of the areas graphically. Fig. VII shows the accuracy of the method when compared to the analytical Fourier Method. The two curves are almost coinciding because the harmonics are of negligible amplitude.

(d) In the case of a cubic approximated nonlinearity, with prodominant third harmonic, the error in the describing function is 10 per cent. But this corresponds to an error of only 5.6 per cent in the amplitude of limit cycle, if any exists. To illustrate this, consider an open loop transfer function $G(s) = 10/s(1+s)(0.5s+1)$ together with a cubic nonlinearity. The Nyquist plot shows the existence of a limit cycle with an amplitude of 0.718 and $\omega = \sqrt{2}$ while the equal area method gives the amplitude of oscillation to be 0.756 and $\omega = \sqrt{2}$, an error of 5.3 per cent. Further, when the Nyquist locus is modified to take into account the predominant third harmonic, we get the *Tsypkin* locus giving the amplitude of the limit cycle as 0.745 thus differing by only 1.5 percent from that obtained by the equal area method.

(e) *Asymmetrical nonlinearity* :—Consider the asymmetrical nonlinearity shown in Fig. VIII, in which the characteristic of the nonlinearity is different for positive and negative values of the input. We have

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

where
$$f(\theta) = A\sin\theta \qquad \pi \geqq \theta \geqq 0$$
$$= \alpha A \sin\theta \quad 2\pi \geqq \theta \geqq \pi$$

The following Fourier coefficients are obtained :

$$a_0 = A(1+\alpha)/\pi,\ a_1, b_2, b_3, \cdots \text{ are all zero}$$

$$b_1 = \frac{A(1+\alpha)}{2},\ a_2 = \frac{-2A(1+\alpha)}{3\pi},\ a_4 = \frac{-2}{15\pi}A(1+\alpha), \cdots$$

The describing function by conventional method

$$= A(1+\alpha)/2A = (1+\alpha)/2$$

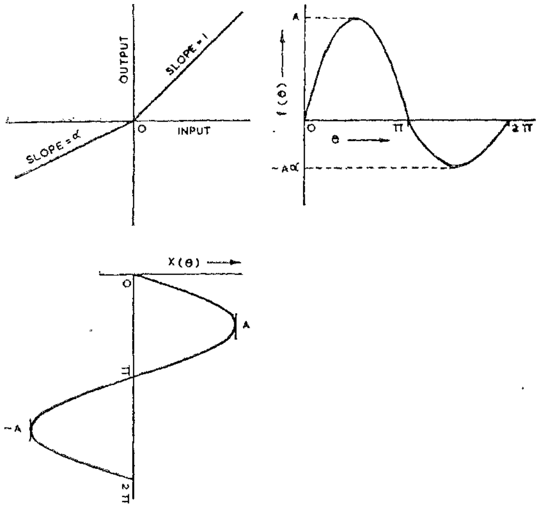

FIG. VIII
Output waveform for an asymmetrical nonlinearity

The D.F. by the equal area method is obtained by considering the total area in both loops of the output. Hence

$$2 \times 2K = \int_0^{\pi} A \sin\theta \, d\theta + \int_0^{\pi} LA \sin\theta \, d\theta = 2A\,(1+\alpha)$$

or $K/A = (1+\alpha)/2$ which is same as that obtained by Fourier Analysis. This is because b_3, b_5, b_7 etc. are all zero in equation (7) of the Appendix.

CONCLUSIONS

The describing function for a nonlinearity is obtained graphically as a ratio of the output area to the input area in one period. It is concluded that the values of the describing function for an actual nonlinear characteristic evaluated by the equal area method will equal that obtained by conventional methods with negligible approximation. The method has been extended satisfactorily to evaluate describing functions for successive nonlinearities.[4] The error introduced by the use of this method is in the values of the amplitudes of limit cycles (if they exist) to a maximum of 8 per cent and negligible (if at all it affects) in the frequencies of limit cycles. In most cases, from the shape of the output waveform it is possible to know the presence of odd sine harmonics and whether they are positive or negative. This gives a rough indication of the error of the equal area method when compared with the conventional method.

APPENDIX

Mathematical Basis for the Method

The output waveform $f(\theta)$ obtained from a nonlinear element can be represented in the interval $0-2\pi$ by Fourier Series of the form

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \qquad [1]$$

This represents a general case of asymmetrical nonlinearity. To find the amplitude K of the equivalent sinusoid, we are to equate the total area in one period of the output to the total area in one period of the equivalent sinusoid which is $4K$ units. As we are only interested in areas, the phase shift has no effect and we can consider

$$\int_{\theta_1}^{2\pi+\theta_1} f(\theta)\, d\theta = \int_0^{2\pi} f(\theta)\, d\theta \qquad [2]$$

$f(\theta)$ is negative in the interval $\pi+\theta_1$ to $2\pi+\theta_1$ and because we are interested in the arithmetic sum of areas, we interpret

$$\int_0^{2\pi} f(\theta)\, d\theta = \int_0^{\pi} f(\theta)\, d\theta - \int_{\pi}^{2\pi} f(\theta)\, d\theta \qquad [3]$$

Hence

$$\begin{aligned} 4K &= \int_0^{\pi} f(\theta)\, d\theta - \int_{\pi}^{2\pi} f(\theta)\, d\theta \\ &= \int_0^{\pi} a_0\, d\theta - \int_{\pi}^{2\pi} a_0\, d\theta + \int_0^{\pi} a_1 \cos\theta\, d\theta - \int_{\pi}^{2\pi} a_1 \cos\theta\, d\theta \\ &\quad + \int_0^{\pi} a_2 \cos 2\theta\, d\theta - \int_{\pi}^{2\pi} a_2 \cos 2\theta\, d\theta + \int_0^{\pi} a_3 \cos 3\theta\, d\theta - \int_{\pi}^{2\pi} a_3 \cos 3\theta\, d\theta + \cdots \end{aligned}$$

$$+\int_0^{\pi} b_1 \sin d\theta - \int_{\pi}^{2\pi} b_1 \sin\theta\, d\theta + \int_0^{\pi} b_2 \sin 2\theta\, d\theta - \int_{\pi}^{2\pi} b_2 \sin 2\theta\, d\theta$$

$$+\int_0^{\pi} b_3 \sin 3\theta\, d\theta - \int_{\pi}^{2\pi} b_3 \sin 3\theta\, d\theta + \cdots \tag{4}$$

$$= 4b_1 + \frac{4b_3}{3} + \frac{4b_5}{5} \tag{5}$$

or $$K = b_1 + \frac{b_3}{3} + \frac{b_5}{5} \tag{6}$$

and the describing function by the equal area method is given by

$$\frac{K}{A} = \frac{b_1 + \frac{b_3}{3} + \frac{b_5}{5} + \cdots}{A} \tag{7}$$

If $f(\theta)$ is antisymmetrical about the origin as for symmetrical nonlinearities, then a_0 will be absent and the integration can be done over the range 0 to π and doubled. Thus

$$2 \times 2K = 2\int_0^{\pi} f(\theta)\, d\theta$$

giving $$K = b_1 + \frac{b_3}{3} + \frac{b_5}{5} + \cdots \tag{8}$$

which is same as equation [7].

In both cases, it is of interest to note that the amplitude of the equivalent sinusoid is equal to the fundamental amplitude plus or minus (depending on the signs of b_3, b_5, etc.) a term accounting for harmonics. In the usual describing function $K = b_1$, as b_3, b_5, etc., are filtered off. Hence we are justified in in saying that the equal area method takes into account a term for harmonics and will approximate to the conventional describing function only if the harmonics are small.

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