THE SECONDARY FLOWS INDUCED IN A NON-NEWTONIAN FLUID BETWEEN TWO PARALLEL INFINITE OSCILLATING PLANES

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Abstract

The flow of a non-Newtonian fluid between two parallel infinite planes, separated by a distance d, is investigated in the two specific cases: (i) one plane held at rest and the other oscillating with a large frequency n, (ii) both the planes oscillating with the same frequency n and the same angular amplitude Ω but in the opposite sense. Two parameter family of solutions depending on the Reynolds number $R=(nd^2\rho/\Phi_1)$, and the cross-viscosity parameter $S=(\Phi_3/\rho d^2)$ and the visco-elastic parameter $K=(\Phi_2/\rho d_2)$ are obtained correct to the order of Ω/n. The steady component of the secondary flow is discussed for both small and large Reynolds numbers. The breaking of the steady component of the secondary flow for critical values of S or K depending on R and then its reversal for higher values is a characteristic feature of the non-Newtonian fluids. Finally a suggestion is made about some experiments which can enable the measurement of the visco-elasticity or the cross-viscosity of a fluid.

1. Introduction

Recently Rosenblat[1] has investigated the problem of flow of a Newtonian fluid between two parallel, infinite oscillating planes. He has pointed out the interesting fact that the rotating planes behave like centrifugal pumps throwing the fluid near them away from the axis of rotation and this motion in turn induces a radial axial flow. In recent communications the present authors[2,3] have studied the same problem respectively for an electrically conducting Newtonian fluid in the presence of a uniform axial magnetic field and for a Reiner-Rivlin[4] visco-inelastic fluid for which the constitutive equation is of the following form

$$T = -PI + \Phi_1 A + \Phi_3 A^2, \qquad [1.1]$$

Φ_1 and Φ_3, the coefficients of viscosity and cross-viscosity being assumed constant and A being as usual the rate of strain tensor:

$$A_{ij} = u_{i,j} + u_{j,i}. \qquad [1.2]$$

In the present paper we consider the same problem for the Rivilin-Ericksen fluid for which the constitutive equation is

$$T = -pI + \Phi_1 A + \Phi_2 B + \Phi_3 A^2, \qquad [1.3]$$

where $$B_{ij} = a_{i,j} + a_{j,i} + v_{m,i}\, v^m{}_{,j} + v_{m,j}\, v^m{}_{,i} \qquad [1.4]$$

a being the acceleration.

The fluid [1.1] is a particular case of this general fluid which takes account of visco-elasticity through the Φ_2-term. The main aim of the present discussion is to indicate another interesting phenomenon, namely that when the magnitudes of the Reynolds number R and the dimensionless parameters S and K representing the effects of cross-viscosity and visco-elasticity respectively are suitably adjusted by altering the distance between the planes and the frequency of the oscillating planes, the radial-axial flow breaks down and then its sense is reversed. For small values of S and K, the fluids behave like the Newtonian fluid and possess radial-axial flows similar to the one predicted by Rosenblat. There exist critical values of S and K depending on the Reynolds number R above which the flow reverses in direction. In fact, for a prescribed value of S (or K) R can always be found for which the flow field divides into sub-fields having distinct flow characteristics. For example, in the case when only one plane oscillates, there are two sub-fields. In the sub-field nearer to the oscillating plane the fluid is drawn inwards and thrown out at some height; in the sub-field nearer the stationary plane again the flow is drawn inwards and thrown out at the same hieght. In the case when both the planes are oscillating with the same frequency the flow field originally consists of two sub-fields symmetric about the plane $y=0.5$, in the scale on which the distance between the planes is 1. For a prescribed value of S (or K) we can determine a value of R for which each of the sub-fields are further broken into two domains. In these domains the fluid is drawn inwards near the oscillating planes and also the plane $y=0.5$ and thrown out at some height in between them.

2. EQUATIONS OF THE PROBLEM

We shall work through the cylindrical coordinates $(\bar{r}, \theta, z)$ with the z-axis being taken along the axis of vibration. We shall render the physical quantities dimensionless according to the following scheme:

Physical Quantity	Standard Magnitude	Dimensionless quantity
1. Linear distance $\bar{r}, z$	d, distance between the planes	r, y
2. Time, T	$1/n$, reciprocal of the frequency of vibration.	t
3. Radial and axial velocity components U, W	$d\Omega^2/n$, Ω being the angular amplitude of the vibration of a plane.	u, w
4. Azimuthal component of velocity, V	$d\Omega$	v

Setting

$$p=\rho\, d^2\, \Omega^2 P, \quad \Phi_1 = n\, d^2 \rho\, R^{-1}, \quad \Phi_2 = \rho\, d^2 K, \quad \Phi_2 = \rho\, d^2 S, \qquad [2.1]$$

the equations of the problem reduce to the following forms:

Continuity equation:

$$\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\,u)+\frac{\partial w}{\partial y}=0. \qquad [2.2]$$

Momentum equations:

$$u_t+(\Omega/n)^2\,[u\,u_r+w\,u_y]-v^2/r$$
$$= -P_r+(1/R)\,[\Delta\, u-u/r^2]+(K K_1+S S_1)+(\Omega/n)^2\,(K K_1'+S S_1'), \qquad [2.3]$$

$$v_t+(\Omega/n)^2\,[u\,v_r+w\,v_y+u\,v/r]$$
$$=(1/R)[\Delta\, v-v/r^2]+K K_2+(\Omega/n)^2\,[K K_2'+S S_2'], \qquad [2.4]$$

and

$$w_t+(\Omega/n)^2\,[u\,w_r+w\,w_y]$$
$$= -P_y+(1/R)\,\Delta w+[K K_3+S S_3]+(\Omega/n)^2\,[K K_3'+S S_3'], \qquad [2.5]$$

where

$$K_1=(\Delta u-u/r^2)_t-2\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial}{\partial r}-\frac{1}{r^2}+\frac{1}{2}\frac{\partial^2}{\partial y^2}\right)(v^2/r)$$
$$+\;2\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)v_r^2+2\,(v_r\,v_y)_y-(2/r^3)\,v^2, \qquad [2.6]$$

$$K_1'=2\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial}{\partial r}-\frac{1}{r^2}+\frac{1}{2}\frac{\partial^2}{\partial y^2}\right)(u\,u_r+w\,u_y)-(2/r^3)\,u^2$$
$$+\;2\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)(u_r^2+w_r^2)+(u\,w_r+w\,w_y)_{ry}+2\,(u_r\,u_y+w_r\,w_y)_y, \qquad [2.7]$$

$$K_2=(\Delta v-v/r^2)_t, \qquad [2.8]$$

$$K_2'=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial}{\partial r}-\frac{1}{r^2}+\frac{\partial^2}{\partial y^2}\right)(u\,v_r+w\,v_y+u\,v/r)$$
$$+\;2\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)(1/r)\,(u\,v_r-v\,u_r)-(2/r)\,(v\,u_y-u\,v_y)_y, \qquad [2.9]$$

$$K_3 = (\Delta w)_t - \left(\frac{\partial^2}{\partial r\,\partial y} + \frac{1}{r}\cdot\frac{\partial}{\partial_y}\right)\left(\frac{v^2}{r}\right) + 2\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)(v_r\, v_y) + 2\,(v_y^2)_y, \qquad [2.10]$$

$$K_3' = \left(\frac{\partial^2}{\partial r\,\partial y} + \frac{1}{r}\cdot\frac{\partial}{\partial y}\right)(u\,u_r + w\,u_y) + \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + 2\frac{\partial^2}{\partial y^2}\right)(u\,w_r + w\,w_y)$$

$$+ 2\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)(u_r\,u_y + w_r\,w_y) + 2\,(u_y^2 + w_y^2)_y, \qquad [2.11]$$

$$S_1 = \left[\,2\,(v_r - v/r)\frac{\partial}{\partial r} + v_y\frac{\partial}{\partial y} + v_{yy}\right][v_r - v/r] - (1/r)\,v_y^2, \qquad [2.12]$$

$$S_1' = 4\,u_r\,u_{rr} + (u_y + w_r)(u_{ry} + w_{rr}) - (u/r)\,w_{ry} - (u/r)\,u_{yy}$$

$$+ (1/2\,r)\,(w_r^2 - u_y^2) + (2/r)\,u_r^2 - (2/r^3)\,u^2, \qquad [2.13]$$

$$S_2' = \left[2\,v_{ry} + v_y\frac{\partial}{\partial r} + \left(v_r - \frac{v}{r}\right)\frac{\partial}{\partial y} + \frac{v_y}{r}\right](u_y + w_r)$$

$$+ 2\left[w_y\frac{\partial}{\partial r} - w_{ry} - \frac{2}{r}\,w_y\right]\left[v_r - \frac{v}{r}\right] + 2\,u_{ry}\,v_y - 2\,u_r\,v_{yy}, \qquad [2.14]$$

$$S_3 = v_y\,(2v_{yy} + v_{rr}) + v_{ry}\,(v_r - v/r), \qquad [2.15]$$

$$S_3' = 4\,w_y\,w_{yy} - (u/r)\,u_{ry} + (u_y + w_r)(u_{yy} + w_{ry})$$

$$- (u/r)\,w_{rr} - (1/r)\,u_r\,u_y - (1/r)\,u_r\,w_r, \qquad [2.16]$$

where a suffix ξ denotes partial differentiation with respect to ξ and

$$\Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{\partial^2}{\partial y^2}.$$

We choose the following expressions for the velocity and pressure so that the continuity equation is automatically satisfied and the momentum equations are reduced to the forms in which the radial distance r does not occur:

$$u = r\,F_y(y, t)$$

$$v = Rl\,[r\,e^{it}\,G(y)]$$

$$w = -2\,F(y, t)$$

and

$$P = (r^2/2)\,\bar{p}\,(y, t) + 2\,q\,(y, t). \qquad [2.17]$$

Even with this assumption the resultant equations are too complicated to yield to an analytical treatment and hence following Rosenblat [loc. cit] we assume that the frequency of the vibration of the planes is so large that we can neglect terms of the order of $(\Omega/n)^2$. With this assumption the equations reduce to

$$F_{yt} - G^2 \exp.(2it) = -\bar{p} + (1/R) F_{yyy} + K F_{yyyt} - S G_y^2 \exp.(2it), \quad [2.18]$$

$$iG = (1/R) G_{yy} + iK G_{yy}, \quad [2.19]$$

$$F_t = q_y + (1/R) F_{yy} + K F_{yyt}, \quad [2.20]$$

and

$$\bar{p}_y = 4(2K + S) G_y G_{yy} \exp.(2it). \quad [2.21]$$

It is clear from [2.19] that cross-viscosity does not affect the azimuthal velocity to the present approximation.

Boundary Conditions :—We have the following boundary conditions in terms of the physical variables:

$$U = W = 0, \ V = Rl(\bar{r}\,\Omega\, e^{inT}) \text{ at } z = 0$$

and at $z = d$, $\quad U = W = 0$;

$V = 0$, when the upper plane is fixed [Case (i)]

$= Rl(\Omega\,\bar{r}\, e^{-inT})$, when the upper plane is vibrating in the opposite sense [Case (ii)].

In terms of our new variables these conditions reduce to :

at $y = 0$, $\quad F = F_y = 0, \ G = 1$

and at $y = 1$, $\quad F = F_y = 0,$

and $\quad G = 0$ (Case i) $\quad G = -1$ (Case ii). [2.22]

3. Solution of the Equations

The solution of [2.19] for a visco-elastic fluid is given by

$$G(y) = \frac{\text{sh}\,\xi(1-y)}{\text{sh}\,\xi} \quad \text{(Case i)} \quad [3.1]^*$$

$$= \frac{\text{sh}\,\xi(1-y) - \text{sh}\,\xi y}{\text{sh}\,\xi} \quad \text{(Case ii)} \quad [3.2]$$

* ch θ = cosh θ ; sh θ = sinh θ.

where

$$\xi^2 = \frac{R(RK+i)}{R^2K^2+1}. \qquad [3.3]$$

For a visco-inelastic Reiner-Rivlin fluid the azimuthal velocity $G(y)$ is the same as for the Newtonian fluid, namely that given by [3.1] and [3.2] with

$$\xi^2 = Ri. \qquad [3.4]$$

Consequently, it is advisable at this stage to separate the considerations of visco-inelastic and visco-elastic fluids. From the form of $G(y)$ it is clear that we must take

$$F(y,t) = f(y) + h(y) \exp.(2it)$$
$$q(y,t) = q_0(y) + q_1(y) \exp.(2it) \qquad [3.5]$$

(*a*) *Visco-Inelastic fluids*:—In the case (i) the transverse component of the velocity and the function $f(y)$ and $h(y)$ determining the radial-axial flow are:

$$v = r[\text{ch}\,\lambda_1 - \cos\lambda_1]^{-1}[\{\cos(\tfrac{1}{2}\lambda_1 y)\,\text{ch}(\tfrac{1}{2}\lambda_1(2-y)) - \text{ch}(\tfrac{1}{2}\lambda_1 y)\cos(\tfrac{1}{2}\lambda_1(2-y))\}\cos t + \{\sin(\tfrac{1}{2}\lambda_1 y)\,\text{sh}(\tfrac{1}{2}\lambda_1(2-y)) - \sin(\tfrac{1}{2}\lambda_1(2-y))\,\text{sh}(\tfrac{1}{2}\lambda_1 y)\}\sin t], \qquad [3.6]$$

and

$$f(y) = \tfrac{1}{4}[\text{ch}\lambda_1 - \cos\lambda_1]^{-1}[\{(1/\lambda_1)\langle\text{sh}(\lambda_1(1-y)) + \sin(\lambda_1(1-y))\rangle + y(1-y)^2(\text{ch}\,\lambda_1 + \cos\lambda_1) - (1/\lambda_1)(1-3y^2+2y^3)(\text{sh}\,\lambda_1 + \sin\lambda_1) - 2y^2(1-y)\} + (\tfrac{3}{2}S\lambda_1)\{(2y^3-3y^2+1)(\text{sh}\,\lambda_1 - \sin\lambda_1) - \langle\text{sh}(\lambda_1(1-y)) - \sin(\lambda_1(1-y))\rangle\}] - (\tfrac{3}{8}S\lambda_1^2)\,y(1-y)^2, \qquad [3.7]$$

with $h(y)$ as given in Ref. 3 [equation 2.30],

and $$\lambda_1^2 = 2R. \qquad [3.8]$$

The stream function for the secondary flow is given by

$$\psi = r^2 f(y). \qquad [3.9]$$

In the case (ii) the corresponding solutions are:

$$v = r[\text{ch}\,\lambda_1 - \cos\lambda_1]^{-1}\langle\{\text{ch}[\tfrac{1}{2}\lambda_1(2-y)]\cos(\tfrac{1}{2}\lambda_1 y) - \text{ch}(\tfrac{1}{2}\lambda_1 y)\cos[\tfrac{1}{2}\lambda_1(2-y)] + \text{ch}[\tfrac{1}{2}\lambda_1(1-y)]\cos[\tfrac{1}{2}\lambda_1(1+y)] - \text{ch}[\tfrac{1}{2}\lambda_1(1+y)]\cos[\tfrac{1}{2}\lambda_1(1-y)]\}\cos t + \{\text{sh}[\tfrac{1}{2}\lambda_1(2-y)]\sin\tfrac{1}{2}(\lambda_1 y) - \text{sh}(\tfrac{1}{2}\lambda_1 y)\sin[\tfrac{1}{2}\lambda_1(2-y)] + \text{sh}[\tfrac{1}{2}\lambda_1(1-y)]\sin[\tfrac{1}{2}\lambda_1(1+y)] - \text{sh}[\tfrac{1}{2}\lambda_1(1+y)]\sin[\tfrac{1}{2}\lambda_1(1-y)]\}\sin t\rangle, \qquad [3.10]$$

and

$$f(y) = -\tfrac{1}{2}\left[\frac{\operatorname{ch}\frac{1}{2}\lambda_1 + \cos\frac{1}{2}\lambda_1}{\operatorname{ch}\lambda_1 - \cos\lambda_1}\right]\left\{\left(\frac{3S\lambda_1^2}{2} - 1\right)[(3y^2 - 2y^3)\times\right.$$
$$\{(2/\lambda_1)\operatorname{sh}\tfrac{1}{2}\lambda_1 - \operatorname{ch}\tfrac{1}{2}\lambda_1\} + (1/\lambda_1)\operatorname{sh}\tfrac{1}{2}\lambda_1(1-2y) + y\operatorname{ch}\tfrac{1}{2}\lambda_1 - (1/\lambda_1)\operatorname{sh}\tfrac{1}{2}\lambda_1]$$
$$+[(3S\lambda_1^2/2)+1)][(\cos\tfrac{1}{2}\lambda_1 - (2/\lambda_1)\sin\tfrac{1}{2}\lambda_1)(3y^2 - 2y^3) - y\cos\tfrac{1}{2}\lambda_1$$
$$\left.-(1/\lambda_1)\sin\tfrac{1}{2}\lambda_1(1-2y) + (1/\lambda_1)\sin\tfrac{1}{2}\lambda_1]\right\}, \qquad [3.11]$$

with $h(y)$ as given in Ref. 3 [equation 3.11].

Here again λ_1 is given by [3.8] and the stream function by [3.9] with $f(y)$ as defined in [3.11].

(*b*) *Visco-elastic fluids*:—We note below the corresponding solutions for a visco-elastic fluid in the two cases.

Case (i)

$$v = r[\operatorname{ch}(\lambda_2\cos\mu) - \cos(\lambda_2\sin\mu)]^{-1}\langle\{\operatorname{ch}(\tfrac{1}{2}\lambda_2(2-y)\cos\mu)\times$$
$$\cos(\tfrac{1}{2}\lambda_2 y\sin\mu) - \operatorname{ch}(\tfrac{1}{2}\lambda_2 y\sin\mu)\cos[\tfrac{1}{2}\lambda_2(2-y)\sin\mu]\}\cos t$$
$$-\{\operatorname{sh}(\tfrac{1}{2}\lambda_2 y\cos\mu)\sin[\tfrac{1}{2}\lambda_2(2-y)\sin\mu]$$
$$-\operatorname{sh}[\tfrac{1}{2}\lambda_2(2-y)\cos\mu]\sin(\tfrac{1}{2}\lambda_2 y\sin\mu)\}\sin t\rangle, \qquad [3.12]$$

$$(2/R)[\operatorname{ch}(\lambda_2\cos\mu) - \cos(\lambda_2\sin\mu)]f(y)$$
$$= (y^3 - 2y^2 + y)\left[(\lambda_2^2K+1)\frac{\cos(\lambda_2\sin\mu)}{\lambda_2^2\sin^2\mu} - (\lambda_2^2K-1)\frac{\operatorname{ch}(\lambda_2\cos\mu)}{\lambda_2^2\cos^2\mu}\right]$$
$$+(2y^3 - 3y^2 + 1)\left[(\lambda_2^2K-1)\frac{\operatorname{sh}(\lambda_2\cos\mu)}{\lambda_2^3\cos^3\mu} - (\lambda_2^2K+1)\frac{\operatorname{sh}(\lambda_2\sin\mu)}{\lambda_2^3\sin^3\mu}\right]$$
$$+y^2(y-1)\left[\frac{(\lambda_2^2K+1)}{\lambda_2^2\sin^2\mu} - \frac{(\lambda_2^2K-1)}{\lambda_2^2\cos^2\mu}\right]$$
$$+\left[(\lambda_2^2K+1)\frac{\operatorname{sh}[\lambda_2(1-y)\sin\mu]}{\lambda_2^3\sin^3\mu} - (\lambda_2^2K-1)\frac{\operatorname{sh}[\lambda_2(1-y)\cos\mu]}{\lambda_2^3\cos^3\mu}\right] \qquad [3.13]$$

and

$$h(y) = (1/m)[Ae^{my} - Be^{-my}] + L_1 y/(2i) + M$$
$$-\frac{\lambda_2^2Ke^{2i\mu}+1}{4i[\operatorname{ch}(\lambda_2e^{i\mu})-1]} - \frac{W}{4i}\cdot\frac{\operatorname{sh}[\lambda_2(1-y)e^{i\mu}]}{\lambda_2e^{i\mu}[\operatorname{ch}(\lambda_2e^{i\mu})-1]}, \qquad [3.14]$$

where A, B, L_1 and M are given by the equations :

$$2A\left[\operatorname{ch}(\lambda_2 e^{i\mu})-1\right] = -(L_1/2i+mM)\left[\operatorname{ch}(\lambda_2 e^{i\mu})-1\right] - \frac{W}{4i}\operatorname{ch}(\lambda_2 e^{i\mu}) + \frac{mW}{4i}\cdot\frac{\operatorname{sh}(\lambda_2 e^{i\mu})}{\lambda_2 e^{i\mu}} + \frac{\lambda_2^2 K e^{2i\mu}+1}{4i}, \quad [3.15]$$

$$2B\left[\operatorname{ch}(\lambda_2 e^{i\mu})-1\right] = (-L_1/2i+mM)\left[\operatorname{ch}(\lambda_2 e^{i\mu})-1\right] - \frac{W}{4i}\operatorname{ch}(\lambda_2 e^{i\mu}) - \frac{mW}{4i}\cdot\frac{\operatorname{sh}(\lambda_2 e^{i\mu})}{\lambda_2 e^{i\mu}} + \frac{\lambda_2^2 K e^{2i\mu}+1}{4i}, \quad [3.16]$$

$$\left[\operatorname{ch}(\lambda_2 e^{i\mu})-1\right]\left[2-2\operatorname{ch} m+m\operatorname{sh} m\right]L_1 = \frac{\lambda_2^2 K e^{2i\mu}+1}{2}\left[2-2\operatorname{ch} m+m\operatorname{sh} m\right] - \frac{W}{2}(1-\operatorname{ch} m) \times\left[1+\operatorname{ch}(\lambda_2 e^{i\mu})\right] - \frac{m\,W\operatorname{sh} m}{2\,\lambda_2 e^{i\mu}}\operatorname{sh}(\lambda_2 e^{i\mu}), \quad [3.17]$$

and

$$\left[\operatorname{ch}(\lambda_2 e^{i\mu})-1\right]\left[2-2\operatorname{ch} m+m\operatorname{sh} m\right] m M i = (W/4)\operatorname{ch}(\lambda_2 e^{i\mu})\left[\operatorname{sh} m-m\operatorname{ch} m\right] - (W/4)(\operatorname{sh} m-m) - (mW/4)\left[\operatorname{ch} m-1-m\operatorname{sh} m\right]\frac{\operatorname{sh}(\lambda_2 e^{i\mu})}{\lambda_2 e^{i\mu}}, \quad [3.18]$$

with

$$\lambda_2^2 = \frac{4R}{(R^2K^2+1)^{1/2}}; \quad \mu = \tfrac{1}{2}\tan^{-1}(1/RK), \quad m^2 = \frac{2R(2RK+i)}{4R^2K^2+1}$$

and

$$W = 1 + \frac{2(2iRK-1)e^{2i\mu}}{8K^2R^2e^{2i\mu} - (1+K^2R^2)^{1/2}(i+2KR) + 2e^{2i\mu}}. \quad [3\ 19]$$

Case (ii)

$$\begin{aligned} v = r \left\langle \operatorname{ch}[\lambda_2\cos\mu] - \cos[\lambda_2\sin\mu]\right\rangle^{-1} & \left\langle \left\{\operatorname{ch}\left[\tfrac{1}{2}\lambda_1(2-y)\cos\mu\right]\times\right.\right. \\ & \times\cos\left[\tfrac{1}{2}\lambda_2 y\sin\mu\right] - \operatorname{ch}\left[\tfrac{1}{2}\lambda_2 y\cos\mu\right]\cos\left[\tfrac{1}{2}\lambda_2(2-y)\sin\mu\right] \\ & - \operatorname{ch}\left[\tfrac{1}{2}\lambda_2(1+y)\cos\mu\right]\cos\left[\tfrac{1}{2}\lambda_2(1-y)\sin\mu\right] + \operatorname{ch}\left[\tfrac{1}{2}\lambda_2(1-y)\cos\mu\right]\times \\ & \times\cos\left[\tfrac{1}{2}\lambda_2(1+y)\sin\mu\right]\right\}\cos t - \left\{\operatorname{sh}\left[\tfrac{1}{2}\lambda_2 y\cos\mu\right)\sin\left[\tfrac{1}{2}\lambda_2(2-y)\sin\mu\right] \\ & - \operatorname{sh}\left[\tfrac{1}{2}\lambda_2(2-y)\cos\mu\right]\sin\left[\tfrac{1}{2}\lambda_2 y\sin\mu\right] - \operatorname{sh}\left[\tfrac{1}{2}\lambda_2(1-y)\cos\mu\right]\times \\ & \times\sin\left[\tfrac{1}{2}\lambda_2(1+y)\sin\mu\right] - \operatorname{sh}\left[\tfrac{1}{2}\lambda_2(1+y)\cos\mu\right]\times \\ & \left.\times\sin\left[\tfrac{1}{2}\lambda_2(1-y)\sin\mu\right)\right\}\sin t\right\rangle, \end{aligned} \quad [3.20]$$

$$\tfrac{1}{4}f(y)=(4y^3-6y^2+1)\left[(\lambda_2^2K-1)\frac{\operatorname{sh}(\frac{1}{2}\lambda_2\cos\mu)}{\lambda_2^3\cos^3\mu}-(\lambda_2^2K+1)\frac{\operatorname{sh}(\frac{1}{2}\lambda_2\sin\mu)}{\lambda_2^3\sin^3\mu}\right]$$
$$+(2y^3-3y^2+y)\left[(\lambda_2^2K+1)\frac{\cos(\frac{1}{2}\lambda_2\sin\mu)}{\lambda_2^2\sin^2\mu}-(\lambda_2^2K-1)\frac{\operatorname{ch}(\frac{1}{2}\lambda_2\cos\mu)}{\lambda_2^2\cos^2\mu}\right]$$
$$-\left[(\lambda_2^2K-1)\frac{\operatorname{sh}[\frac{1}{2}\lambda_2(1-2y)\cos\mu]}{\lambda_2^3\cos^3\mu}-(\lambda_2^2K+1)\frac{\operatorname{sh}[\frac{1}{2}\lambda_2(1-2y)\sin\mu]}{\lambda_2^3\sin^3\mu}\right], \quad [3.21]$$

and

$$\frac{\operatorname{ch}(\lambda_2e^{i\mu})-1}{\operatorname{ch}(\frac{1}{2}\lambda_2e^{i\mu})+1}\,h(y)$$
$$=(1/m)\left[A\,e^{my}-B\,e^{-my}\right]+\frac{L_1}{2i}\left[\frac{\operatorname{ch}(\lambda_2e^{i\mu})-1}{\operatorname{ch}(\frac{1}{2}\lambda_2e^{i\mu})+1}\right]-\frac{\lambda_2^2Ke^{2i\mu}+1}{2i}\,y$$
$$+M-[W/(2i\lambda_2e^{i\mu})]\cdot\operatorname{sh}[\tfrac{1}{2}\lambda_2(1-2y)\,e^{i\mu}], \quad [3.22]$$

where A, B, L_1 and M are given by the following relations:

$$2A\frac{\operatorname{ch}(\lambda_2e^{i\mu})-1}{\operatorname{ch}(\frac{1}{2}\lambda_2e^{i\mu})+1}=-\left(mM+\frac{L_1}{2i}\right)\frac{\operatorname{ch}(\lambda_2e^{i\mu})-1}{\operatorname{ch}(\frac{1}{2}\lambda_2e^{i\mu})+1}$$
$$+\frac{\lambda_2^2Ke^{2i\mu}+1}{2i}-\frac{W}{2i}\operatorname{ch}(\tfrac{1}{2}\lambda_2e^{i\mu})+\frac{mW}{2i\lambda_2e^{i\mu}}\operatorname{sh}(\tfrac{1}{2}\lambda_2e^{i\mu}), \quad [3.23]$$

$$2B\frac{\operatorname{ch}(\lambda_2e^{i\mu})-1}{\operatorname{ch}(\frac{1}{2}\lambda_2e^{i\mu})+1}=\left(Mm-\frac{L_1}{2i}\right)\frac{\operatorname{ch}(\lambda_2e^{i\mu})-1}{\operatorname{ch}(\frac{1}{2}\lambda_2e^{i\mu})+1}$$
$$-\frac{W}{2i}\operatorname{ch}(\tfrac{1}{2}\lambda_2e^{i\mu})-\frac{mW}{2i\lambda_2e^{i\mu}}\operatorname{sh}(\tfrac{1}{2}\lambda_2e^{i\mu})$$
$$+(1/2i)(\lambda_2^2Ke^{2i\mu}+1), \quad [3.24]$$

$$\frac{\operatorname{ch}(\lambda_2e^{i\mu})-1}{\operatorname{ch}(\frac{1}{2}\lambda_2e^{i\mu})+1}\left[2-2\operatorname{ch}m+m\operatorname{sh}m\right]L_1/2$$
$$=\tfrac{1}{2}(\lambda_2^2Ke^{2i\mu}+1)(2-2\operatorname{ch}m+m\operatorname{sh}m)$$
$$-W\operatorname{ch}(\tfrac{1}{2}\lambda_2e^{i\mu})(1-\operatorname{ch}m)+[mW/(\lambda_2e^{i\mu})]\operatorname{sh}m\operatorname{sh}(\tfrac{1}{2}\lambda_2e^{i\mu}), \quad [3.25]$$

and

$$\frac{\operatorname{ch}(\lambda_2 e^{i\mu}) - 1}{\operatorname{ch}(\frac{1}{2}\lambda_2 e^{i\mu}) + 1}[2 - 2\operatorname{ch} m + m \operatorname{sh} m] M m i$$

$$= \tfrac{1}{2} W m (1 - \operatorname{ch} m)(\operatorname{ch}\tfrac{1}{2}\lambda_2 e^{i\mu}) + [m^2 W/(2\lambda_2 e^{i\mu})] \operatorname{sh} m \operatorname{sh}(\tfrac{1}{2}\lambda_2 e^{i\mu}), \quad [3.26]$$

with the same definitions for λ_2, μ, m and W as in [3.19].

In both these cases the stream function is given by [3.9] with the proper $f(y)$. We may note that in deriving the above solutions we have made no assumptions about the magnitudes of R and S (or K) and they are, therefore, exact except that we have neglected the terms of the order of $(\Omega/n)^2$.

4. DISCUSSION OF THE RESULTS

In the case when R is small, we give below the simplified expressions for the velocity components which we need for discussing the secondary flows: We have checked that the values given by these formulae differ insignificantly from the exact values for $R = 5$ or so.

(*a*) *Visco-inelastic fluids*:—

Case (*i*)

$$v = r(1-y)[\{1 - (R^2/360)\, y(8 + 8y - 12y^2 + 3y^3)\}\cos t$$
$$+ (R/6)\, y(2-y)\sin t] + 0(R^4), \quad [4.1]$$

and

$$f(y) = R y^2(1-y)^2 [(1/120)(3-y) - (8/10\,!)\, R^2 (21 + 133y - 175y^2$$
$$+ 105y^3 - 35y^4 + 5y^5) - (6/7\,!)\, R^2 S (10 - 10y + 5y^2 - y^3)]$$
$$+ 0(R^5). \quad [4.2]$$

Case (*ii*)

$$v = r(1-2y)[\{1 - (R^3/360)\, y(1-y)(1 + 3y - 3y^2)\}\cos t$$
$$+ (1/6) R y (1-y)\sin t] + 0(R^4), \quad [4.3]$$

and

$$f(y) = (1/60) R y^2 (1-y)^2 (1-2y)\{1 - [R^2/(7!\cdot 3)](3 + 20y - 40y^2$$
$$+ 40y^3 - 20y^4) - (1/28) R^2 S(3 - 4y + 4y^2)\} + 0(R^5). \quad [4.4]$$

(*b*) *Visco-elastic fluids* :—

Case (*i*)

$$v = r(1-y)\,[\{1-(R^2/360)\,y(2-y)(4+6y-3y^2+60K)\}\cos t$$
$$+(R/6)\,y(2-y)\{1-(1/90)R^2-(1/30)R^2K(30K+16-6y+3y^2)\}\sin t]$$
$$+0(R^4), \qquad [4.5]$$

and

$$f(y) = (R/120)\,y^2(1-y)^2\,[(3-y)-(R^2/3780)(21+133y-175y^2+105y^3$$
$$-35y^4+5y^5)-(R^2K/21)\{41-27y+10y^2-2y^3+84K(3-y)\}]$$
$$+0(R^5). \qquad [4.6]$$

Case (*ii*)

$$v = r(1-2y)\,[\{1-(R^2/360)\,y(1-y)(1+3y-3y^2+60K)\}\cos t$$
$$+(R/6)\,y(1-y)\{1+R^2/90+(R^2K/30)(19-3y+3y^2-30K)\}\sin t]$$
$$+0(R^4), \qquad [4.7]$$

and

$$f(y) = (R/60)\,y^2(1-y)^2(1-2y)\,[1-(R^2/(7!.3))(3+20y-40y^2$$
$$+40y^3-20y^4)-(R^2K/84)(13-8y+8y^2+336K)]+0(R^5). \qquad [4.8]$$

(*a*) *Visco-inelastic fluids*:—For the visco-inelastic fluid the transverse velocity is the same as for the Newtonian fluids. The graphs for $|v|$ have been drawn by Rosenblat[1]. The typical streamlines for the steady part of the secondary motion and the steady radial volocity have been plotted in Ref. 3. These graps show that for small values of S, the flow resembles that for the Newtonian fluids, but when S is large the sense of the flow is reversed. We have examined the flow characteristics for the intermediate values of S in the present paper.

From [4.2], we find that the axial velocity satisfying the no slip conditions at the planes $y=0$ and $y=1$, vanishes at some height $y=y_0$, $0<y_0<1$ provided R and S are chosen according to the following relation:

$$R^2 = \frac{3780(3-y_0)}{(21+133y_0-175y_0^2+105y_0^3-35y_0^4+5y_0^5)+540S(10-10y_0+5y_0^2-y_0^3)} \qquad [4.9]$$

The dashed lines is Fig. I represent the relation between R and S for $y_0 = 0.1$, 0.5 and 0 9. We have retained only those parts of the curve for which both R and S

are small in view of our approximation. For large values of R we have to consider the exact solution [3.7]. Fig. II shows that for $R=50$, the reversal of the flow takes place between $S=0$ and $S=0.1$. To examine what happens at the critical values of S and R, we have taken a typical pair of values of R and S, $R=5$, $S=0.12$ and drawn the typical streamline $\psi=0.005$ in Fig. III, with the help of the exact formula [3.7]. We find that the flow breaks at $y_0 \simeq 0.38$. The fluid is drawn inwards near the planes more or less parallel to them at large distances from the axis and thrown away approximately at some height $y=0.38$. Complete reversal of the flow takes place for $S=0.15$ as shown in Fig. III.

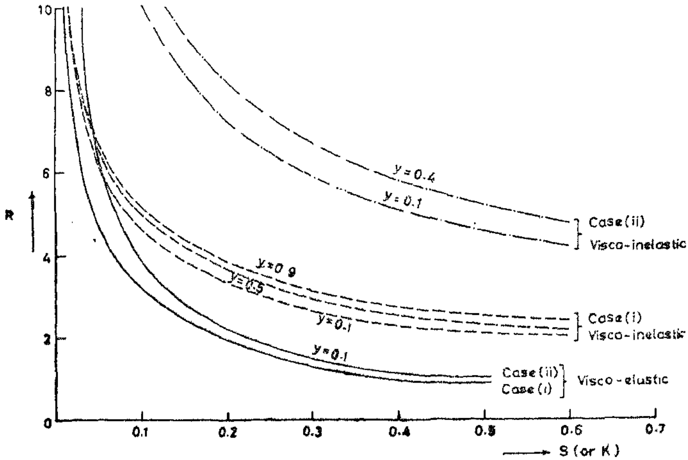

FIG. I

The relation between R and K (or S) for which the breaking of flow takes place at $y=y_0$; the value of y_0 is indicated on each curve

In the case (ii), when both the planes are vibrating the dot-dash curves in Fig. I, represent the relation between R and S for which the flow breaks at heights $y_0=0.1$ and 0.4, which are drawn with the help of the following equation obtained from [4.4]:

$$R^2 = \frac{15120}{(3+20\,y_0-40\,y_0^2+40\,y_0^3-20y_0^4)+540\,S\,(3+4\,y_0+4\,y_0^2)}. \qquad [4.10]$$

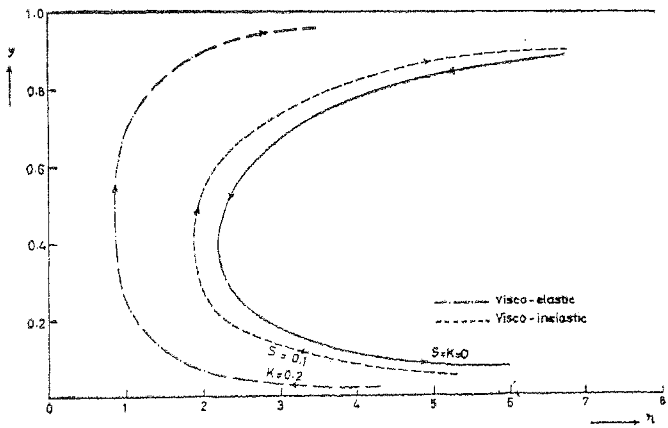

FIG. II
One disc oscillating: For $R=50$ from the exact expressions

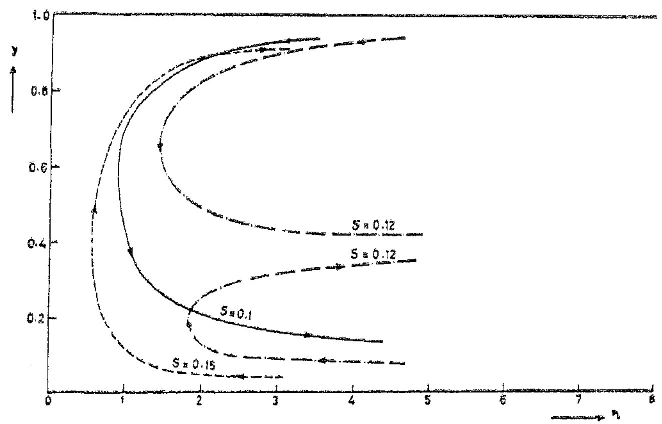

FIG. III
A typical streamline of the steady part of the flow for small Reynolds number 5, calculated from the exact expressions

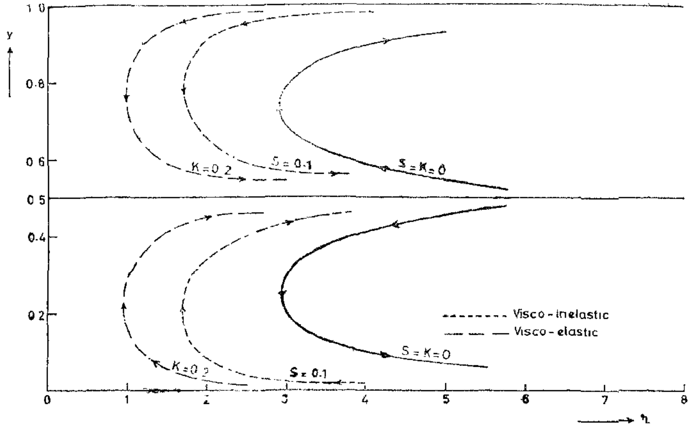

FIG. IV

Two discs oscillating: $R=98$; from the exact expression

The flow between $y=0.5$ and $y=1.0$ can be inferred from the flow between $y=0$ and $y=0.5$ from the considerations of symmetry. The Fig. IV gives the typical streamlines for $R=98$ and $S=0$ and $S=0.1$ which are drawn with the help of the exact solution [3.11]. From the figure it is clear that the reversal of the secondary flow takes place between $S=0$ and $S=0.1$. Once again we have shown in Fig. V the typical streamlines for $R=10$, $S=0.12$. We find that the flow field between $y=0$ and $y=0.5$ is broken into two parts. The fluid near the planes $y=0$ and $y=0.5$ is drawn inwards nearly parallel to the planes at large distances from the axis and thrown away at the height $y_0 \simeq 0.25$. Correspondingly the flow between the planes $y=0.5$ and $y=1.0$ breaks at $y_0 \simeq 0.75$. The total reversal of the flow takes place for $S=0.15$, $R=10$ between $y=0$ and $y=0.5$ and symmetrically between $y=0.5$ and $y=1.0$ as shown by the Fig. V.

(*b*) *Visco-elastic fluids*:—The Fig. VI plots the modulus of the transverse velocity for $K=0.1$ and $K=-0.1$ to assess the effect of visco-elasticity on the azimuthal velocity component of the flow in comparison with the corresponding curve for the Newtonian fluids ($K=0$) for $R=5$ for the cases (i) and (ii). The Fig. VII gives the steady part of the radial velocity for $R=5$ and $K=0.04$ and 0.05 for the case (i) and for $K=0.04$ and 0.1 for the case (ii) to show the reversal of the flow in these cases. For comparison we have drawn the

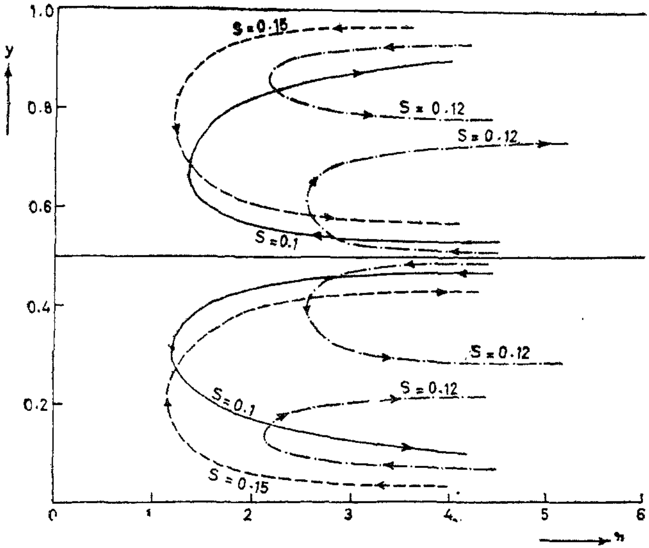

FIG. V

Two Discs Oscillating:
Visco-inelastic [Approximate Expression] $R=10$ showing the flow reversal ($\psi = -0.001$)

corresponding curves for the Newtonian fluid $(K=0)$. We have represented the typical streamline for large Reynolds number $(R=50)$ for case (i) in Fig. II from the exact solution [3.13], showing the reversal of the flow. Similarly for the case (ii) we have drawn in Fig. IV the typical streamline for large Reynolds number $(\mathbf{R}=98)$ from the exact solution [3.21] to show the reversal of the flow.

The Fig. VIII gives the typical streamlines in the case (i) for $R=5$ and $K=0.045, 0.0457, 0.046$, and 0.047. We find that the flow for 0.045 is similar to that for Newtonian fluids, while for 0.047 the sense of the flow is reversed. We have drawn the streamlines for the remaining three values of K to show how the height at which the flow breaks shifts upwards as K increases. From the approximate expression [4.6] we find that R, K and y_0, the height at which the flow breaks are connected by the following relation:

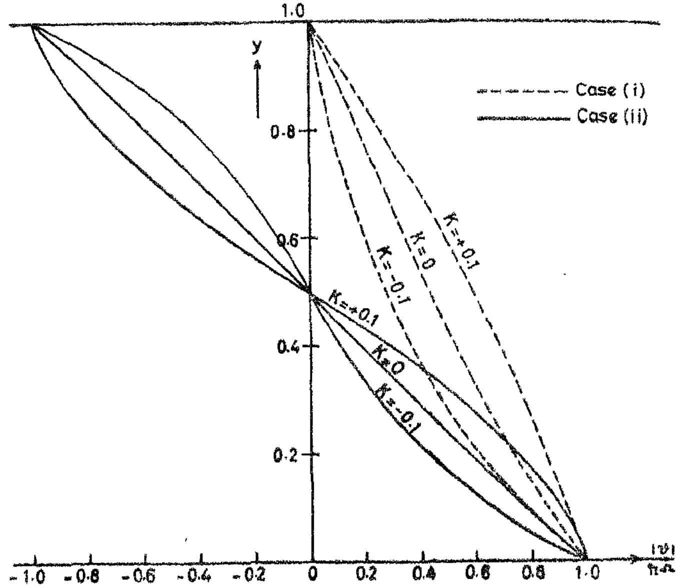

FIG. VI
Amplitude of the transverse component of the velocity

$$R^2 = 3780\,(3 - y_0)D^{-1},$$

where $D = (21 + 133\,y_0 - 175\,y_0^2 + 105\,y_0^3 - 35\,y_0^4 + 5\,y_0^5) + 180\,K(41 - 27\,y_0 + 10\,y_0^2 - 2\,y_0^3) - 15120\,K^2\,(3 - y_0),$ [4.11]

which is graphically represented in Fig. I, for $y_0 = 0.1$.

The Fig. IX gives the typical streamlines for $R = 5$ and $K = 0.083$, 0.084 and 0.085 in the case (ii). We find that for $K = 0.083$, the flow resembles that for the Newtonian fluids ($K = 0$), while $K = 0.085$ the sense of the flow is completely reversed. For $K = 0.084$, the flow breaks at $y_0 \simeq 0.3$ and symmetrically at $y_0 \simeq 0.7$. In the present case the relation between R, K and y_0 is given by

$$R^2 = \frac{15120}{(3 + 20\,y_0 - 40\,y_0^2 + 40\,y_0^3 - 20\,y_0^4) + 180\,K(13 - 8\,y_0 + 8\,y_0^2) + 60480K^2} \quad [4.12]$$

which once again is represented graphically in Fig. I.

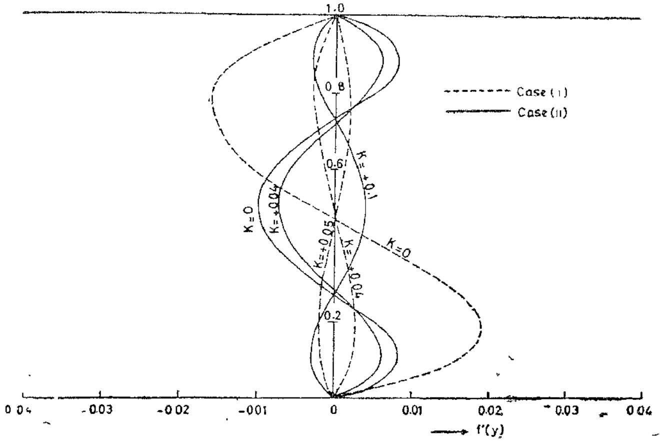

FIG. VII
Steady component of the radial velocity

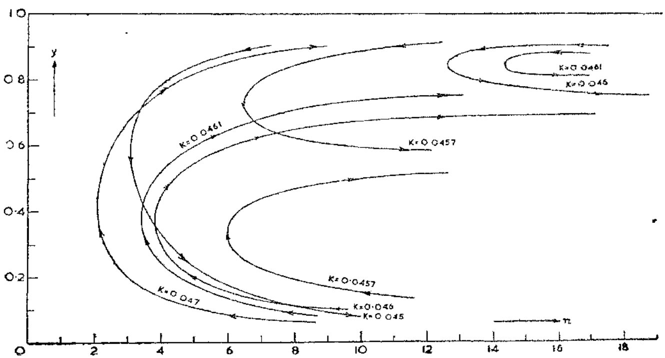

FIG. VIII
One disc oscillating
Typical streamline ($\psi = \pm 0.001$) of the steady component of the flow in the critical region of K

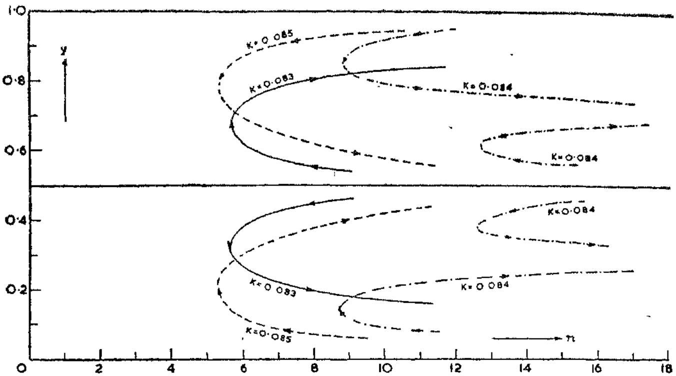

FIG. IX

Two discs Oscillating:
Typical Streamline ($\psi = \pm 0.001$) of the steady component of the flow in the critical region of K

5. STRESSES ON THE PLANES

In this section we shall calculate the torque on the planes $y=0$ and $y=1$ in both the cases (i) and (ii). We can easily show that the $T_{y\theta}$ component of the stress on the planes is given by the following:

(*a*) *Visco-inelastic fluids*:

$$T_{y\theta}]_{y=0} = (-1)\,\Phi_1\, r\,\Omega\, Rl\,[\sqrt{(R\,i)}.\ e^{it}\coth h\sqrt{(R\,i)}], \qquad \text{(case } i) \quad [5\ 1]$$

$$= (-1)\,\Phi_1\, r\,\Omega\, Rl\,\{\sqrt{(R\,i)}.\ e^{it}\coth h\,[\tfrac{1}{2}\sqrt{(R\,i)}]\}, \qquad \text{(case } ii) \quad [5.2]$$

which are the same as for a Newtonian fluid. Neglecting the edge effects the frictional torque M on the wet side of the plane of radius A will be

$$M = \frac{\pi A^4 \Omega\, \Phi_1\, \lambda_1}{4d\,[\mathrm{ch}\,\lambda_1 - \cos\lambda_1]}\{[\mathrm{sh}\,\lambda_1 + \sin\lambda_1]\cos t - [\mathrm{sh}\,\lambda_1 - \sin\lambda_1]\sin t\}, \text{(case i)} \quad [5.3]$$

$$= \frac{\pi A^4 \Omega\, \Phi_1\, \lambda_1}{2d\,[\mathrm{ch}\,\frac{1}{2}\lambda_1 - \cos\frac{1}{2}\lambda_1]}\{[\mathrm{sh}\tfrac{1}{2}\lambda_1 + \sin\tfrac{1}{2}\lambda_1]\cos t - [\mathrm{sh}\tfrac{1}{2}\lambda_1 - \sin\tfrac{1}{2}\lambda_1]\sin t\},$$
$$\text{(case ii)} \quad [5.4]$$

where $\lambda_1^2 = 2R$.

(*b*) *Visco-elastic fluids:*—

$$T_{y\theta}]_{y=0} = (-1)\,\Phi_1\, r\,\Omega\, Rl\,[(iRK+1)\,e^{it}\,\xi \coth \xi], \qquad \text{(case i)} \quad [5.5]$$

$$= (-1)\,\Phi_1\, r\,\Omega\, Rl\,[(iRK+1)\,e^{it}\,\xi \coth (\xi/2)], \quad \text{(case ii)} \quad [5.6]$$

where ξ is given by [3.3].

We can easily calculate the torque M, on a plane of radius A, neglecting the edge effects.

$$M = \frac{\pi \Phi_1 \Omega A^4 \lambda_2}{4d[\operatorname{ch}(\lambda_2 \cos\mu) - \cos(\lambda_2 \sin\mu)]} \{[\cos\mu \operatorname{sh}(\lambda_2\cos\mu)$$

$$+ \sin\mu \sin(\lambda_2 \sin\mu)]\,[\cos t + RK \sin t]$$

$$- [\sin\mu \operatorname{sh}(\lambda_2\cos\mu) - \cos\mu \sin(\lambda_2 \sin\mu)]\,[\sin t + RK\cos t]\}, \qquad \text{(case i)} \quad [5.7]$$

$$M = \frac{\pi \Phi_1 \Omega A^4 \lambda_2}{2d[\operatorname{ch}(\frac{1}{2}\lambda_2 \cos\mu) - \cos(\frac{1}{2}\lambda_2 \sin\mu)]} \{[\cos\mu \operatorname{sh}(\tfrac{1}{2}\lambda_2\cos\mu)$$

$$+ \sin\mu \sin(\tfrac{1}{2}\lambda_2 \sin\mu)]\,[\cos t + RK \sin t]$$

$$- [\sin\mu \operatorname{sh}(\tfrac{1}{2}\lambda_2\cos\mu) - \cos\mu \sin(\tfrac{1}{2}\lambda_2 \sin\mu)]\,[RK\cos t + \sin t]\} \ \text{(case ii)}, [5.8]$$

where $\mu = \frac{1}{2}\tan^{-1}(1/RK)$,

$$\lambda_2^2 \cos^2\mu = \frac{2R}{(R^2K^2+1)^{1/2}}\left[1 + \frac{RK}{(R^2K^2+1)^{1/2}}\right],$$

and

$$\lambda_2^2 \sin^2\mu = \frac{2R}{(R^2K^2+1)^{1/2}}\left[1 - \frac{RK}{(R^2K^2+1)^{1/2}}\right].$$

We notice that unlike the visco-inelastic fluid, the torque in the case of a visco-elastic fluid, is affected by the visco-elasticity.

6. Conclusion

We have noticed above that the breaking and the reversal of the flow are the characteristic features of the class of non-Newtonian fluids which we have considered in the present note and that these features, along with others, provide a basis for distinguishing them from the Newtonian fluids.

Let us first consider a visco-elastic fluid and subject it to plane viscometer. We know that[6] the skin friction is not affected by the visco-elasticity and thus the experiment will help us in measuring the co-efficient Φ_1. The formula [5.7] for torque deduced in the present paper will then enable us to determine K and hence Φ_2. In fact to find an experimental set up for the determination of Φ_2 was one of the motives in taking up this problem.

Let us now consider a visco-inelastic fluid. Once again the coefficient Φ_1 can be determined by the ordinary plane viscometer as the skin friction in the rectilinear laminar flow is not affected by the cross-viscosity. The torque formula for a visco-inelastic fluid is independent of cross-viscosity and hence the measurement of torque does not provide us with a means to measure S and hence Φ_3. However, the point at which the reversal of the flow takes place helps us in measuring S, according to the formulae deduced above.

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