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Convolution and Dilworth truncation of submodular functions

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Abstract

In this paper we survey the many applications of the operations of convolution and Dilworth truncation of submodular functions. Among other things we discuss in detail the strong analogies that exist between structural results related to the two operations, in particular those that exist between the principal partition of a submodular function with respect to a positive weight function and the principal lattice of partitions of a submodular function.

Keywords: Submodular functions, polymatroids, matroids, convolution, Dilworth truncation.

1. Introduction

In combinatorial mathematics submodular functions are a relatively recent phenomenon. Systematic interest in this area perhaps began with the work of Edmonds[1] in 1970. By then, matroids were well studied with numerous applications to engineering systems being found[2]. Submodular functions could be regarded as a generalization of matroid rank functions and it is natural to wonder whether the generalization is really required. The answer is that even if we ignore considerations of theory we come across them far more often in practical problems than matroids. The method of attack for these problems using submodular function theory is usually quite simple and the algorithms generated very efficient. A study of basic 'submodular' operations such as convolution and Dilworth truncation is likely to prove fruitful for practical algorithm designers since, in addition to completely capturing the essence of many practical situations, they also allow us to give acceptable approximate solutions to several intractable problems.

In this paper we sketch the many applications of the convolution and the truncation operations. We also emphasize, through a number of instances, the strong analogy that exists in their properties and in the problems involving them.

The outline of the paper is as follows: Section 2 describes some preliminary notions. Sections 3 and 4 describe convolution and truncation operations, respectively, along with their applications. Section 5 is on the analogy between the principal partition, which is naturally associated with convolution, and the principal lattice of partitions, which is associated with truncation. Section 6 deals with conclusions.

2. Preliminaries

If X is a *subset* (*proper subset*) of Y, we denote it by $X \subseteq Y$ ($X \subset Y$).

A set function $\rho : 2^S \to \Re$ is a *submodular* (*supermodular*) function iff

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y),$$

$$(\rho(X) + \rho(Y) \leq \rho(X \cup Y) + \rho(X \cap Y)) \quad \forall X, Y \subseteq S.$$

If further $\rho(\Phi) = 0$ and is monotone increasing then it is a *polymatroid rank function*. If, in addition, ρ is integral and $\rho(A \cup e) \leq \rho(A) + 1$ then ρ is a *matroid rank function*. Let $\rho(.)$ be a submodular function on subsets of S with $\rho(\Phi) = 0$. A *separator* of $\rho(.)$ is a subset X of S such that $\rho(X) + \rho(S - X) = \rho(S)$. It can be easily shown that unions and intersections of separators yield separators. A minimal nonnull separator is an *elementary separator*.

A *matroid M* is alternatively defined as a pair (S, I) where S is a set and I a collection of subsets called independent sets such that

- $X \subseteq Y,\ \ Y \in I \Rightarrow X \in I.$
- $X, Y \in I, |X| < |Y| \Rightarrow \exists\, e \in (Y - X)$ s.t. $(X \cup e) \in I$.

The *rank* of a set X, denoted by $r(X)$, can be identified with the (unique) cardinality of the maximal independent subset contained in it. It can be shown that $r(.)$ would be a matroid rank function as defined earlier. A maximal independent subset of S in M is called a *base* of M. A minimal nonindependent set is called a *circuit* of M. An element of M that is present in no circuit is called a *coloop* of M. If b is a base of M and $e \notin b$ then $e \cup b$ contains a unique circuit called the *fundamental circuit* $C(e, b)$. The *closure* of a set X in M is the maximal superset of X of the same rank as X. A set that is its own closure is called a *flat*. (For definitions and proofs regarding submodular functions and matroids see Welsh[3].)

A *bipartite graph* $(V_L^+\ V_R, E)$ has V_L as its left set of vertices and V_R as its right set of vertices and E as the set of its edges. All the edges have one end point in the left set and the other in the right set. A *matching* of a graph is a set of edges no two of which have common end points.

A *preorder* on S is a transitive relation ($\geq$) on S such that $a \geq a\ \forall a \in S$ (but $a \geq b$, $b \geq a$ does not imply $a = b$). A *partial order* is a preorder in which $a \geq b$, $b \geq a$ implies $a = b$. An *ideal* I of a preorder ($\geq$) on S is a subset of S that satisfies $a \in I$, $a \geq b \Rightarrow b \in I$. With every collection C of subsets of S closed under union and intersection we can associate a preorder ($\geq_C$) by the rule $a \geq b$ iff it is true that whenever a is in a member set of C, b is also in it.

A *partition* of a set S is a collection of subsets (called its *blocks*) of S no two of which intersect and whose union is S. A partition Π_1 of S is said to be *coarser* (*finer*) than a partition Π_2 of S, denoted by $\Pi_1 \geq \Pi_2$ (denoted by $\Pi_1 \leq \Pi_2$), iff every block of the latter (former) is contained in a block of the former (latter). If Π_1, Π_2 are two partitions of S

then the finest (coarsest) partition coarser (finer) than both is denoted by $\Pi_1 \vee \Pi_2$ ($\Pi_1 \wedge \Pi_2$). The collection of all partitions of S is denoted by P_S.

We abuse the notation in the following instance: A modular function ω with $\omega(\Phi) = 0$ is treated simultaneously as a vector $\omega : S \to \Re$ and a set function $\omega : 2^S \to \Re$ with $\omega(X) = \Sigma_{e \in X} \omega(e)$. Such a modular function is called a *weight function*.

3. Convolution

Definition 3.1. Let $f(.), g(.) : 2^S \to \Re$. The lower convolution of $f(.)$ and $g(.)$, denoted by $f * g(.)$, is defined through

$$f * g(X) \equiv \min_{Y \subseteq X} (f(Y) + g(X - Y)).$$

The collection of subsets Y at which $f(Y) + g(X-Y) = f * g(X)$ is denoted by $M_{f,g}(X)$, while if $X = S$, we will simply write $M_{f,g}$. The upper convolution of $f(.)$ and $g(.)$, denoted by $f \mathbin{\bar{*}} g(.)$, is defined through

$$f \mathbin{\bar{*}} g(X) \equiv \max_{Y \subseteq X} (f(Y) + g(X - Y)).$$

We then have the following basic result (see, Lovasz[4] for instance).

Theorem 3.1. If $f(.)$ is submodular (supermodular) and $g(.)$ is modular then $f * g(.)$ ($f \mathbin{\bar{*}} g(.)$) is submodular (supermodular).

Remark 3.1. Henceforth, we will confine our attention to lower convolution of submodular functions with submodular or modular functions. The results can be appropriately translated for upper convolution in the supermodular case.

Remark 3.2. If $f(.), g(.)$ are both submodular, $f * g(.)$ is not always submodular.

We now list from the literature a number of examples which are related to the notion of convolution, more specifically, to that of principal partition.

1. (a) *Hall's Theorem* (Hall[5]). Hall's Theorem on systems of distinct representatives states the following in the language of bipartite matching: 'Let $B \equiv (V_L \uplus V_R, E)$ be a bipartite graph. There exists a matching meeting all the vertices in V_L iff for no subset X of V_L we have $|\Gamma(X)| < |X|$'. This condition is equivalent to saying '... iff $(\Gamma * |.|)(V_L) = |V_L|$'.

   (b) *Dulmage–Mendelsohn decomposition of a bipartite graph* (Dulmage and Mendelsohn[6,7]. The above-mentioned authors made a complete analysis of all min covers and max matchings in a bipartite graph through a unique decomposition into derived bipartite graphs. We present their decomposition using the language of convolution.

Let $B \equiv (V_L \uplus V_R, E)$ be a bipartite graph. Let M_1 denote the collection of subsets of V_L which minimize $h_1(X) \equiv \Gamma_L(X) + |V_L - X|$, where $\Gamma_L(X) \equiv |\Gamma(X)|$, $X \subseteq V_L$, with $\Gamma(X)$ denoting the set of vertices adjacent to vertices in X. Thus, $\min_{X \subseteq V_L} h_1(X) = (\Gamma_L * |.|)(V_L)$. It is easily seen that M_1 is closed under union and intersection. Let X_{min} and X_{max} be the minimal and maximal sets which are members of M_1. Then $X_{max} - X_{min}$ can be partitioned into sets N_i such that each N_i is either contained in a given member of M_1 or does not intersect it and further the partition is the coarsest with this property. Let Π be the partition whose blocks are X_{min}, all the N_i and $V_L - X_{max}$. Let us define a partial order ($\geq$) on the blocks of Π as follows: $N_i \geq N_j$ iff N_j is present in a member of M_1 whenever N_i is present. For all N_i, $N_i \geq X_{min}$ and $V_L - X_{max} \geq N_i$. Next, for each block K of Π we build the bipartite graph B^K as follows: Let I_K be the principal ideal of K (*i.e.* the collection of all elements (blocks of Π) that are 'less than or equal to' K) in the partial order. Let J_K be the union of all the elements in I_K. Then B^K is the subgraph of B on $K \uplus (\Gamma(J_K) - \Gamma(J_K - K))$. The partial order ($\geq$) induces a partial order ($\geq_B$) on the collection of bipartite graphs B^K, $K \in \Pi$. The Dulmage–Mendelsohn decomposition is the collection of all B^K's together with the partial order ($\geq_B$).

We now list the important properties of this decomposition.

- A set $X \uplus Y$, $X \subseteq V_L$, $Y \subseteq V_R$, is a minimum *cover* of B (*i.e.* every edge of B is incident on some vertex of the set) iff $V_L - X$ is the union of blocks in an ideal contained in X_{max} of the partial order ($\geq$) and $Y = \Gamma(V_L - X)$.
- A set of edges P is a maximum matching of B iff $P = \biguplus_{K \in \Pi} P^K$, where P^K is a maximum matching of B^K.
- Every maximum matching is incident on all the vertices in $\Gamma(X_{max})$ and $V_L - X_{min}$.

2. (a) *Decomposition of a graph into minimum number of subforests* (Tutte[8], Nashwilliams[9]). Tutte and Nashwilliams characterized graphs which can be decomposed into k disjoint subforests as those which satisfy $kr(X) \geq |X|$ $\forall X \subseteq E(G)$. This condition can be shown to be equivalent to $kr * |.|(E(G)) = |E(G)|$.

We next list four problems and give their (combined) solution.

(b) *Tree of minimum size hybrid representation* (Kishi and Kajitani[10]). Let a tree t be represented by a pair of sets (A_t, B_t), where $A_t \subseteq t$, $t \cap B_t = \emptyset$ such that $(A_{t_1}, B_{t_1}) = (A_{t_2}, B_{t_2})$ iff $t_1 = t_2$. Note that there can be several pairs representing the same tree, for instance, (t, ϕ), $(\phi, E(G) - t)$ both represent t. We call $|A_t \cup B_t|$ the *size of the representation*. Find a tree which has the representation of minimum size.

(c) *Maximum distance between two trees* (Kishi and Kajitani[10]). Find two trees in a given graph which have the maximum distance between them (distance between two trees t_1 and t_2 is $|t_1 - t_2|$), *i.e.*, the size of their union is the largest possible.

(d) *The topological degree of freedom of an electrical network* (Ohtsuki *et al.*[11]) Select a minimum sized set of branch voltages and branch currents from which, by using Kirchhoff's voltage equations and Kirchhoff's current equations, we can find either the voltage or the current associated with each branch. The minimum size is called the *topological degree of freedom of the network*, or equivalently, *hybrid rank of the graph.*

(e) *The Shannon switching game*[12]. G is a graph with one of its edges, say e_M, 'marked'. There are two players—a 'cut' player and a 'short' player. The cut player during his turn removes (opens) an edge leaving the end points in place. The short player during his turn fuses the end points of an edge and removes it. Neither player is allowed to touch the e_M. The aim of the cut player is to destroy all the paths between the end points of e_M (equivalently, destroy all circuits containing e_M). The aim of the short player is to fuse the end points of e_M (equivalently, destroy all cut sets containing e_M). The problem is to analyse this game to characterize situations where the cut or short player playing second can always win and to determine the winning strategy.

(f) *The maximum rank of a cobase submatrix* (Iri[13]). For a rectangular $(m \times n)$ matrix with linearly independent rows, let us call an $m \times (n - m)$ submatrix a *cobase* submatrix iff the remaining set of columns correspond to an identity matrix. The *term rank* of a matrix is the maximum number of nonzero entries in the matrix which belong to distinct rows and distinct columns. Find

- a cobase matrix of maximum rank, and
- a cobase matrix of minimum term rank.

Solution. For the above four problems the solution involves essentially the same strategy: Find a set X (or a minimal set $X_{\min}$ or a maximal set $X_{\max}$) such that $(2r * |\cdot|)(E(G)) = 2r(X) + |E(G) - X|$. Select a tree t which has maximal intersection with X. The representation $(t \cap X,\ (E(G) - t) \cap (E(G)–X))$ has the least size among all representations of all trees.

The maximum distance turns out to be the same as the above minimum size of representation. Kishi and Kajitani[10] gave an algorithm for building a pair of maximally distant trees which is essentially the well-known algorithm for building a base of the union of two matroids[14].

Let t_X be a tree of the subgraph on X. Let $L_{\bar{X}}$ be a cotree of the graph on $G \times (E(G) - X)$ (the graph obtained by fusing the end points of edges in X and removing them). Select the branch voltages of t_X and the branch currents of $L_{\bar{X}}$ as the desired set of variables. As is easily seen, the topological degree of freedom is also the same as the minimum size of representation of a tree.

If $e_M \in X_{min}$, the short player can always win. If $e_M \in E(G) - X_{max}$ the cut player can always win. If $e_M \in X_{max} - X_{min}$, whoever plays first can always win. The winning strategies involve the construction of appropriate maximally distant trees during every turn.

The solution is similar for the last problem. Let S be the set of all columns and let $r(.)$ be the rank function on the collection of subsets of S. Then the maximum rank of a cobase matrix = the minimum term rank of a cobase matrix = $(2r * |.|)\ (S) - r(S)$. Select two maximally distant bases (bases ≡ maximally independent columns). Perform row operations so that an identity matrix appears corresponding to one of these. The submatrix corresponding to the complement of this base is the desired cobase matrix which has both maximum rank as well as minimum term rank.

3. (a) *The matroid intersection problem* (Edmonds[15]). Given two matroids M_1, M_2 on S find a maximum cardinality subset which is independent in both matroids.

Solution. The size of the maximum-cardinality common independent set = $(r_1 * r_2)$ (S). To find this set one can either use Edmond's algorithm for this purpose or find bases b_1, b_2^* of M_1, M_2^*, which are maximally distant. (Hence M^* denotes the dual of M.)

(b) *The matroid union*. Given two matroids M_1, M_2, find the maximum-cardinality union of an independent set in M_1 and an independent set in M_2.

Solution. The collection of all unions of two independent sets, one independent in M_1 and the other in M_2, is also a matroid denoted by $M_1 \vee M_2$. Thus, the maximum-cardinality union of an independent set of M_1 and one of M_2 is a base of $M_1 \vee M_2$. There is the well-known[14] matroid union algorithm for constructing this set. The rank function of this matroid is $(r_1 + r_2) * |.|\ (.)$. The union of all circuits of this matroid is the minimal set X which satisfies $(r_1 + r_2) * |.|\ (S) = (r_1 + r_2)\ (X) + |S - X|$.

4. *Representability of matroids* (Horn[16]). Horn showed that k independent sets of columns can cover the set of all columns of a matrix iff there exists no subset A of columns such that $|A| > kr(A)$. He conjectured that this might be correct only for representable matroids. If the conjecture had been true then there would have been a nice characterization of representability. However, Edmonds[14] showed that this result is true for all matroids. He gave an algorithm for constructing k bases of a matroid whose union has the maximum cardinality. His results are equivalent to saying that k bases will cover the underlying set S of a matroid M iff M^k, the union of M with itself k times, has no circuits. The rank function of this matroids is $(kr * |.|)\ (.)$. So the result can be stated equivalently as 'covering is possible iff $(kr * |.|)\ (S) = |S|$'.

5. *Polyhedral interpretation for convolution* (Edmonds[1], also Lovasz[4] and Cunningham[17])

Definition 3.2. Let $f(.)$ be a real-valued set function on subsets of $S \equiv \{e_1, \ldots, e_n\}$.

Let χ_X denote the characteristic vector of $X \subseteq S$. Let

$$x(X) \equiv (\chi_X)^{\mathrm{T}} x \quad \forall X \subseteq S.$$

Then the polyhedron denoted by P_f is defined as follows: A vector $x \in \Re^S$ belongs to P_f iff $x(X) \le f(X)\ \forall X \subseteq S$. We say $f(.)$ is polyhedrally tight iff for each $X \subseteq S$ there exists a vector $x \in P_f$ such that $x(X) = f(X)$.

(a) (i) Let $f(.)$, $g(.)$ be set functions on subsets of S. Then $P_f \cap P_g = P_{f*g}$.

(ii) If $f(.)$, $g(.)$ are submodular functions that take zero value on Φ then $f * g(.)$ is polyhedrally tight. Equivalently,

$$f * g(X) \equiv \min_{Y \subseteq X} (f(Y) + g(X - Y)) = \max(x(X)),$$

where x is a vector satisfying $x(Z) \le f(Z)$, $x(Z) \le g(Z)\ \forall Z \subseteq X$. Further, if $f(.)$, $g(.)$ are integral, x can be chosen to be integral.

6. *Submodular function minimization* (see Lawler[18]). Let $\mu(.)$ be a submodular function. Let $g(.)$ be the weight function defined through $g(e) \equiv \mu(S - e) - \mu(S)$. Let $f(.) \equiv \mu(.) + g(.)$. It can be shown that $f(.)$ is a polymatroid rank function and that $\mu(.)$ reaches a minimum at $X \subseteq S$ iff $X \in M_{f,g}$ (see Definition 3.1). Thus, minimization of a submodular function is equivalent to convolving a polymatroid rank function with a weight function.

7. *New matroids* (Edmonds[1]). The following is one of the most important ways of generating new matroids.

Theorem 3.2. Let $f(.)$ be an integral polymatroid rank function, k, an integer and let $g(.) \equiv |(.)|$. Then $kf * g(.)$ is a matroid rank function.

8. *The principal partition.* The idea of convolution has found perhaps its strongest application in the study of the *principal partition*, which concerns itself with the sets $Y \subseteq X$ at which we have $\lambda f * g(X) = \lambda f(Y) + g(X - Y)$, $\lambda \ge 0$. Indeed, one is tempted to conclude that the function $f * g(X)$ is often less important than such subsets of X. This study started with the work of Kishi and Kajitani[10]. It was completed for the case of a matroid with respect to the weight function $|.|$ independently by Tomizawa[19] and Narayanan[20]. The case of two graphs was studied by Ozawa[21] and that of two polymatriods by Nakamura and Iri[22]. Much work has been done by Iri[13], Tomizawa[23] and Fujishige[24, 25] and others clarifying the underlying notions. A survey of applications may be found in Iri and Fujishige[2].

Definition 3.3. Let $f(.)$, $g(.)$ be polymatroids on the subsets of S. The collection of all subsets of S which belong to some $M_{\lambda f, g}$, $\lambda \in \Re$ (see Definition 3.1) is called the *principal partition* of $(f(.), g(.))$.

As can be seen below, one of the interesting features of the principal partition is that one need only examine a finite number of λs in order to solve the optimization problems for all the λs.

We will confine ourselves to the case where $g(.)$ is strictly increasing, more particularly, to the case where $g(.)$ is a positive weight function.

We list below the main properties of the principal partition.

(a) For the case where $f(.)$, $g(.)$ are submodular, but $g(.)$ not necessarily strictly increasing, we have:

Property PP1. The collection $M_{\lambda f, g}$, $\lambda \geq 0$, is closed under union and intersection and thus has a unique maximal and a unique minimal element.

Let X^{λ}, X_{λ}, denote, respectively, the maximal and minimal elements of $M_{\lambda f, g}$.

Remark 3.3. For the remaining properties we assume $f(.)$ to be submodular and $g(.)$ to be a strictly increasing (*i.e.*, $g(Y) < g(X)$, $\forall Y \subset X \subseteq S$) polymatroid rank function.

(b) *Property PP2.* If $\lambda_1 > \lambda_2 \geq 0$, then $X^{\lambda_1} \subseteq X_{\lambda_2}$.

(c) *Definition 3.4.* A nonnegative value λ for which $M_{\lambda f, g}$ has more than one subset as a member is called a critical value of $(f(.), g(.))$.

Property PP3. The number of critical values of $(f(.), g(.))$ is finite.

(d) *Property PP4.* Let $\lambda_1, \ldots, \lambda_t$ be the decreasing sequence of critical values of $(f(.), g(.))$. Then $X^{\lambda_i} = X_{\lambda_{i+1}}$ for $i = 1, \ldots, t-1$.

(e) *Property PP5.* Let (λ_i) be the decreasing sequence of critical values. Let $\lambda_i > \sigma > \lambda_{i+1}$. Then $X^{\lambda_i} = X^{\sigma} = X_{\sigma} = X_{\lambda_{i+1}}$.

Remark 3.4. Each of the collection $M_{\lambda f, g}$ is closed under union and intersection. Hence one can define preorders for each of them as follows: $e_1 \geq_p e_2$, $e_1, e_2 \in S$, iff whenever e_1 is in a member set of $M_{\lambda f, g}$ so is e_2. For a given λ, it is clear that X^{λ}, the maximal element of $M_{\lambda f, g}$ is partitioned by the equivalence classes of the corresponding preorder.

Here is a simple result on 'density' which suggests usefulness of principal partitions. The routine proof is omitted.

Definition 3.5. Let $f(.)$ be a polymatroid rank function and $g(.)$ a positive weight function on subsets of S. The density of $X \subseteq S$ with respect to $(f(.), g(.))$ is the ratio $g(X)/f(X)$.

Theorem 3.3. Let $f(.)$ be a polymatroid rank function and let $g(.)$ be a weight function on the subsets of S. Let X be a member of $M_{\lambda f, g}$ for some λ. Then among all sets which have the same $g(.)$ value as itself, X has the maximum density.

4. Dilworth truncation

In this section we study the *Dilworth truncation* (*truncation* for short) operation on submodular functions, some of its many theoretical and practical applications and some structural results analogous to those on the convolution operation.

Definition 4.1. Let $f(.)$ be a real set function on the subsets of S. The partition associate of $f(.)$, defined on the collection P_S of all partitions of S, is denoted by $\bar{f}(.)$ and is defined through $\bar{f}(\Pi) \equiv \Sigma_{N_i \in \Pi} f(N_i)$. The lower (upper) Dilworth truncation of $f(.)$ is denoted by $f_t(.)$ ($f^t(.)$) and is defined through

$$f_t(\Phi) \equiv 0, \quad f_t(X) \equiv \min_{\Pi \in P_X}\left(\sum_{X_i \in \Pi} f(X_i)\right)$$

$$\left(f^t(\Phi) \equiv 0, \quad f^t(X) \equiv \max_{\Pi \in P_X}\left(\sum_{X_i \in \Pi} f(X_i)\right)\right)$$

Remark 4.1. As in the case of convolution, perhaps even more important than the function $f_t(.)$ ($f^t(.)$) are the partitions at which $\bar{f}(.)$ reaches its optimum.

When $f(.)$ is submodular (supermodular) $\bar{f}(.)$ has the following attractive property.

Theorem 4.1[30]. *Let Π_N denote a partition of S such that N is a block and all other blocks are singletons. Then $f(.)$ is submodular (supermodular) iff*

$$\bar{f}(\Pi) + \bar{f}(\Pi_N) \geq \bar{f}(\Pi_N \vee \Pi) + \bar{f}(\Pi_N \wedge \Pi)$$

$$\left(\bar{f}(\Pi) + \bar{f}(\Pi_N) \leq \bar{f}(\Pi_N \vee \Pi) + \bar{f}(\Pi_N \wedge \Pi)\right) \quad \forall \Pi \in P_S$$

Using Theorem 4.1 we can prove the following two results.

Theorem 4.2[30]. *Let $f(.)$ be submodular (supermodular) over subsets of S and let Π_1, Π_2 minimize (maximize) $\bar{f}(.)$ over P_S. Then:*

1. *$\Pi_1 \vee \Pi_2$ ($\Pi_1 \wedge \Pi_2$) also minimize (maximize) $\bar{f}(.)$.*
2. *If $N_1, \ldots, N_k$ are some of the blocks of Π_1 and $M_1, \ldots, M_r$ are some of the blocks of Π_2 such that $N_i \cap M_j = \emptyset$, $\forall i, j$ and $\cup N_i \cup M_j = S$, then the partition $\{N_1, \ldots, N_k, M_1, \ldots, M_r\}$ minimizes (maximizes) $\bar{f}(.)$.*

Theorem 4.3. Let $f(.)$ be submodular (supermodular) over subsets of S and let $X \subseteq Y \subseteq S$. Let Π minimize (maximize) $\bar{f}(.)/P_X$. Then there exists a Π' in P_Y such that the blocks of Π are contained in the blocks of Π' and Π' minimizes (maximizes) $\bar{f}(.)/P_Y$.

The following well known and fundamental result[4, 26] can be proved using the above results.

Theorem 4.4. If $f(.)$ is submodular (supermodular) on subsets of S then $f_t(.)$ ($f^t(.)$) is submodular (supermodular).

We now list a number of examples from the literature relevant to the Dilworth truncation operation.

1. *Truncation of matroids* (Dilworth[27]). Let M be a matroid on S. Let S_k be the collection of k-rank flats of M. Build a matroid M_k on S_k such that

   - each element of S_k has rank 1;
   - if A is a flat of M with rank $p > k$ then $\hat{A}$, the collection of all k-rank flats of M contained in A, is a flat of M_k with rank $p-(k-1)$.

Solution. Let P_X denote the collection of all partitions of $X \subseteq S_k$. Define the rank function $r_k(.)$ on subsets of S_k as follows:

$$r_k(\Phi) \equiv 0, \quad r_k(X) \equiv \min_{\Pi \in P_X} \sum_{X_i \in \Pi} (r-(k-1))(X_i).$$

It can be shown that

- $r_k(.)$ is a matroid rank function,
- if A is a flat of M with rank $p > k$ then $\hat{A}$ is a flat of M_k with rank $p-(k-1)$.

2. *Intersecting submodular function* (Lovasz [28]). A set function $f(.)$ on subsets of S is said to be *intersecting submodular* iff

   $$f(X)+f(Y) \geq f(X \cup Y)+f(X \cap Y) \quad \forall X, Y \text{ s.t. } X-Y,\ Y-X,\ X \cap Y \neq \emptyset.$$

Find a submodular function $g(.)$ such that $P_f = P_g$.

Solution. It turns out that $f_t(.)$ is submodular. This is also the desired function since for any set function $f(.)$ we have $P_f = P_{f_t}$.

3. *Hybrid rank relative to a partition of the edges of a graph* (Narayanan[29]). The problem described below arises when we attempt to solve an electrical network by decomposing it. First we define two operations on graphs. A *node pair fusion* means fusing two specified vertices v_1, v_2 into a single vertex v_{12}, while a node fission means splitting a node v_1 into v_{11}, v_{12} making some of the edges incident at v_1 now incident at v_{11} and the remaining at v_{12}. We are given a partition Π of the edge set $E(G)$ of a graph G such that the subgraph on each block of the partition is connected. Find a sequence of fusion and fission operations least in number such that the resulting graph has no circuit intersecting more than one block of Π. (The hybrid rank problem given in the description of *topological degree of freedom* in Section 3 is a special case of this problem corresponding to the edge partition with singleton blocks.)

Solution. It is easy to see that one cannot lose if one performs fusion operations first and then fission operations. Let $I(X)$, $X \subseteq V(G)$, be the set of branches incident on vertices in X. Let Π^V be a partition that minimizes $\overline{|I(.)|-2}$. The best sequence is the following: Fuse each block of Π^V into a single node. (If k nodes are in a single block, this involves k–1 operations.) In the resulting graph, which we shall call G', perform the minimum number of node fissions required to destroy all circuits intersecting more than one block of Π. This is relatively easy to do and the number of such fission operations is $\Sigma_{N_i \in \Pi} r'(N_i) - r'(E(G))$, where $r'(.)$ is the rank function of G'.

4. *New matroids.* A well-known method for generating new matroids from polymatroid rank functions is the following[30–32]. Let $\mu(.)$ be an integral polymatroid rank function with $\mu(e) = k$, $e \in S$. Let $pk - q = 1$. The $(p\mu - q)_t$ is a matroid rank function.

Example. Let $V(.)$ be the polymatroid rank function on the subsets of $E(G)$ (where G is a self-loop free graph) such that $V(X) \equiv$ number of vertices incident on edges in X. Clearly, $V(e) = 2$. Then $(kV(.) - (2k-1))_t$ is a matroid rank function. In particular, $(V(.)-1)_t$ is the rank function of the graph and $(2V(.)-3)_t$ is the rank function of the *rigidity matroid* associated with the graph.

5. *Posing convolution problems as truncation problems*[30, 33, 34]. We give an example. Consider the convolution problem: Find $\min_{X \subseteq E(G)} \lambda r(X) + \omega(E(G) - X)$, $\lambda \geq 0$, where $E(G)$ is the edge set, $r(.)$ the rank function of the graph G and $\omega(.)$, a weight function on $E(G)$. Let $I(X) \equiv$ set of edges incident on vertices in $X \subseteq V(G)$, let $E(X) \equiv$ set of edges incident only on vertices in $X \subseteq V(G)$ and let $\omega(I(X))$, $\omega(E(X))$ denote the sum of the weights of edges in the corresponding sets. Then one can show that $X \subseteq E(G)$ solves the above convolution problem iff $X = \cup_{N_i \in \Pi'} E(N_i)$, where Π' solves the truncation problem: Find $\min_{\Pi \in P_{V(G)}} \overline{|\omega(I(.))| - \lambda}(\Pi)$ or, *equivalently*, find $\max_{\Pi \in P_{V(G)}} \overline{|\omega(E(.))| + \lambda}(\Pi)$. Thus, the principal partition of the rank function of a graph can be determined equivalently by solving either of the above-mentioned truncation problems for appropriate values of λ. Indeed, this approach yields the fastest algorithm currently known for this principal partition problem ($O(EV^2 \log^2(V)$ for the unweighted case and $O(EV^3 \log(V)$ for the weighted case).

6. *The principal lattice of partitions of a submodular function* (Narayanan[30]). The natural optimization problems associated with convolution are over the collection of subsets of the underlying set S while those associated with Dilworth truncation are over the collection P_S of partitions of S. The principal partition finds its analogue in the principal lattice of partitions.

Definition 4.2. Let $f(.)$ be submodular on the subsets of S. Let L_λ denote the collection of partitions of S that minimize $\overline{f - \lambda}$. The collection of all partitions of S which belong to some L_λ, $\lambda \in \Re$, is called the principal lattice of partitions of $f(.)$.

As can be seen below, one of the interesting features of the principal lattice of partitions is that one need only examine a finite number of λs in order to solve the optimization problems for all the λs.

We list below the main properties of the principal lattice of partitions. The reader might like to compare them with those of the principal partition given in Section 3.

(a) *Property PLP1.* The collection L_λ is closed under join ($\vee$) and meet ($\wedge$) operations and thus has a unique maximal and a unique minimal element.

(b) *Property PLP2.* If $\lambda_1 > \lambda_2$, then $\Pi^{\lambda_1} \leq \Pi_{\lambda_2}$, where Π^λ, Π_λ denote, respectively, the maximal and minimal elements of L_λ.

(c) *Definition 4.3.* A number λ for which L_λ has more than one partition as a member is called a *critical value* of $f(.)$. ⊕

Property PLP3. The number of critical values of $f(.)$ is finite.

(d) *Property PLP4.* Let $\lambda_1, \ldots, \lambda_t$ be the decreasing sequence of critical values of $f(.)$. Then $\Pi^{\lambda_i} = \Pi_{\lambda_{i+1}}$ for $i = 1, \ldots, t-1$.

(e) *Property PLP5.* Let $\lambda_1, \ldots, \lambda_t$ be the decreasing sequence of critical values. Let $\lambda_i > \sigma > \lambda_{i+1}$. Then $\Pi^{\lambda_i} = \Pi^\sigma = \Pi_\sigma = \Pi_{\lambda_{i+1}}$.

Definition 4.4. If $\Pi \equiv \{S - N_1, \ldots, S - N_k\}$ is a partition of S, the collection $\{N_1, \ldots, N_k\}$, denoted by Π^*, is called a *copartition* of S. Further it is said to be the *dual copartition* to Π. We say $\Pi_1^* \geq \Pi_2^*$ iff $\Pi_1 \geq \Pi_2$.

We now have the following lemma whose routine proof is omitted.

Lemma 4.1. Let $f(.)$ be a set function on subsets of S. Let $f'(X) \equiv f(S - X)$, $\forall X \subseteq S$. Then

(a) $(f')'(.) = f(.)$,

(b) $f'(.)$ is submodular iff $f(.)$ is submodular, *and*

(c) $\overline{f' - \lambda}(\Pi^*) = \overline{f - \lambda}(\Pi)$, *where* $\overline{f' - \lambda}(\Pi^*) \equiv \Sigma_{N_i \in \Pi^*}(f' - \lambda)(N_i)$.

Remark 4.2. It is clear from the above lemma that we can define a 'principal lattice of copartitions (PLC)' for a submodular function $f(.)$ by considering the dual copartitions to the partitions in the PLP of $f'(.)$. We say that λ is a critical value of the PLC of $f(.)$ iff it is a critical value of the PLP of $f'(.)$.

7. *Optimal partitions*[30, 35, 36]. Most large-scale problems are solved by taking recourse to partitioning. This naturally gives rise to the problem of determining the optimal

partition relevant in the context. Usually, this amounts to determining a partition which minimizes 'interaction' between blocks. As is to be expected, this partitioning problem, even if it can be stated precisely, is invariably NP-hard. A promising approach to tackle this issue is to check whether the problem (or one near enough) can be stated as a truncation problem on an appropriate submodular function. The starting point is the following simple result[30] (the reader might like to compare with Theorem 3.3).

Theorem 4.5. Let $f(.)$ be real-valued set function on subsets of S. Let Π be any partition that minimizes $\overline{f(.)-\lambda}$ for some λ. Let Π have n blocks. Then among all n block partitions of S, Π minimizes $\overline{f(.)}$. If $f(.)$ is submodular then every partition in the principal lattice of partitions of $f(.)$ has this property.

It must, however, be remembered that there may be no partition in the PLP with the desired number, say k, of blocks. A simple way to get 'near optimal' partitions is to first find two partitions $\Pi_1 \geq \Pi_2$ in some L_λ whose numbers of blocks are on either side of k. By judiciously 'mixing' the two partitions, one can get a partition whose $\overline{f(.)}$ value is worse than the optimal by no more than a fixed factor.

Example. A standard problem is to partition the vertices of a graph G (whose edges are weighted by $\omega(.)$) such that the sum of the weights of edges lying between blocks is minimized. It is easily seen that this is equivalent to minimizing $\overline{\omega(I(.))}$ (see the definition in '*posing convolution problems as truncation problems*'). This approach yields a partition which is worse than the optimal partition atmost by a factor $2 - k'/n'$, where $k' \equiv k-(|\Pi_1| - 1)$ and $n' \equiv |E(G)| - (|\Pi_1| - 1)$.

5. The PP–PLP analogy

In this section we present a number of additional results which emphasize the analogy already suggested between the principal partition and the principal lattice of partitions. We adopt the technique of presenting a result on principal partition followed by its PLP counterpart.

We begin with a pair of elementary symmetry results whose trivial proofs are omitted. But first some definitions.

Definition 5.1. Let $f(.)$ be a real-valued function on the subsets of S. An automorphism of $f(.)$ is a bijection $\alpha : S \rightarrow S$ such that $f(X) = f(\alpha(X))$ $\forall X \subseteq S$. A set X is invariant under $\alpha(.)$ iff $\alpha(X) = X$. A function $g(.)$ is symmetric with respect to $f(.)$ iff every automorphism of $f(.)$ is also an automorphism of $g(.)$. A partition Π is invariant under $\alpha(.)$ iff $\alpha(X)$ is a block of Π whenever X is. A collection P of partitions of S is invariant under $\alpha(.)$ iff whenever Π is a partition in P the partition $\alpha(\Pi) \equiv \{\alpha(X), X \in \Pi\}$ is also in P.

Theorem 5.1. Let $f(.)$ be a submodular function and $g(.)$ a strictly increasing polymatroid rank function on the subsets of S. If $g(.)$ is symmetric with respect to $f(.)$, every $M_{\lambda f,g}$ and the maximal and minimal sets in any $M_{\lambda f,g}$ are invariant.

Theorem 5.2. Let $f(.)$ be a submodular function on the subsets of S. Every L_λ is invariant and the maximal and minimal partitions in any L_λ are invariant.

The next four results are about changes in the submodular function which leave the PP (PLP) unchanged.

Theorem 5.3. Let $f(.)$ be a polymatroid rank function on subsets of S and $g(.)$ a positive weight function on S.

1. *The principal partition of $(\beta f(.), \alpha g(.))$, where $\beta, \alpha > 0$, is the same as that of $(f(.), g(.))$; λ is a critical value of the principal partition of $(f(.), g(.))$ iff $(\lambda\alpha/\beta)$ is a critical value of the principal partition of $(\beta f(.), \alpha g(.))$.*
2. *The principal partition of $((f+\alpha g)(.), g(.))$, where $\alpha \geq 0$, is the same as that of $(f(.), g(.))$; λ is a critical value of the principal partition of $(f(.), g(.))$ iff $\lambda/(1+\lambda\alpha)$ is a critical value of the principal partition of $((f+\alpha g)(.), g(.))$.*

Proof.

1. We have

$$\lambda f(X) + g(S-X) = (1/\alpha)[(\lambda\alpha/\beta)\beta f(X) + \alpha g(S-X)].$$

Clearly, therefore, we must have

$$M_{\lambda f,g} = M_{(\lambda\alpha/\beta)f,\alpha g}.$$

The required result is now immediate.

2. We see that

$$\begin{aligned}\lambda f(X) + g(S-X) &= \lambda(f(X) + \alpha g(X)) + g(S-X) - \lambda\alpha(g(S) - g(S-X)) \\ &= \lambda(f(X) + \alpha g(X)) + (1+\lambda\alpha)g(S-X) - \lambda\alpha g(S).\end{aligned}$$

Clearly, therefore, we must have

$$M_{\lambda f,g} = M_{\lambda(f+\alpha g),(1+\lambda\alpha)g} = M_{(\lambda/(1+\lambda\alpha))(f+\alpha g),g}.$$

The required result is now immediate.

The analogous PLP result is presented below. We omit the routine proof.

Theorem 5.4. Let $f(.)$ be a submodular function on subsets of S and let $g(.)$ be a weight function on S. Then:

1. *The principal lattice of partitions of $\beta f(.)$, where $\beta > 0$, is the same as that of $f(.)$; λ is a critical value of the principal lattice of partitions of $f(.)$ iff $\lambda\beta$ is a critical value of the principal lattice of partitions of $\beta f(.)$.*
2. *The principal lattice of partitions of $(f+g)(.)$ is the same as that of $f(.)$; λ is a critical value of the principal lattice of partitions of $f(.)$ iff it is a critical value of the principal lattice of partitions of $(f+g)(.)$.*

Theorem 5.5. Let $f_0(.)$, $f_1(.)$ be polymatroid rank functions on subsets of S and let $g(.)$ be a positive weight function on S. Let $(f_0(.), g(.))$, $(f_1(.), g(.))$ have the same principal partition with decreasing sequence of critical values $\lambda_{01}, \ldots, \lambda_{0t}$ and $\lambda_{11}, \ldots, \lambda_{1t}$, respectively. The $n((f_0+f_1)(.), g(.))$ has the same principal partition with decreasing sequence of critical values $\lambda_{31}, \ldots, \lambda_{3t}$, where $\lambda_{3i} = ((\lambda_{0i})^{-1} + (\lambda_{1i})^{-1})^{-1}$, $i = 1, \ldots, t$.

Proof. Let M^3_λ denote the collection of minimizing sets corresponding to λ in the principal partition of $(f_0(.) + f_1(.), g(.))$. Let X be a set in $M^0_{\lambda_0}$ as well as in $M^1_{\lambda_1}$. We claim that $X \in M^3_{\lambda_3}$, where $\lambda_3 = ((\lambda_0)^{-1} + (\lambda_1)^{-1})^{-1}$. We have

$$f_i(X) + ((\lambda_i)^{-1})g(S-X) \le f_i(Y) + ((\lambda_i)^{-1}g(S-Y), \quad i = 0,1 \quad \forall Y \subseteq S.$$

Hence,

$$f_0(X) + f_1(X) + ((\lambda_0)^{-1} + (\lambda_1)^{-1})g(S-X)$$
$$\le f_0(Y) + f_1(Y) + ((\lambda_0)^{-1} + (\lambda_1)^{-1}g(S-Y) \quad \forall Y \subseteq S.$$

This proves the claim. In the above proof note that the final inequality reduces to an equality iff the former inequalities do so for $i = 0, 1$. So, if X is a maximal (minimal) member of $M^0_{\lambda_0}$ then X is a maximal (minimal) member of $M^3_{\lambda_3}$ and further if $M^0_{\lambda_0} = M^1_{\lambda_1}$ then $M^3_{\lambda_3} = M^1_{\lambda_1}$. The required result is now immediate.

On the same lines as the above (indeed by a line-by-line translation) we can also prove the following.

Theorem 5.6. Let $f_0(.)$, $f_1(.)$ be submodular functions on subsets of S. Let $f_0(.)$, $f_1(.)$ have the same principal lattice of partitions with decreasing sequence of critical values $\lambda_{01}, \ldots, \lambda_{0t}$ and $\lambda_{11}, \ldots, \lambda_{1t}$ respectively. Then $(f_0+f_1)(.)$ has the same principal lattice of partitions with decreasing sequence of critical values $\lambda_{31}, \ldots, \lambda_{3t}$, where $\lambda_{3i} = \lambda_{0i} + \lambda_{1i}$, $i = 1, \ldots, t$.

The next couple of results are on the principal partition of $(f(.) * g(.), g(.))$ and the principal lattice of partitions of $(f-\sigma)_t$. Once again the result about the latter and its proof are literal 'translations' of those pertaining to the former.

Definitions 5.2. Let $f(.)$ be a submodular function on subsets of S with $f(\Phi) = 0$. Let $g(.)$ be a positive weight function on S with $f(e) \le g(e)$ $\forall e \in S$. If $\{e\}$ is a separator of $f(.)$ with $f(e) = g(e)$ then e is called a coloop with respect to $g(.)$.

Theorem 5.7. Let $f(.)$ be a polymatroid rank function on subsets of S and let $g(.)$ be a positive weight function on S such that $f(e) \leq g(e)$ $\forall e \in S$. Let $p(X)$ denote $\lambda(f * g)(X) + g(S - X)$ and let $h(X)$ denote $\lambda f(X) + g(S - X)$ $\forall X \subseteq S$.

1. When $\lambda \geq 1$

- *The minimum values of $p(.)$ and $h(.)$ over subsets of S are equal. If Y minimizes $p(.)$ then it contains a subset Z that minimizes $h(.)$.*
- *Any set that minimizes $h(.)$ also minimizes $p(.)$.*

2. When $\lambda > 1$, Y minimizes $p(.)$ iff it minimizes $h(.)$.

3. There is a unique minimal set that minimizes both $p(.)$ and $h(.)$ and when $\lambda = 1$ its complement is the set of coloops of $f * g(.)$ with respect to $g(.)$.

Proof.

1. $\lambda \geq 1$. By the definition of convolution,

$$(f * g)(X) \leq f(X) \quad \forall X \subseteq S.$$

Hence, since $\lambda \geq 1$, $p(X) \leq h(X)$ $\forall X \subseteq S$ and $\min_{X \subseteq S} p(X) \leq \min_{X \subseteq S} h(X)$. Next, for any subset X of S, when $\lambda \geq 1$, we have

$$\begin{aligned} p(X) &\equiv \lambda(f * g)(X) + g(S - X) \\ &= \lambda(f(Z) + g(X - Z)) + g(S - X) \quad \text{for some } Z \subseteq X, \\ &\geq \lambda f(Z) + g(S - Z) \equiv g(Z). \end{aligned}$$

We conclude that

$$\min_{X \subseteq S} p(X) = \min_{X \subseteq S} h(X)$$

and that any set that minimizes $p(.)$ contains a subset that minimizes $h(.)$. Let m denote this minimum value. Suppose Y minimizes $h(.)$. We then have

$$m = \lambda f(Y) + g(S - Y) \geq \lambda(f * g)(Y) + g(S - Y) \geq m.$$

Thus, Y must minimize $p(.)$.

2. $\lambda > 1$: We need to show that if Y minimizes $p(.)$ it also minimizes $h(.)$. We claim that in this case $f * g(Y) = f(Y)$, from which it would follow that $m = h(Y)$. Suppose otherwise. Then we must have

$$\begin{aligned} m = p(Y) &\equiv \lambda(f * g(Y)) + g(S - Y) \\ &= \lambda(f(Z) + g(Y - Z)) + g(S - Y) \quad \text{for some } Z \subset Y, \\ &> \lambda(f(Z)) + g(S - Z) \equiv h(Z) \geq m, \end{aligned}$$

which is a contradiction. Thus, we must have $f * g(Y) = f(Y)$ and hence Y minimizes $h(.)$.

3. Since $p(.)$ is clearly submodular (it is the sum of the submodular function $\lambda(f * g)(Y)$ and the submodular function $g(S - Y)$), we must have the minimal minimizing set to be unique since the minimizing sets of $p(.)$ are precisely the minimizing sets in the principal partition of $(f * g(.), g(.))$ and property PP1 can be used. From the first part of the present theorem it follows that this set is also the unique minimal set minimizing $h(.)$. Consider the situation when $\lambda = 1$. Let X minimize $f * g(Y) + g(S - Y)$ and let $Z \supseteq X$. We show that Z also minimizes this expression. We have $f * g(X) + f * g(Z - X) \geq f * g(Z)$ ($f * g(.)$ is submodular by Theorem 3.1 and its value on the null set is zero). So $f * g(X) + g(Z - X) \geq f * g(X) + f * g(Z - X) \geq f * g(Z)$. Hence $f * g(X) + g(S - X) \geq f * g(Z) + g(S - Z)$. Thus, Z and, in particular, S minimizes the expression as claimed. Further since $f(e) \leq g(e)$ $\forall e \in S$, $f * g(Z) + g(S - Z) \geq f * g(Z) + f * g(S - Z)$. The latter is greater or equal to $f * g(S)$. The only way these inequalities can be satisfied is by having equalities throughout. Thus, every superset of X is a separator of $f * g(.)$, from which we conclude that $S - X$ must be a set of coloops of $f * g(.)$. Next if K is a set of coloops of $f * g(.)$, we have $f * g(S - K) + g(K) = f * g(S - K) + f * g(K) = f * g(S)$. This completes the proof.

Now the PLP version.

Theorem 5.8. Let $f(.)$ be a submodular function on subsets of S. Let $p(.)$ denote $((f - \sigma)_t - \lambda)(.)$ and let $h(.)$ denote $(f - (\sigma + \lambda))(.)$. Then:

1. When $\lambda \geq 0$

- *The minimum values of $\overline{p}(.)$ and $\overline{h}(.)$ over partitions in P_S are equal. If Π minimizes $\overline{p}(.)$ then there exists a finer partition Π' that minimizes $\overline{h}(.)$.*
- *Any partition that minimizes $\overline{h}(.)$ also minimizes $\overline{p}(.)$.*

2. When $\lambda > 0$, Π minimizes $\overline{p}(.)$ iff it minimizes $\overline{h}(.)$.

3. There is a unique minimal partition that minimizes both $\overline{p}(.)$ and $\overline{h}(.)$ and when $\lambda = 0$ its blocks are the elementary separators of $(f - \sigma)_t(.)$.

The next couple of results are on the PP and PLP associated with duals.

Definition 5.3. Let $f(.)$ be a submodular function on the subsets of S and let $g(.)$ be a positive weight function on S. The comodular dual of $f(.)$ with respect to $g(.)$ is denoted by $f^*(.)$ and defined through $f^*(.) \equiv g(X) - (f(S) - f(S - X))$ $\forall X \subseteq S$.

Remark 5.1. If $f(\Phi) = 0$ then $(f^*)^*(.) = f(.)$. If $f(.)$ is a polymatroid rank function and $f(e) \leq g(e)$ $\forall e \in S$ the $f^*(.)$ is also a polymatroid rank function.

Theorem 5.9. Let $f(.)$ be a polymatroid on the subsets of S and let $g(.)$ be a positive weight function on S. Let M_λ, M^*_λ denote, respectively, the collection of minimizing sets corresponding to λ in the principal partitions of $(f(.), g(.))$, $(f^*(.), g(.))$, where $f^*(.)$ denotes the comodular dual of $f(.)$ with respect to $g(.)$. Let $d(\lambda)$ denote $(1-(\lambda)^{-1})^{-1}$ $\forall \lambda \in \Re$. Then:

1. *A subset X of S is in M_λ iff $S-X$ is in $M^*_{d(\lambda)}$.*
2. *If $\lambda_1, \ldots, \lambda_t$ is the decreasing sequence of critical values of $(f(.), g(.))$ then $d(\lambda_t), \ldots, d(\lambda_1)$ is the decreasing sequence of critical values of $(f^*(.), g(.))$.*

Proof. We begin by observing that $f^{**}(.) = f(.)$ and if $g(e) \geq f(e)$ $\forall e \in S$ then we must have $_g(e) \geq f^*(e)$ $\forall e \in S$.

1. We will show that Y minimizes $\lambda f(X) + g(S-X)$ iff $S-Y$ minimizes $d(\lambda) f^*(X) + g(S-X)$. We have

$$d(\lambda) f^*(X) + g(S-X) = d(\lambda)\left[\sum_{e\in X} g(e) - (f(S) - f(S-X))\right] + g(S-X)$$
$$= d(\lambda) f(S-X) + (d(\lambda)-1) g(X) - d(\lambda) f(S) + g(S).$$

This is equivalent to minimizing the expression $d(\lambda)(d(\lambda)-1)^{-1} f(S-X) + g(X)$. Noting that $d(\lambda)(d(\lambda)-1)^{-1} = \lambda$ we get the desired result.

The corresponding result for PLP is immediate from the definition of PLC (see Remark 4.2, Lemma 4.1, Theorem 5.4).

Theorem 5.10. Let $f(.)$ be submodular on subsets of S and let $f^*(.)$ be its comodular dual with respect to the weight function $g(.)$. Then Π belongs to the PLP of $f(.)$ $(f^*(.))$ iff Π^* belongs to the PLC of $f^*(.)$ $(f(.))$. Further, λ is a critical value of the PLP of $f(.)$ $(f^*(.))$ iff $\lambda + f(S)(\lambda + f^*(S))$ is a critical value of the PLC of $f^*(.)$ $(f(.))$.

We sketch the nature of the algorithms[19, 20, 30] for constructing the PP of $(f(.), g(.))$ and the PLP of $f(.)$, where $f(.)$ is submodular and $g(.)$ is a positive weight function. In both cases we make use of the following idea: Suppose we have found two sets $Y_1 \subseteq Y_2$ (partitions $\Pi_1 \leq \Pi_2$) in the PP (PLP). Suppose these minimize $\lambda_1 f(X) + g(S-X)$, $\lambda_2 f(X) + g(S-X)$ $(\overline{f-\lambda_1}(\Pi), \overline{f-\lambda_2}(\Pi))$. If these minimize $\lambda f(X) + g(S-X)(\overline{f-\lambda}(\Pi))$ then $\lambda(f(Y_2) - f(Y_1)) = (g(Y_2) - g(Y_1))$ $(\bar{f}(\Pi_2) - \bar{f}(\Pi_1) = \lambda(|\Pi_2| - |\Pi_1|))$. So in order to find more critical values, if they exist, we minimize appropriate expressions involving λ, calculated as above. If the values on the two sets (partitions) turn out to be the same as this minimum value, no further critical value can be found between λ_1 and λ_2. Repetition of this process gives us the list of critical values and, after pruning, the maximal and minimal sets (partitions) corresponding to these critical values. For each critical value the collection of all sets (partitions) which minimize the appropriate expression is found essentially by repeated minimization using the partial order (multiple partial order) representation of the distributive lattice (partition lattice) of minimizing sets (partitions). In the PP case, each minimization is of a submodular function. In many practical situations this reduces to a min cut problem or transforms to a matroid union problem. In the PLP case the key step is to find a fusion set, *i.e.*, a nonsingleton set which is contained in a block of the minimizing partition. This involves a submodular function minimization-cut minimization in many practical problems. Once such a set is found, it is 'fused' into a single element and we work with a 'fused' submodular function which agrees with the previous function modulo an appropriate weight function in all supersets of the fusion set. Repetition of this process ultimately leads to a situation where there are no fusion sets, which means the partition into singletons is the required minimizing partition. This partition is blown up enlarging the fusion sets in the reverse order in which they were formed to get the minimizing partition corresponding to the original submodular function.

6. Conclusions

We have described a number of instances in the literature where the notions of convolution and Dilworth truncation are implicitly or explicitly used both for generating new results as well as for applications. Although the truncation operation was explicitly described much earlier in the literature, it is curious that theoretical and practical applications of the convolution operation have been studied so far with much greater thoroughness. In our view, research into the former operation promises to be equally fruitful, particularly in view of the strong analogies that, as we have shown in this paper, exist between the principal partition and the principal lattice of partitions.

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Appendix

Some proofs omitted in the main text

Proof of Theorem 5.4.

1. A partition $\Pi \in P_S$ minimizes $\overline{f-\lambda}(.)$ iff it minimizes $\overline{\beta f-\beta\lambda}(.)$. The result follows.

2. If $g(.)$ is a weight function and $\Pi \in P_S$ then
$$\overline{f+g-\lambda}(\Pi)=\overline{f-\lambda}(\Pi)+g(S).$$

The result follows.

Proof of Theorem 5.6. Let $L^0_\lambda, L^1_\lambda, L_\lambda$ denote the collection of minimizing partitions corresponding to λ in the principal lattice of partitions of $f_0(.)$, $f_1(.)$, $f_0(.)+f_1(.)$, respectively. Let Π be a partition in $L^0_{\lambda_0}$ as well as in $L^1_{\lambda_1}$. We claim that $\Pi \in L^3_{\lambda_3}$, where $\lambda_3 = \lambda_0 + \lambda_1$. We have

$$\overline{f_i - \lambda_i}(\Pi) \leq \overline{f_i - \lambda_i}(\Pi') \quad \forall \Pi' \in P_S.$$

Hence,

$$\overline{f_0 + f_1 - (\lambda_0 + \lambda_1)}(\Pi) \leq \overline{f_0 + f_1 - (\lambda_0 + \lambda_1)}(\Pi') \quad \forall \Pi' \in P_S.$$

This proves the claim. In the above proof note that the final inequality reduces to an equality iff the former inequalities do so for $i = 0, 1$. So, if Π is a maximal (minimal) member of $L^0_{\lambda_0}$ then it is also a maximal (minimal) member of $L^3_{\lambda_3}$ and, further, if $L^0_{\lambda_0} = L^1_{\lambda_1}$ then $L^3_{\lambda_3} = L^1_{\lambda_1}$. The required result is now immediate.

Proof of Theorem 5.8.

1. By the definition of truncation,

$$\overline{(f - \sigma)_t}(\Pi) \leq \overline{f - \sigma}(\Pi) \quad \forall \Pi \in P_S.$$

Hence, $\bar{p}(\Pi) \leq \bar{h}(\Pi)$ $\forall \Pi \in P_S$ and $\min_{\Pi \in P_S} \bar{p}(\Pi) \leq \min_{\Pi \in P_S} \bar{h}(\Pi)$. Next, for any partition Π of S, when $\lambda \geq 0$, we have

$$\bar{p}(\Pi) \equiv \overline{(f - \sigma)_t}(\Pi) - \lambda|\Pi| = \overline{f - \sigma}(\Pi_1) - \lambda|\Pi|$$

and, for some $\Pi_1 \leq \Pi$,

$$\bar{p}(\Pi) \geq \overline{f - \sigma}(\Pi_1) - \lambda|\Pi_1| = \bar{h}.$$

We conclude that $\min_{\Pi \in P_S} \bar{p}(\Pi) = \min_{\Pi \in P_S} \bar{h}(\Pi)$ and that if Π minimizes $\bar{p}(.)$ then there exists a finer partition Π' that minimizes both $\bar{h}(.)$ and $\bar{p}(.)$. Let m denote this minimum value. Suppose Π minimizes $\bar{h}(.)$. We then have

$$m = \overline{f - \sigma}(\Pi) - \lambda|\Pi| \geq \overline{(f - \sigma)_t}(\Pi) - \lambda|\Pi| \geq m.$$

Thus, Π minimizes $\bar{p}(.)$.

2. $\lambda > 0$: We need to show that if Π minimizes $\bar{p}(.)$ then it also minimizes $\bar{h}(.)$. We claim that in this case

$$\overline{(f - \sigma)_t}(\Pi) = \overline{f - \sigma}(\Pi),$$

from which it would follow that $m = \bar{h}(\Pi)$. Suppose otherwise. Then we must have

$$m = \overline{f - \sigma}(\Pi) - \lambda|\Pi| = \overline{(f - \sigma)_t}(\Pi_1) - \lambda|\Pi|,$$

and, for some $\Pi_1 < \Pi$,

$$m > \overline{(f-\sigma)_t}(\Pi_1) - \lambda|\Pi_1| \geq m,$$

which is a contradiction. Thus, we must have

$$\overline{(f-\sigma)_t}(\Pi) = \overline{f-\sigma}(\Pi),$$

and that Π minimizes $\overline{f-\sigma}(.)$.

3. Since $p(.)$ is clearly submodular, we must have the minimal minimizing partition to be unique since the minimizing partitions of $\overline{p}(.)$ are precisely the minimizing partitions in the principal lattice of partitions of $(f-\sigma)(.)$ and property PLP1 can be used. From the first part of the present theorem it follows that this partition is also the unique minimal partition that minimizes $\overline{h}(.)$. Consider the situation when $\lambda = 0$. Let Π minimize $\overline{p}(.)$. Now if M is any union of blocks of Π, we have, by the submodularity of $(f-\sigma)_t$ and the fact that it takes value zero on Φ, $(f-\sigma)_t(M) \leq \Sigma(f-\sigma)_t(N_i)$, where N_i are the blocks of Π contained in M. Thus, if $\Pi' \geq \Pi$ then Π' (in particular, Π_S) also minimizes $\overline{p}(.)$. It follows that the blocks of $\Pi_{\min}$, the minimal minimizing partition of $\overline{p}(.)$ must be separators of $(f-\sigma)_t$. On the other hand, if Π has its blocks as separators of $(f-\sigma)_t$, by the definition of separators, we must have $\overline{(f-\sigma)_t}(\Pi) = \overline{(f-\sigma)_t}(\Pi_S)$. This completes the proof.