J. Indian Inst. Sci., Mar.–Apr. 1995, 75, 141–174.

A survey of solution concepts in multicriteria games

L. ANAND[†], N. SHASHISHEKHAR*, D. GHOSE[†] AND U. R. PRASAD*
[†]Department of Aerospace Engineering, *Department of Computer Science and Automation, Indian Institute of Science, Bangalore 560 012, India.

Received on December 16, 1993; Revised on June 17, 1994.

Abstract

In this paper, a survey of the main results in the theory of multicriteria games is presented. The primary objective of the paper is to trace the major developments of the theory in four directions: (*i*) approachability–excludability theory, (*ii*) multicriteria cooperative games, (*iii*) equilibrium solutions in multicriteria games, and (*iv*) security strategies in multicriteria games. Potential applications of the thoery of problems of practical interest are also discussed. The paper also discusses several possible future directions of research.

Keywords: Games with vector payoffs, multicriteria games, non-cooperative games, cooperative games, approachability–excludability theory, security strategies, Pareto optimality, equilibrium strategies, dominance structures.

1. Introduction

Multicriteria optimization forms a substantial part of the general optimization theory. Its evolution was motivated by the need to model multiple objectives of a decision maker and identify acceptable decisions and solution concepts. Since the late fifties till present time this line of research has spawned an enormous number of books and research monographs, in addition to papers, special issues, and survey articles in scientific periodicals. The primary reason for the popularity of such a topic is that decisions in real life are seldom motivated by a single objective or goal. Usually, any decision has several different (and sometimes apparently incompatible) consequences and therefore has to be evaluated against several different criteria.

Another important extension of optimization theory is known as the theory of games[1], which models and analyses conflict situations involving more than one decision maker each with his own objective function to optimize. This theory is motivated by the fact that many real-life situations are influenced by the decisions of more than one decision maker. Perhaps the most obvious example of this kind of situation is the economic market place which is influenced by the decisions of several producers and consumers.

The theory of multicriteria games (also known as games with vector payoffs) is a confluence of game theory and multicriteria optimization. It is applicable to a situation in which the system of interest in influenced by more than one decision maker (or player) each of them having more than one objective to fulfil. A general model of such a game will consist of n players, each having an l_i-dimensional payoff vector with $i = 1, 2, \ldots, n$. The payoff spaces of the ith players (Pi) will be l_i-dimensional and may not have any payoff in common with other players. The game could be zero-sum or nonzero-sum, cooperative or non-cooperative, and modelled in normal form or in extensive form, in the same sense as in single-criterion games, which is the domain of conventional game theory. Depending on the application, the theory of multicriteria games has evolved in four main directions: (*i*) approachability–excludability theory, (*ii*) cooperative games, (*iii*) equilibrium solutions, and (*iv*) security strategies. Most of these developments have adopted a normal-form game as their basic model and proposed feasible solution concepts.

The published literature of multicriteria game theory is somewhat scattered and important contributions have appeared in journals of engineering science, economics, optimization, statistics, and pure mathematics. Renewed interest in the theory of games (especially in microeconomics literature) in recent years has been triggered by the realization that realistic game-theoretic models can explain, to a large extent, a decision makers' real-life behaviour and also many previously unexplained market phenomena. It is felt that the theory of multicriteria games, by the very fact of its realistic premise, can effectively embellish these results and make them more acceptable. Hence, collating the various developments in this area at one place will be of help to researchers in the area of multicriteria games.

The objective of this paper is to trace the development of multicriteria games through each of the above approaches. One of the major objectives here is to identify links (if they exist) between the four different approaches and also with other branches of game theory which are of current interest. It is also attempted to identify potential future directions of research and some non-trivial open problems in this area.

2. The approachability–excludability theory

2.1. *Introduction*

The intuitively appealing elegance of von Neumann's minimax theorem for single criterion zero-sum games had initially led researchers to welcome it as the long-awaited solution to general conflict-resolution problems. That it was not so became clear as the theory failed to give satisfactory solutions to most real-life problems modelled in the game-theoretic framework. Some researchers even went so far as to suggest[2] that the importance given to the minimax theorem has actually hindered developments in game theory. This set the stage for the emergence of the theory of nonzero-sum games and its general acceptability. However, the criticism levelled against the minimax theorem was somewhat unjustified since there is always some amount of pure antagonism between players in any game and this is the situation that zero-sum games attempt to model. In this context, the minimax theorem can be considered to be one of the vital cornerstones in the theory of games. This conviction has led to several generalizations and extensions of the minimax theorems.

One such extension is Blackwell's[3] approachability–excludability theory, which was motivated by the desire to obtain an analogous result for multicriteria zero-sum games. In recent times this theory has moved away from the confines of pure mathematical theory and has been applied to many practical problems. The essential idea behind this theory is to define a set (a desirable set) in the payoff space and say whether or not, through repeated play of a zero-sum game, a player can force the average payoffs to approach this set. Similarly, it is also required to know whether or not, through repeated play of a zero-sum game, a player can force the average payoff to approach this set. Similarly, it is also required to know whether a player can avoid a similar set (an undesirable one) in the payoff space.

In repeated games with vector payoffs, the extent to which a player controls the trajectory of the average payoff (that is, of the centre of gravity of the actual payoffs) determines the control the player has over the game. In a scalar game, von Neumann's minimax theorem gives an idea of the control each player could have over the repeated game.

In Section 2.2, we discuss Blackwell's definition of approachability and excludability and the sufficiency condition (which becomes necessary under certain conditions) for approachability. Blackwell assumes that the payoff space is bounded. Extending the results to approachability this condition is relaxed in Section 2.3. In Section 2.4, the notions of weak approachability and weak excludability are introduced and the collection of sets that are weakly approachable are studied. Approachability and excludability theory in infinitely repeated games is now generalized to a stochastic game and discussed in Section 2.5. How the information available to the players influences the class of approachable sets is discussed in Section 2.6. Section 2.7 provides a brief discussion of a few of the applications of approachability–excludability theory.

2.2. *The basic results*

A finite two-person zero-sum game is represented by an rxs matrix, $M = \{(m_{ij})\}_{rxs}$, each element of which is a probability distribution over a closed bounded convex set X in N-dimensional Euclidean space. The strategy for the minimizing player, $P1$, is defined as a sequence $f = \{f_n\}$, $n = 0, 1, 2, \ldots$ of functions where f_n is a mapping from the set of n-tuples $(x_1, x_2, \ldots, x_n)$, $x_i \varepsilon X$ to the set p of vectors, $p = (p_1, p_2, \ldots, p_r)$ with $p_i \geq 0$ for all i and $\Sigma_{i=1}^{r} p_i = 1$. Here p is a vector of probability measures on the set of pure strategies of $P1$, with p_i denoting the probability with which the ith pure strategy is chosen by $P1$. Similarly, the strategy for the maximising player, $P2$, is defined as a sequence $g = \{g_n\}$, $n = 0, 1, 2, ..$ of functions, where g_n is a mapping from the set of n-tuples $(x_1, x_2, \ldots, x_n)$, $x_i \varepsilon X$ to the set Q of vectors $q = (q_1, q_2, \ldots, q_s)$, with $q_j \geq 0$ for all j and $\Sigma_{j=1}^{s} q_j = 1$. Like p, q is a vector of probability measures on the set of pure strategies of $P2$, with q_j being the probability with which the jth pure strategy is chosen by $P2$. In the above formulation, obviously, the past actions of each player are known to both.

Blackwell[3] raises the following question—"Given a matrix M and a set S in an N-dimensional space, can $P1$ guarantee that the average payoff is forced arbitrarily close to S, with probability approaching 1 as the number of plays become infinite?" He terms such a property of set S as approachability (by $P1$). Similarly, the property of ensuring (by $P2$) that the average payoff lies outside S by a certain positive distance, no matter how many times the game is played, is termed as excludability. These terms could be formally described as under.

Let S be any set in the N-dimensional space. S is said to be *approachable* in M, if there exists f^* such that for every $\varepsilon > 0$ there exists an N_0 such that, for every g, Prob$\{\delta_n \geq \varepsilon$ for some $n \geq N_0\} < \varepsilon$, where δ_n is the distance of the point $\overline{X}_n = \Sigma_{i=1}^{n}(x_i/n)$ from S, $x_1, x_2, \ldots \varepsilon X$ are the variables determined by f^*, g. On the other hand, S is said to be *excludable* in M if there exists a g^* such that there exists a $d > 0$ such that for every $\varepsilon > 0$ there is an N_0 such that, for every f, Prob$\{\delta_n \geq d$ for all $n \geq N_0\} > 1-\varepsilon$, where $x_1, x_2, \ldots \varepsilon X$ are variables determined by f, g^*. Approachability and excludability are the same for S and its closure and so without loss of generality we can assume, S to be closed. Clearly, any superset of an approachable set is approachable and any subset of an excludable set is excludable.

The payoffs, in general, are vectors belonging to the N-dimensional Euclidean space. When $N = 1$, it reduces to games with scalar payoffs. In such a game with $N = 1$, associated with every M are a number v and vectors $p \varepsilon P$, $q \varepsilon Q$ such that the set $S = \{x \geq t\}$ is approachable for $t \leq v$ with $f : f_n \equiv p$ and excludable for $t \leq v$ with $g : g_n \equiv q$. Here S is considered to be a convex set. The result that S is either approachable or excludable for S nonconvex also holds so long as $N = 1$. More of this will be seen later. It will also be shown that for $N > 1$, S is either approachable or excludable, if S is convex. To reiterate the fact that it is not true for S convex when $N > 1$, Blackwell[3] gives an illustrative example with $N = 2$ and S nonconvex.

Example 2.1. Let $r = s = 2$; $m(1, 1) = m(1, 2) = (0, 0)$, $m(2, 1) = (1, 0)$, $m(2, 2) = (1, 1)$. $S = I_1 \cup I_2$, where I_1 is the set of points $(\frac{1}{2}, y)$, $0 \leq y \leq \frac{1}{4}$ and I_2 is the set of points $(1, y)$, $\frac{1}{4} \leq y \leq 1$. Suppose $f_j \equiv (0, 1)$ for $j \leq n$ so that $\overline{x}_n = (1, u)$, $0 \leq u \leq 1$. If we consider the following strategy for $P1$: for $j > n$, $f_j \equiv (0, 1)$ if $u \geq \frac{1}{2}$, else $f_j \equiv (1, 0)$, then for $u \geq \frac{1}{2}$, $\overline{x}_{2n} \varepsilon I_2$ and for $u < \frac{1}{2}$, $\overline{x}_{2n} \varepsilon I_1$. So, given any N_0 we can find some $n > N_0$ such that $\overline{x}_n \varepsilon S$. So S is not excludable. On the other hand, let $\overline{x}_n = (a_n, b_n)$ for some n. If $P2$ employs the following strategy: if $a_n \geq \frac{3}{4}$, $g_n \equiv (1, 0)$, else $g_n \equiv (0, 1)$, then, as $n \to \infty$ (a_n, b_n) tends either to the line joining $(0, 0)$ and $(\frac{3}{4}, \frac{3}{4})$ or to the line joining $(\frac{3}{4}, 0)$ and $(1, 0)$ but never stays near S. Thus, S is not approachable.

For the matrix M, let $\overline{M}$ denote the matrix whose elements $\overline{m}\ (i, j)$ are the respective means of $m(i, j)$ of M. If $p \varepsilon P$ is a strategy of $P1$ then $\Sigma_{i=1}^{r} p_i \overline{m}\ (i, j)$, $1 \leq j \leq s$ are the s expected pure-strategy payoffs for $P2$ when $P1$ plays p. The convex hull, $R(p)$, of the s points is the region in which $P2$'s average payoffs are expected to lie for any mixed strategy of $P2$ corresponding to the mixed strategy p of $P1$. Similarly, we could define for $q \varepsilon Q$. $T(q)$ as the convex hull of the r points $\Sigma_{j=1}^{s} q_j \overline{m}\,(i, j)$, $1 \leq i \leq r$. $T(q)$ is the convex region in which $P1$'s average payoffs are expected to lie for any mixed strategy of $P1$ corresponding to the mixed strategy q of $P2$.

Blackwell[3] gives a sufficient condition for approachability, which under certain conditions also becomes necessary.

Theorem 2.1. *Let S be any closed set. If for every $x \notin S$ there is a $p(x) \in P$ such that the hyperplane through y, the closest point to x in S, perpendicular to the line segment xy, separates x from R(p), then S is approachable with the strategy $f: f_n$, where*

$$f_n = \begin{cases} p(\bar{x}_n) \text{ if } n > 0 \text{ and } \bar{x}_n = \sum_{i=1}^{n} (x_i/n) \notin S, \\ \text{arbitrary if either } n = 0 \text{ or } \bar{x}_n \in S. \end{cases}$$

If y_n is the point in S closest to $\bar{x}_n$ then we can define $\delta_n = |\bar{x}_n - y_n|^2$. When the hypotheses of the theorem are satisfied, the sequence $\{\delta_n\}$ also satisfies the following:

$$E[\delta_n, \delta_1, \delta_2, \ldots, \delta_{n-1}] \le \delta_{n-1}(1 - 2/n) + c/n^2. \tag{1}$$

$$0 \le \delta_n \le a, \tag{2}$$

$$\delta_n - \delta_{n-1} \le b/n. \tag{3}$$

where a, b and c are constants. To prove the theorem, we need to show that a sequence $\{\delta_n\}$ satisfying (1), (2) and (3) above, tends to zero as n becomes infinite. This follows from the lemma stated below.

Lemma 2.1. *A sequence of chance variables $\delta_1, \delta_2, \ldots$ satisfying* (1), (2) *and* (3) *converges to zero with probability* 1 *at a rate depending only on a, b and c; that is, for every $\varepsilon > 0$, there is an N_0 depending only on ε, a, b and c, such that for any $\{\delta_n\}$ satisfying* (1), (2) *and* (3), *we have Prob $\{\delta_n \ge \varepsilon$ for some $n \ge N_0\} < \varepsilon$.*

If a closed set S is approachable in the $s \times r$ matrix M'(the transpose of M) then any closed set S_1 not intersecting S is excludable in M with any strategy with which S is approachable in M'. Hence, it follows that the sufficient condition for approachability in Theorem 2.1 also results in a sufficient condition for excludability.

The theorem gives only a sufficient condition for any closed set S to be approachable. Under certain other constraints, there could be a necessary condition too.

Theorem 2.2. *A closed convex set S is approachable by P*1 *if and only if it intersects $T(q)$ for every $q \in Q$. If it fails to intersect $T(q_0)$ for some $q_0 \in Q$, then S is excludable by P2 with $g: g_n \equiv q_0$.*

Suppose S intersects $T(q)$ for all $q \in Q$; let $x_0 \notin S$ and $y \in S$ be the point in S which is closest to x_0. For S convex, it is possible to derive the value of the game with matrix $A = \{(a_{ij})\}$, $a_{ij} = \langle y - x_0, \bar{m}(i, j)\rangle$ ($\langle x, y\rangle$ denotes the inner product of x and y), to be

$$\min_q \max_i \left\langle y - x_0, \sum_{j=1}^{s} q_j \bar{m}(i, j) \right\rangle \ge \min_{s \in S} \langle y - x_0, s\rangle.$$

Consequently, there exists a $p \in P$ such that for all $r = \Sigma_{i=1}^{r} p_i \bar{m}(i, j) \in R(p)$, $\langle y - x_0, r\rangle \ge \langle y - x_0, x_0\rangle$. It is thus possible to find a hyperplane $\langle y - x_0, x\rangle = \langle y - x_0, y\rangle$ that separates x_0 from $R(p)$. From Theorem 2.1, it follows that S is approachable.

On the other hand, $T(q_0)$, for any $q_0 \varepsilon Q$, satisfies the hypotheses of Theorem 2.1 in M' with $f : f_n \equiv q_0$ and so is approachable in M' with this f. Consequently, if S fails to intersect $T(q_0)$ for some $q_0 \varepsilon Q$, S is excludable in M with $g : g_n \equiv q_0$.

From Theorem 2. 2, it follows that if S is convex, then S is either approachable by one player or is excludable by the other player.

Corollary 2. 1. *A closed convex set S is approachable if and only if, for every vector* $u \varepsilon X$, $v(u) \geq \min_{s \varepsilon S} \langle u, s \rangle$, *where* $= v(u)$ *is the value of the game with matrix* $B = \{(b_{ij})\}$, $b_{ij} = \langle u, \bar{m}(i, j) \rangle$.

Using this corollary, a result is proved for the case $N = 1$ that even if S is nonconvex, S is either approachable or excludable.

Theorem 2. 3. *For* $N = 1$, *let* v *and* v' *be the values of the games with matrices* M *and* M', *respectively. If* $v' \leq v$, *a closed set S is approachable if it intersects the closed interval* $v'v$ *and excludable otherwise. If* $v' \geq v$, *a closed set S is approachable if it contains the closed interval* vv' *and excludable otherwise.*

When the result of Corollary 2. 1 is applied to a closed interval AB, $A < B$, with $u = \pm 1$, it follows that AB is approachable if and only if $v \geq A$ and $-v' \geq -B$. If $v' \leq v$, these conditions imply that AB has to intersect the closed interval $v'v$, and if $v' \geq v$, the above conditions imply that AB should contain vv'. Thus, if $v' \leq v$, then any set S intersecting with $v'v$ is approachable in M, and if S does not intersect with $v'v$, it is excludable in M'. If $v' \geq v$, any point in vv' is approachable in M so that any closed set not containing vv' is disjoint from the approachable set in M and hence is excludable in M'.

In the example of Blackwell discussed earlier, for $N = 2$, it was seen that S is neither approachable not excludable, because of the nonconvexity of set S. If the definitions of approachability and excludability are relaxed to weak approachability and weak excludability, respectively, then for $N \geq 1$, any set S(whether convex or not) is either weakly approachable in a matrix game M or weakly excludable. This is discussed in Section 2.4.

2.3. *Extensions of the basic results*

Blackwell[3] assumes that the entries (m_{ij}) of the matrix M are probability distributions over a closed bounded convex set X of the N-dimensional Euclidean space. Hou[4] relaxes the constraint of boundedness of X, clamping instead the weaker constraint that the means of the distributions $(m_{ij})_{ij}$ are finite (that is, for $(m_{ij})_{ij}$, $E[\,\cdot\,]^{\alpha}$ is bounded by some $K' < \infty$, for some $\alpha > 1$). He states and proves that for a game with matrix M, the class of approachable sets for a player depends only on $\bar{M}$. Sackrowitz[5] confirms these results subject only to the condition that the mean vectors exist.

Let Ω be the convex hull of the rs elements of $\bar{M}$, and K, its bound. Given a set S, B is defined to be an *insufficient subset* of S if

(i) there exists an open set $U(B)$ such that $S \cap U(B) = B$.

(ii) a d can be found such that if $x_n \varepsilon U(B)$ for some $n \geq 10\, K/d$, then there is a series of strategies $q_{n+1}, q_{n+2},\ldots$ for the second player such that irrespective of the strategy of the first player, for some integer $m > n$, $\mathrm{Prob}\{\delta(x_m, S) \geq d\} = 1$.

If the first of the above condition is to be satisfied then, if $B \subseteq \text{int}(S)$ (where int(S) denotes the interior of the set S), then B is open; else, that part of B which is contained in int(S) is open and for the remainder we can find some open set $U(B)$ that contains B and satisfies the second condition. According to the second condition, whenever the centre of gravity, $\bar{x}_n$, enters $U(B)$, the second player has some strategy, irrespective of the strategy adopted by the first player, by which $\bar{x}_m$ can be moved away from S such that $\delta(\bar{x}_m, S) \geq d$ for some $m > n$ and $d > 0$.

If $\mathcal{B}$ is the collection of all insufficient subsets B of S, let $B^* = \bigcup_{B \in \beta} B$. Then a set $\tilde{S} = S/B^*$ (that is, the set of all elements in S but not belonging to B^*) is called the *sufficient subset* of S. $\tilde{S}$ does not contain any insufficient subsets. Note that, using Blackwell's[3] notation, a sufficient subset should be such that it intersects $T(q)$ for every $q \in Q$. Hou[4] gives the necessary and sufficient condition for approachability through the following theorem.

Theorem 2.4. *A set $S \subseteq R^N$ is approachable by player P1 if and only if $\tilde{S}$, the sufficient subset of $\bar{S} \cap \Omega$, is nonempty.*

Without loss of generality, S can be considered to be a closed subset of Ω (see Hou[4]) and hence $\bar{S} \cap \Omega = S$. From the theorem and the earlier definitions it follows that if $\tilde{S}$ is nonempty then $\tilde{S} \not\subseteq \text{int}(S)$, for if $\tilde{S} \subseteq \text{int}(S)$ then there is no way of approaching $\tilde{S}$, thereby making $\tilde{S}$ and hence S unapproachable (if $\tilde{S} \subseteq \text{int}(S)$ then $\tilde{S}$ is surrounded by B^*, which means that whenever $\tilde{S}$ is to be approached, $\bar{x}_n$ must pass through B^*, thereby making S unapproachable).

The following examples illustrate sufficient and insufficient subsets and the condition for approachability. For the following set of examples, the matrix $\bar{M}$ considered is $\bar{m}_{11} = (1, 1)$, $\bar{m}_{21} = (1, 0)$, $\bar{m}_{12} = \bar{m}_{22} = (0, 0)$. The average payoff lies only in the closed region enclosed by the triangle (0, 0), (1, 0) and (1, 1).

Example 2.2. Let I_1^1 be the line segment OA and I_1^2 be AB as in Fig. 1(a). Let a set S_1 be defined as $S_1 = I_1^1 \cup I_1^2$. It can be seen that I_1^2 is an insufficient subset of S_1 and the suf-

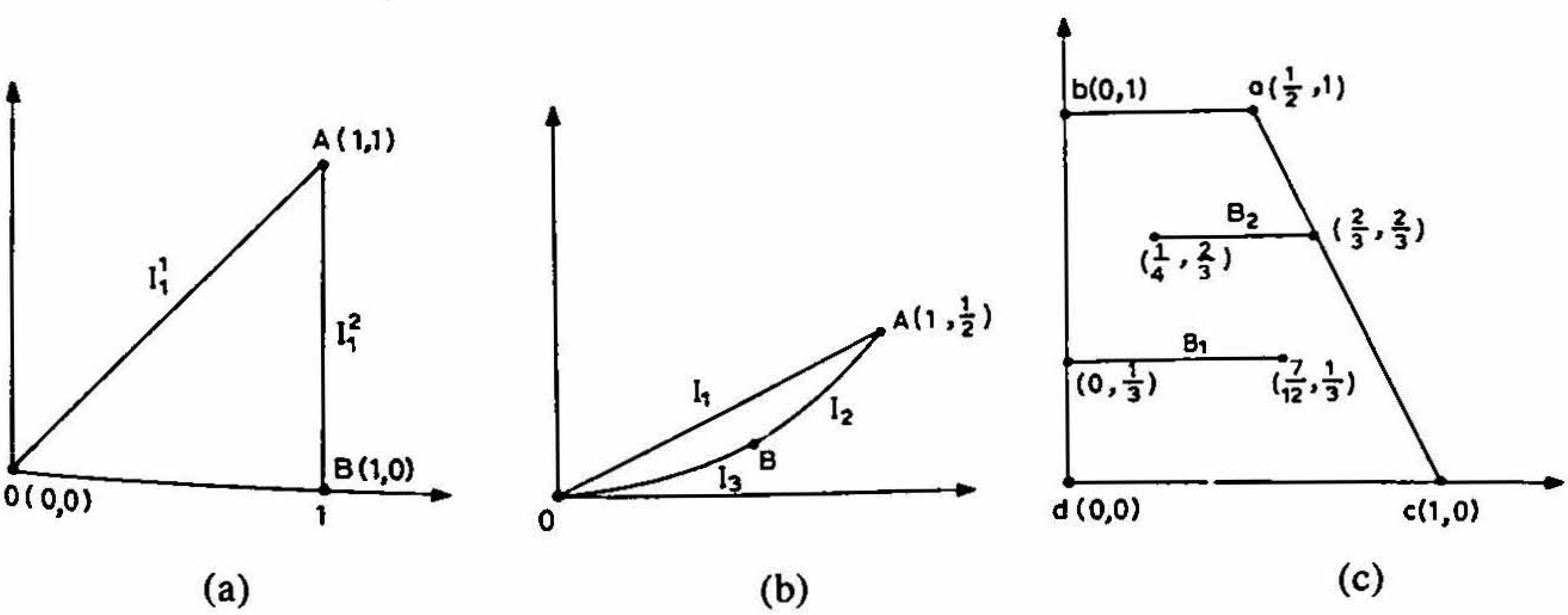

FIG. 1. Examples (a) 2.2, (b) 2.3, and (c) 2.4.

ficient subset $\tilde{S}_1$ is $\tilde{S}_1 = I_1^1$. Since $\tilde{S}_1$ is nonempty, S_1 is approachable. I_1^1 is $R(p)$ for the strategy $p = (1, 0)$ of the first player. So I_1^1 is approachable and every superset S_1 of I_1^1 is approachable.

Example 2.3. Considering I_1, I_2 and I_3 as shown in Fig. 1(b), let $S_2 = I_1 \cup I_2$. I_2 is an insufficient subset of S_2 while I_1 is again an $R(p)$ for the strategy $p = \left(\frac{1}{2}, \frac{1}{2}\right)$ for the first player and is a sufficient subset of S_2. Hence, S_2 is approachable. Define $S_3 = I_2 \cup I_3$. S_3 is approachable and contains no insufficient subsets. S_3 does not satisfy the sufficiency condition of Blackwell[3] but is still approachable (for any $\bar{x}_n$ above S_3, the sufficiency condition is not satisfied).

2.4. *Weak approachability*

According to the definition of approachability, a set S in R^N is approachable if there exists an N_0 such that the trajectory of the centre of gravity, $\bar{x}_n$, remains within the ε-neighbourhood of S for all $n > N_0$. On the other hand, if we could relax this condition by requiring the trajectory of $\bar{x}_n$ to be in the ε-neighbourhood of S at least for some $n > N_0$, then S is said to be *weakly approachable*. Formally, weak approachability and weak excludability are defined by Blackwell[3] as follows.

A set S is said to be *weakly approachable* in a game with matrix M if, for every $\varepsilon > 0$, there is an N_0 such that for every $n \geq N_0$ there is a strategy f for $P1$ such that, for all g, Prob $\{\delta_n > \varepsilon\} < \varepsilon$, where δ_n is the distance of $\bar{x}_n$ from S. Similarly, S is *weakly excludable* in M if there is a $d > 0$ such that for every $n \geq N_0$ there is a strategy g for $P2$ such that, for all f, Prob$\{\delta_n < d\} < \varepsilon$.

As with approachability, weak approachability and weak excludability for a set S are the same for its closure and hence, without loss of generality, S can be considered to be closed. Any superset of a weakly approachable set is weakly approachable and every subset of a weakly excludable set is weakly excludable. No set which is weakly approachable by a player can also be weakly excludable by the other player. For two sets S and S' that are disjoint, if S is weakly approachable by a player, then S' is weakly excludable by the same player. So, any condition for weak approachability implies a condition for weak excludability. Hou[6] studies the collection of sets in two-dimensional Euclidean space that are weakly approachable by $P1$. These ideas could be extended to any N-dimensional Euclidean space. For the following discussion let the 2×2 matrix $\overline{M}$ be: $\overline{m}_{11} = a$, $\overline{m}_{12} = b$, $\overline{m}_{21} = c$, $\overline{m}_{22} = d$. Let Ω^* denote the convex hull of the points a, b, c, d and $\Omega = \bigcup_p R(p)$. Ω^*, in general, is the convex hull of Ω. If $\Omega = \Omega^*$, then every set in 2-space is either weakly approachable by one player or weakly excludable by the other. The following example illustrates a weakly approachable set.

Example 2.4. Let $a = (1/2, 1)$, $b = (0, 1)$, $c = (1, 0)$, $d = (0, 0)$. Every continuous graph from bd to ac in the trapezoid $abdc$ is weakly approachable by player $P1$ (see Fig. 1 (c)). Besides, many nonconnected graphs are also weakly approachable. One such nonconnected graph is $B_1 \cup B_2$, where B_1 is the segment $(1,1/3)(7/12,1/3)$ and B_2 is the segment $(1/4, 2/3)(2/3, 2/3)$. It can be very easily verified that, for every N, $\bar{x}_{3N} \in B_1 \cup B_2$ with the following strategy: $p_n = (0, 1)$ for $1 \le n \le N$ and $p_n = (1, 0)$ for $N < n \le 2N$ so that $\bar{x}_{2N} = (u, 1/2)$; if $u \le 3/8$, $p_n = (0, 1)$ for $2N < n \le 3N$, in which case $\bar{x}_{3N} \varepsilon B_1$; else, if $u > 3/8$, $p_n = (1, 0)$ for $2N < n \le 3N$, whence $\bar{x}_{3N} \varepsilon B_2$.

As already mentioned, if $\Omega = \Omega^*$, every set is either weakly approachable by one player or weakly excludable by the other. In such a case, without loss of generality, we can assume that $d = (0, 0)$, $b = (0, b^y)$, $b^y \ge 0$, a^x, $c^x > 0$, where w^x and w^y are the x and y coordinates of $w = (w^x, w^y)$.

Let F be the graph of a continuous function f with one terminal point on $T(0)$ and the other on $T(1)$ and the slope of any chord of the graph (the linear joint of two points on the curve) be between the slopes of the diagonals bc and da of the quadrilateral $abdc$. Let $\mathcal{F}$ denote the collection of all such subsets F on Ω. Since every continuous graph that has its terminals points on $T(0)$ and $T(1)$, respectively, is weakly approachable, if a set S in 2-space contains an $F \varepsilon \mathcal{F}$, then S is weakly approachable by $P1$. From Example 2.4 it is to be expected that certain nonconnected graphs are also weakly approachable. We shall construct a collection of the nonconnected sets which are weakly approachable. For each closed set $S \subseteq \Omega$, let $\delta^l_S = \delta(s, T(0))$, $\delta^r_S = \delta(S, T(1))$ and l_s, r_s be the points in S closest to $T(0)$ and $T(1)$, respectively. Let $u_f = (u^x_f, f(u^x_f)) \varepsilon T(1)$ and $E_1 = \{B{:}B = \{(x, f(x)){:}\ 0 \le \alpha \le x \le \beta \le u^x_f$ for some $F \varepsilon \mathcal{F}$ and some $0 \le \alpha \le \beta \le u^x_f\}\}$. We can now define $E^*_m = \{D = \bigcup_{i=1}^m B_i,\ B_i \varepsilon E_1$ for all $i = 1, 2, \ldots, m$, $l_{Bi} \varepsilon T(0)$, $r_{B_m} \varepsilon T(1)$, and for any $1 < i \le m$, $\delta^l_{B_{i-1}} \le \delta^l_{B_i}$, $\delta^r_{B_{i-1}} \ge \delta^r_{B_i}\}$. Let $E = \bigcup_{m=1}^\infty E^*_m$. If a set S in 2-space contains a set of E, then $P1$ has a pure strategy such that S is weakly approachable by him.

Let $E^* = \{D^*\}$ be the collection of sets in 2-space generated by E such that for each $D^* \varepsilon E^*$ there exists a sequence of sets $\{D_n\}$ belonging to E, with $\delta^*(D_n) \to 0$ as $n \to \infty$, where

$$\delta^*(D_n) = \max_{w \varepsilon D/D^*} (w, D^*).$$

If a set contains a D^*, then S is weakly approachable by $P1$ and has a pure strategy.

Theorem 2.5. *A set S in 2-space is weakly approachable by $P1$ if and only if S contains a set belonging to E^*.*

When $\Omega \ne \Omega^*$, then Ω is no more convex. If a set f is defined such that it intersects $T(q)$ for all $q \varepsilon Q$ and is continuous from $T(0)$ to $T(1)$, then F is weakly approachable. Any S which contains such an F is weakly approachable. So, to ensure weak approachability the graph of F has to intersect $T(q)$ for every $q \varepsilon Q$. We shall discuss this for the following three cases:

(1) $a = d$ or $b = c$;

(2) $R(0) \cap R(1) \ne \emptyset$ or $T(0) \cap T(1) \ne \emptyset$;

(3) $R(0) \cap R(1) = T(0) \cap T(1) = \emptyset$.

Case 1: $\Omega \ne \Omega^*$, $a = d$ or $b = c$.
Let $a = d = (0, 0)$. By a simple transformation we could have $b^x < 0$, $c^x > 0$, $-b^y/b^x = c^y/c^x$. $T(q)$ are lines from ab to cd, with $T(0) = bd$ and $T(1) = ac$, shown in Fig. 2(a).

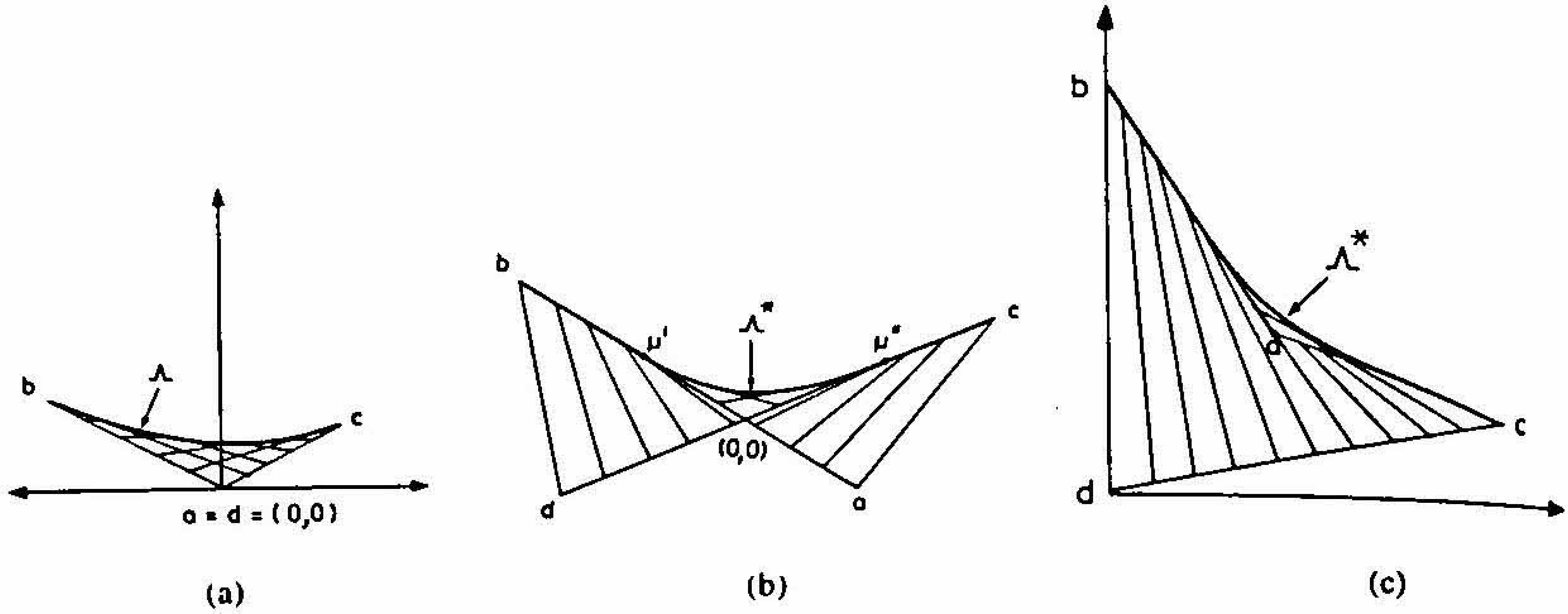

FIG. 2. Cases (a) 1, (b) 2, and (c) 3.

$T(q)$ are tangents to some nonlinear curve Λ. If F should cut $T(q)$ for every $q \varepsilon Q$, F should have an intermediary point on Λ. So, an F that is continuous from $T(0)$ to $T(1)$ with an intermediary point on Λ is weakly approachable.

Case 2: $\Omega \neq \Omega^*$, $R(0) \cap R(1) \neq \emptyset$ or $T(0) \cap T(1) \neq \emptyset$.

Let $R(0) \cap R(1) \neq \emptyset$ and let $R(0)$ and $R(1)$ intersect at (0, 0). Without loss of generality, we assume that a^x, $c^x > 0$, $b^x < 0$, $a^x/(a^x-b^x) \leq c^x/(c^x-d^x)$ and that the absolute values of the slopes of $R(0)$ and $R(1)$ are equal. $T(q)$ for all $q \varepsilon Q$ is shown in Fig 2(b). If F should cut $T(q)$ for every $q \varepsilon Q$, then F should be continuous with an intermediary point on Λ^*. If in addition, $a^x/(a^x-b^x) = c^x/(c^x-d^x)$, then it can be shown that $\mu' = \mu''$ (see Fig. 2(b)) which means that Λ^* is just the point (0, 0). Then every set in 2-space is either weakly approachable by one player or weakly excludable by the other. It is also possible to construct a collection of sets E^* in 2-space such that a set S is weakly approachable by $P1$ if and only if S contains an element of E^*.

Case 3: $\Omega \neq \Omega^*$, $R(0) \cap R(1) = T(0) \cap T(1) = \emptyset$.

In this case one element of $\overline{M}$ is an interior point of the convex hull of the other elements of $\overline{M}$. Let a be this point and d =(0, 0), b = (0, b^y), $b^y > 0$, $c^x > 0$. Λ^*, as in earlier case, is the nonlinear curve that bounds $T(q)$ for all $q \varepsilon Q$ that lie outside Ω. Any F that is continuous from bd to ac with an intermediary point on Λ^* cuts $T(q)$ for every $q \varepsilon Q$ (see Fig. 2(c)).

An important recent contribution in this area by Vieille[7] defines weak approachability in the setting of a fixed-duration differential game. A game G, repeated n times, is considered on an interval of fixed duration t, such that each stage k corresponds to the subinterval $[(k-1)\, t/n,\, kt/n]$ and the players $P1$ and $P2$ are allowed to choose actions i and j, respectively, only at the beginning of each subinterval. The average payoff function in such game is obtained as a discrete version of the differential game associated with the differential system

$$\frac{d\bar{x}}{dt} = p(t)Mq(t),$$

with the initial condition that $\bar{x}(0) = x_0$. Using results from the viscosity solutions of differential games with fixed duration, Vieille[7] proves that every set is either weakly approachable or weakly excludable (but not both). This settles a conjecture on such sets first proposed by Blackwell[3].

2.5. *Extensions to stochastic games*

In the discussion so far, it has been implicitly assumed that a game represented by a matrix M is repeatedly played by the two players. Instead, if we allow the players to play a different game at each stage, then the treatment becomes slightly different and we enter the realm of stochastic games with vector payoffs.

A stochastic game is played in stages, such that at each stage the game enters one of the finitely many states and every player observing the current state, s, chooses one of the finitely many actions. The pair of actions at stage i, together with s, determines the payoff x_i at stage i, and the probability distribution, using which the next stable state s is selected. Each state consists of a matrix game. The choice of the next state is independent of the past and depends only on the current state and the actions in the current state. Such games can be modelled as *discrete parameter Markov chains*. The concepts of approachability and excludability have been extended to such stochastic games by Shimkin and Shwartz[8]. The basic assumption underlying their approach is that a fixed state (say state 0) has certain uniform recurrent properties. Under such an assumption, it is possible to obtain results similar to the repeated matrix games, except that the strategies in the one-shot matrix game are now replaced by stationary substrategies which are employed between subsequent visits to state 0. Thus, the basic idea in the construction of approaching strategies is to use a fixed substrategy between visits to state 0 and modify this substrategy according to the current average payoff whenever state 0 is reached. The game, modelled as a controlled Markov chain with a countable state space S, has two players $P1$ and $P2$, each with finite action spaces A_1 and A_2, respectively, who together determine the payoff as given by an R^N-valued function r. The state transition is governed by a probability distribution law, p. At each stage, $n = 0, 1, 2, \ldots$, the current state, s, is observed and the two players simultaneously and independently choose actions $a^1 \varepsilon A_1$ and $a^2 \varepsilon A_2$, respectively. As a result, a payoff vector $r(s, a^1, a^2)$ is collected and the next state s' is chosen according to the probability distribution $p(. \mid s, a^1, a^2)$ on S. Let $r_n = r\left(S_n, a_n^1, a_n^2\right)$ be the payoff vector at the nth stage and let $\bar{r}_n = \Sigma_{m=0}^{n-1} (r_m / n)$ denote the time-averaged payoff vector up to stage n. A randomized, history-dependent strategy π_i for Pi $(i = 1, 2)$ is a sequence $\pi_i = \{\pi_0^i, \pi_1^i, \ldots\}$, $\pi_n^i : H_n \rightarrow P(A_i)$, where $P(A_i)$ is the set of probability vectors over A_i and $H_n = S \times (A_1 \times A_2 \times S)^n$ is the set of possible histories up to stage n. Let Π_i denote the class of all such strategies for the ith player. A stationary strategy for $P1$ is specified by a single function $f: S \rightarrow P(A_1)$ so that $\pi_n^1(h_n) = f(s_n)$, $n \geq 0$. Let F be the class of stationary strategies for $P1$ and G for $P2$.

For every vector $u \,\varepsilon\, R^N$ and initial state s, if we consider the game with a scalar payoff function $r^u = \langle r, u \rangle$ with $P1$ maximizing the average expected payoff and $P2$

minimizing it, the model becomes a zero-sum stochastic game, denoted by $\Gamma_s(u)$. $\Gamma_s(u)$ is the projection of the game with vector payoff function r on to the vector u. From the minimax theorem for scalar games, it follows that the value of $\Gamma_s(u)$, if it exists, is given by

$$\operatorname{val}\Gamma_s(u) = \sup_{\pi_1} \inf_{\pi_2} \lim_n \inf E^S_{\pi_1,\pi_2}\left[\langle \bar{r}_n, u\rangle\right] \tag{4}$$

$$= \inf_{\pi_2} \sup_{\pi_1} \lim_n \sup E^S_{\pi_1,\pi_2}\left[\langle \bar{r}_n, u\rangle\right]. \tag{5}$$

A strategy $\pi_1 \varepsilon \Pi_1$ $\{\pi_2 \varepsilon \Pi_2\}$ is optimal in $\Gamma_s(u)$ if it satisfies the sup in (4) [the inf in (5)]. Let U denote the set of all unit vectors in R^N. As mentioned earlier, the basic assumption involves recurrence conditions for a fixed state 0. Let T denote the first passage time to state 0:

$$T = \inf\{n \geq 1: s_n = 0\}.$$

A strategy $\pi_1 \varepsilon \Pi_1$ is said to be stable if there exist positive constants M_2 and R_2 such that

$$E^0_{\pi_1,\pi_2}[T^2] \leq M_2 \text{ for all } \pi_2 \varepsilon \Pi_2, \tag{6}$$

$$E^0_{\pi_1,\pi_2}\left[\left(\sum_{n=0}^{T-1}|r_n|\right)^2\right] \leq R_2 \text{ for all } \pi_2 \varepsilon \Pi_2. \tag{7}$$

A set $\Pi_1^1 \subseteq \Pi_1$ is uniformly stable if (6) and (7) are satisfied for every $\pi_1 \varepsilon \Pi_1^1$ with the same constants M_2 and R_2.

The following conditions are introduced into the model:

(C1): For every unit vector $u \varepsilon U$, the game $\Gamma_0(u)$ has a value and $P1$ has a stationary optimal strategy $f^*(u)$ in this game. Moreover, the set $\{f^*(u): u \varepsilon U\}$ is uniformly stable.

(C2): Condition (C1) holds. Furthermore, for each $u \varepsilon U$, $P2$ has an optimal strategy $g^*(u)$ in $\Gamma_0(u)$ which is stationary and stable.

It can be shown that if the following hold then (C1) and (C2) are satisfied and, moreover, the entire strategy sets Π_1 and Π_2 are uniformly stable:

(i) The payoff function r is bounded.

(ii) There exists a number M such that $E^s_{f,g}[T] \leq M$ for every $s \varepsilon S$ and all stationary nonrandomized strategies $f \varepsilon F$ and $g \varepsilon G$.

Let $\{X_n, n \geq 0\}$ be a sequence of random variables over some measurable space $(\Omega, \mathcal{G})$ and let $\{P_v, v \varepsilon V\}$ be a collection of probability measures on $(\Omega, \mathcal{G})$. For a fixed $v \varepsilon V$, $X_n \to 0$ P_v-almost-surely is equivalent to

$$\lim_{N\to\infty} P_v\left(\sup_{n\geq N} |X_n| > \varepsilon\right) = 0, \text{ for all } \varepsilon > 0, \tag{8}$$

$X_n \to 0$ P_v-almost-surely, at a uniform rate over V if convergence in (8) is uniform over V, that is,

$$\lim_{N\to\infty} \sup_{v\,\varepsilon\, V} P_v\left(\sup_{n\geq N}|X_n| > \varepsilon\right) = 0.$$

Let the initial state, s, be fixed. A set $B \subseteq R^N$ is approachable from s by $P1$ if there exists a B-approachable strategy $\pi_1^* \ \varepsilon\ \Pi_1$ such that $d(\bar{r}_n$, only $B)$ tends to 0 P-almost-surely for every $\pi_2\ \varepsilon\ \Pi_2$ at a uniform rate over Π_2. B is excludable from s by $P2$ if there exists a B-excluding strategy $\pi_2^*\ \varepsilon\ \Pi_2$ such that for some $\delta > 0$, $d(\bar{r}_n, B_\delta^c)$ tends to 0 P-almost-surely for every $\pi_1\ \varepsilon\ \Pi_1$ at a uniform rate over Π_1, where $B_\delta^c = \{b\ \varepsilon\ R^N\colon d(b, B) \geq \delta\}$. An important aspect of the definition is the uniform rate of convergence. This requirement is essential if the infinite-stage model is considered as an approximation to the model with very long but finite time horizon. Let

$$\phi(\pi_1,\pi_2) = \frac{E^0_{\pi_1,\pi_2}\left[\sum_{n=0}^{T-1} r_n\right]}{E^0_{\pi_1,\pi_2}[T]}$$

denote the average payoff per cycle from state 0 and back. $\phi(\pi_1, \pi_2)$ is well defined if either π_1 or π_2 is a stable strategy.

To the closed set $B \subseteq R^N$, from any point $x \notin B$, let c_x denote the closest point in B to x. Let H_x be the hyperplane through c_x which is perpendicular to $(c_x - x)$ and let u_x be a unit vector in the direction of $(c_x - x)$.

Theorem 2.6. Assume that the following condition is satisfied (SC refers to sufficient condition):

(SC1): *For every $x\ \varepsilon\ B$, there exists a stable strategy $\pi_2 \notin \Pi_2$ (equivalently, $\phi(\pi_1(x), \Pi_2)$ is weakly separated by H_x from x). Furthermore, the set $\{\pi_1(x){:}x \notin B\}$ is uniformly stable.*

Then B is approachable from state 0 by $P1$ and a B-approachable strategy is given as follows: Let $0 < T(1) < T(2) < \ldots$ be the subsequent arrival instants to state 0. Let $\tilde{\pi}_1$ be some fixed stable strategy for $P1$. Then

(*i*) *at stages $0 \leq n < T(1)$ use $\tilde{\pi}_1$;*
(*ii*) *at stages $T(k) \leq n < T(k+1)$, $k \geq 1$, if $\bar{r}_{T(k)} \notin B$ then use $\tilde{\pi}_1(\bar{r}_{T(k)})$ started at $T(k)$, else use $\tilde{\pi}_1$ started at $T(k)$.*

Corollary 2.2. Assume that the condition (SC1) of Theorem 2.6 is satisfied. In addition, assume that for some strategy $\sigma\ \varepsilon\ \Pi_1$,

$$\sup_{\pi_2} E^s_{\sigma,\pi_2}\left[T^2\right] < \infty$$

and

$$\sup_{\pi_2} E^s_{\sigma,\pi_2}\left[\sum_{n=0}^{T-1} r_n^2\right] < \infty.$$

Then B is approachable from state s by P1. An approaching strategy is given as in Theorem 2.6 except that up to time $T = T(1)$ the strategy σ is used by P1.

It can be clearly seen that Theorem 2.6 is the generalization of the sufficiency condition given by Blackwell[3] for repeated games. Similarly, when set B is convex, it is possible to obtain a complete characterization of approachability.

For every stable $g \in G$, let $R(f, g) = \lim_{n\to\infty} E^0_{f,g}[\tilde{r}_n]$, $f \in F$. Define $R(F, g) = \{R(f, g): f \in F\}$ and $\bar{R}(f, g) = \text{conv } R(f, g)$, where conv denotes the closed convex hull. Similarly, we can define $R(f, G)$ and $\bar{R}(f, G)$ for any stable $f \in F$. For each convex set B in R^N define the set $U(B)$ of unit vectors: $U(B) = \{u_x \in U: x \notin B\}$. $U(B)$ represents all the directions in which a point outside B might be projected on to B. If B is bounded, then $U(B) = U$.

Theorem 2.7. Assume that (C2) *holds. Let B be a closed convex set in R^N, and let the initial state $s_0 = 0$.*

- *(i) B is approachable if and only if either one of the following equivalent conditions are satisfied (NSC refers to necessary and sufficient condition):*
  - *(NSC1): There exists a uniformly stable set $\{f(u): u \in U(B)\}$ of stationary strategies for P1 such that every $x \notin B$ is separated from $\bar{R}(f(u_x), G)$ by H_x, that is, $\inf_{g\in G} \langle R(f(u_x), g), u_x\rangle \geq \langle c_x, u_x\rangle$.*
  - *(NSC2): The separation condition in (NSC1) holds for $f(u) \equiv f^*(u)$, the optimal strategy of P1 in $\Gamma_0(u)$, $u \in U(B)$.*
  - *(NSC3): val $\Gamma_0(u) \geq \min_{b\in B} \langle b, u\rangle$ for every $u \in U(B)$.*
  - *(NSC4): $\bar{R}(F, g)$ intersects B for every stable $g \in G$.*
- *(ii) If B is not approachable, then it is excludable by P2 with a stationary strategy.*

An excludable convex set B may be excluded by a stationary strategy of P2. Thus, an excludable convex set will remain so even if P2 is restricted to stationary strategies only (or any superset thereof).

The assumption that the approaching strategies are adapted to the history of the process (that is, the relative position of the average payoff with respect to the set to be approached) only when the fixed state 0 is reached may have the undesirable effect of increasing the variance of the payoff if the recurrence times are far apart. So it would be of interest to construct approaching strategies, which adapt to the current payoff more frequently.

It is clear that some recurrence conditions are required to preserve the basic approach discussed here. But the condition of recurrence of a single fixed state for all relevant strategies is not the only possibility. We could, for example, consider the condition when the recurrent state is allowed to depend on the strategies, within a finite set of states.

2.6. *Information aspects*

Since a sequence of games is considered, the rules of play must specify to what extent a player's decision at any stage may depend on past plays. This leads to the natural question of how the class of approachable sets depends on the type of information available to the players. In the following discussion, we are interested only in the information that $P1$ gets.

Suppose $P1$ receives no information about $P2$. Then a closed set S is approachable if and only if there exists a $p \varepsilon P$ such that $R(p) \subseteq S$. This result is independent of whether $P1$ receives any information about his own past play or not.

If $P1$ is informed of the complete history of $P2$'s choice but receives no information concerning his own past plays, the class of approachable sets is greatly increased. Suppose $P1$ receives complete information pertaining to his past and to $P2$'s past plays, then Katz[9] shows that the class of approachable sets is not increased. This happens since, in the previous case, $P1$ can actually construct his past history from the strategy he adopts and his knowledge of $P2$'s past history. The result obtained in this case is identical to that of Blackwell[3].

2.7. *Applications*

There are several problems of practical interest which can benefit from the approachability-excludability theory of Blackwell. We will discuss some of them below. These discussions also attempt to identify future research areas having definite applications.

(*a*) *Individually rational payoffs*: Vector payoffs become relevant in games with incomplete information. This can be illustrated by a one-sided information game. Let us suppose that two players have to repeatedly play a game picked out of a set H of matrix games, following a distribution known to both players. The exact element $h \varepsilon H$, chosen at the start, is told to $P1$ but not to $P2$ and we assume that both players have perfect recall. $P1$ computes his payoffs in each possible state $h \varepsilon H$, whereas for $P2$ only his expectation over H matters. In other words, $P1$ has a vector payoff whereas $P2$'s payoff is a scalar. In such a game, individually rational payoff (to be understood in the sense of what each player cannot be prevented from obtaining) can be defined as elements of sets which are approachable by $P1$ and $P2$, respectively.

(*b*) *Finitely repeated games*: In the literature on approachability-excludability theory, only infinitely repeated matrix games (with non-cooperative play) have been considered. In the mathematical economics literature finitely repeated games have been studied in recent times and certain interesting results have been obtained (see Benoit and Krishna[10]). In finitely repeated games cooperation is enforced by defining punishments for deviations by players. In such games, the desired payoffs (which are not obtainable in non-cooperative play) may be considered to constitute sets S_1 and S_2 for players $P1$ and $P2$, respectively, such that $S = S_1 \cap S_2 \neq \varnothing$. S_1 and S_2 are excludable by $P2$ and $P2$, respectively. The payoffs of $P2$ in S_{11} and $P1$ in S_{22} are worse than their respective payoffs in S. Then S_{22} (S_{11}) could be used as $P2$'s ($P1$'s) punishment strategy on $P1$ ($P2$) for deviations from agreed play. Thus, by introducing such punishment sets, we can try to make some sets approachable, which are otherwise excludable (and which give better payoffs to both players than their respective approachable sets). This way, using punishment schemes, we may be able to enlarge the class of approachable sets.

(c) Nonzero-sum games: In most literature on approachability–excludability theory only zero-sum games are discussed. Instead, if we consider a nonzero-sum game, then we could have the payoff space $\Omega_1 \subseteq R^M$ and $\Omega_2 \subseteq R^N$ for players *P*1 and *P*2, respectively. Given some $S_1 \subseteq \Omega_1$ and $S_2 \subseteq \Omega_2$, we could discuss, for example, excludability of S_1 and S_2 by *P*1 means that *P*1 (excludability of S_2 by *P*1 means that P1 tries preventing *P*2's payoff from entering S_2). *P*1 may use the same strategy or different strategies to exclude S_1 and S_2. We could study the collection of pairs of sets in Ω_1 and Ω_2, respectively, which are excludable by one player with the same strategy. In similar fashion we could consider other classes: S_1 and S_2 both approachable; S_1 approachable and S_2 excludable; and S_1 excludable and S_2 approachable, by one player with a single strategy.

(d) n-person games: Approachability–excludability theory so far has been considered only for two-person games. If we consider *n*-person zero-sum games, then a simplistic approach would be to concentrate on the approachability/excludability aspects for the player we are interested in and model the remaining *n*-1 players as a single antagonistic player. But in the realm of nonzero-sum game the approach becomes more complicated since we now have *n* different payoff spaces to consider. Then we could consider pairs of spaces in which we could investigate the collection of pairs of sets that are excludable (or any other combination discussed above) with a single strategy of the player under discussion, or we could consider three or more spaces (and, therefore, groups of three or more sets, respectively).

3. *N*-person multicriteria cooperative games

3.1. *Introduction*

The theory of cooperative games is quite well developed in the literature. Its motivation arises from the fact that when there are several players playing a game, some of them may sense the possibility of additional benefit by forging an alliance (or a 'coalition', in game-theoretic terms). The paradigm of cooperative games attempts to explore the potential of such alliances.

In the literature, most of the cooperative-game theory is concerned with several players, each having a single criterion to optimize. The theory of multicriteria cooperative games relaxes this condition and makes it possible for each player to have more than one objective. The discussion in this section is based mainly on the paper by Bergstresser and Yu[11].

3.2. *The normal-form game*

A general formulation of the *N*-person multicriteria game in normal form would associate a vector-valued payoff (multicriteria) function to each player, defined on a joint decision space *W*. Essentially, *W* can be considered as a probability distribution on all possible pure-strategy combinations of the players. Player *Pi*'s criteria are indexed by 1, 2,..., l_i and his payoff function is $J^i : W \to R^{l_i}$. Each of the other players may share none, all or some of *Pi*'s criteria. The payoff space of player *i*, denoted by $P^i = j^i(W)$ has dimension l_i and $P^i \subseteq R^{l_i}$. The full payoff space, denoted by P^F, is the space of dimension $\Sigma_{i=1}^{n} l_i$ and is given by

$$P^F = J(W) = (J^1 . J^2, \ldots, J^n)(W)$$
$$= \{[J^1(w)), (J^2(w)), \ldots, (J^n(w))], w \varepsilon W\}.$$

Example 3.1. Consider the following multicriteria matrix game played by three players:

$$A = \begin{bmatrix} [(5,1), & (1,2), & (2,4)] & [(3,2), & (1,1), & (4,4)] \\ [(2,0), & (3,3), & (5,5)] & [(5,2), & (4,1), & (1,3)] \end{bmatrix}$$

$$B = \begin{bmatrix} [(1,1),(1,2),(2,4)] & [(1,1),(1,2),(3,6)] \\ [(2,3),(2,1),(5,5)] & [(1,0),(0,0),(5,4)] \end{bmatrix}.$$

In this game $P1$ chooses rows, $P2$ chooses columns, and $P3$ chooses one of the two matrices A and B. The first element in a full payoff vector (six-dimensional) is the payoff vector (two-dimensional) of $P1$. Similarly, the second and third element are payoff vectors of $P2$ and $P3$, respectively. Note that here $l_i = 2$, $i = 1, 2, 3$; the number of possible pure-strategy combinations are eight and W is a probability distribution on these eight combinations, which are given by the total number of elements in the two matrices.

Before we discuss relevant solution concepts, we will introduce the notion of domination structures[12]. One of the important elements in multicriteria decision-making concerns the preference ordering (partial or total) on the criteria (or payoff) space. Given u^1 and u^2 in U, we write $u^1 > u^2$ if u^1 is preferred to u^2. With each point $u^0 \varepsilon U$, we can associate a set $D(U^0)$ so that $u \varepsilon u^0 + D(u^0) = \{u^0 + d \mid d \varepsilon D(u^0)\}$and $u \neq u^0$ if and only if $u^0 + > u$. We will assume that $D(u)$ is a convex cone. This is called the *domination* cone for u. The family $\{D(u) \mid u \varepsilon U\}$, denoted by $D(.)$, is called the *domination structure* associated with the problem. Thus, given a set U, a domination structure $D(.)$ defined on U, and $u^1, u^2 \varepsilon U$, we shall say that u^2 is dominated by u^1 if and only if $u^2 \varepsilon u^1 + D(u^1)$ and $u^2 \neq u^1$. A point u^0 is a nondominated solution (or nondominated outcome) if and only if there is no $u^1 \varepsilon U$ such that $u^1 \neq u^0$ and $u^0 \varepsilon u^1 + D(u^1)$. This u^0 is nondominated if and only if it is not dominated by any other outcome in U.

Similarly, in the decision-making space W, a point $w^0 \varepsilon W$ is a nondominated solution (or a nondominated decision) if and only if there is no $w' \varepsilon W$ such that $J(w') \neq J(w^0)$ and $J(w^0) \varepsilon J(w') + D\, J(w'))$. Here J: $W \rightarrow U$ is the payoff function. The set of all nondominated solutions in the decision space and the payoff space are denoted by $N_W(D(.))$ and $N_U(D(.))$, respectively. When the domination structure is obvious, we will use the notation N_W and N_U only.

Based on the above formulation we can propose a number of solution concepts.

(*a*) *Solution concepts in* P^F: In a cooperative model, all players may jointly agree to apply a solution concept to P^F, the full payoff space. It is assumed that each player implicitly respects and considers all of the criteria for all the other players. This is the same as considering P^F as the payoff space for a single-criterion normal-form game with

$\Sigma_{i=1}^{n} l_i$ players. Or, it may also be thought of as a problem with a single decision maker having $\Sigma_{i=1}^{n} l_i$ criteria (that is, a single-player multicriteria problem). With this kind of formulation one can use the available solution concepts of multicriteria optimization[13] to solve this game. In general, the players may jointly decide to use a certain domination structure in P^F and select the final solution from the set of nondominated solutions. Note that a domination structure used jointly by the players is also a measure of the power held by individual players.

(*b*) *Solution concepts in* P^i: If each player *Pi* individually decides upon his own domination structure $D_i(.)$ in his own payoff space P^i, then we have a different solution concept. Let $W_0 \subseteq W$ be the set of decisions such that, for every $w \in W_0$, $J^i(w)$ is nondominated with respect to $D_i(.)$ for all values of $i = 1, 2, \ldots, n$. Thus, any decision in W_0 should be acceptable to all players. Quite naturally, W_0 may be empty and there may exist payoffs $p \in P^F$ which are greater than the minimally acceptable payoff level vector, in which case there will be an incentive for the players to cooperate.

(*c*) *Reduction of each player's payoff to a single criterion*: Here each player *Pi* defines a real-valued utility function u_i on his payoff space. Using this function for each player, the multicriteria normal-form game is reduced to a single-criterion normal-form game and the solution concepts applicable to *N*-player single-criterion games become relevant. One way to define such a utility function would be to specify a weight vector of dimension l_i, thus reducing the multicriteria game to a single-criterion game. A player may not be able to specify a single weight vector since other weight vectors close to it may be equally acceptable to him. In this case, it is reasonable to adopt a convex cone Λ_i of weight vectors. Then each choice of *n* weight vectors $\lambda_i \in \Lambda_i$, $i = 1, 2, \ldots, n$ yields a single-criterion *n*-person normal-form game. Suitable solution concepts can now be applied to this set of induced single-criterion games. In the case when $l_i = 1$ for all *i*, the players may agree (cooperate) to choose a single $\lambda \in \Lambda$.

3.3. *The characteristic* (*or coalitional*) *form game*

A multicriteria *n*-person game in coalitional function form has a player set $N = \{1, 2, \ldots, n,\}$ and a vector-valued characteristic function $v = (v_1, v_2, \ldots, v_l)$. Each v_k, $k = 1, 2, \ldots, l$ is a real-valued function, $v_k = \mathcal{N} \to R$, where $\mathcal{N} = \{S \mid S \subseteq N\}$ such that $v_k(\Phi) = 0$. Each element $S \in \mathcal{N}$ represents a coalition and $v(S)$ is the total payoff guaranteed to the players in the coalition if they cooperate. The characteristic function may or may not have been derived from the underlying normal-form game.

For ease in analysis we usually (0, 1)-normalize a game. Two games given by the characteristic function v and v' are *strategically equivalent* if there exist real numbers $k > 0$ and a_i, $i = 1, 2, \ldots, n$ such that $v'(S) = kv(S) + \Sigma_{i \in S} a_i$ is satisfied for all $S \in \mathcal{N}$. A game v is (0, 1)-*normalized* if $v(\{i\}) = 0$ for all $i \in N$ and $v(N) = 1$. A given game v is strategically equivalent to a (0, 1)-normalized game v' if and only if $\Sigma_{i=1}^{n} v(i) < v(N)$.

In an *n*-person multicriteria game, we separately (0, 1)-normalize each v_k and obtain the corresponding multicriteria characteristic function as $v' = (v_1', v_2', \ldots, v_l')$.

Example 3.2. Consider the game in Example 3.1. The multicriteria characteristic function is given by $v = (v_1, v_2)$ and $v(1) = (1, 1)$, $v(2) = (1, 1)$, $v(3) = (2, 4)$, $v(12) = (4, 3)$, $v(23) = (5, 6)$, $v(13) = (7, 6)$ and $v(123) = (10, 9)$. Then by (0, 1)-normalization of each criterion, we get $v'(1) = v'(2) = v'(3) = (0, 0)$, $v'(12) = (1/3, 1/3)$, $v'(23) = (1/3, 1/3)$, $v'(13) = (1/3, 1/3)$, $v'(123) = (1, 1)$.

In this example, the underlying normal-form game was such that all players had the same set of criteria. With this assumption on the underlying normal-form game, a multicriteria characteristic-function-form game can be naturally induced. Later we will relax this assumption.

With every multicriteria function $v = (v_1, v_2, \ldots, v_l)$, we can associate the following single-criterion characteristic functions:

(i) $\overline{v}$ defined by $\overline{v}(S) = \max_{1 \le k \le l} v_k(S)$ for all $S \varepsilon \mathcal{N}$.

(ii) The (0, 1)-normalization of $\overline{v}$, denoted by $(\overline{v})'$.

(iii) $(\overline{v'})$ defined by $(\overline{v'})(S) = \max_{1 \le k \le l} v'_k(S)$ for all $S \varepsilon \mathcal{N}$.

(iv) $\underline{v}$ defined by $\underline{v}(S) = \max_{1 \le k \le l} v'_k(S)$ for all $S \varepsilon \mathcal{N}$.

(v) The (0, 1)-normalization of v, denoted by $(\underline{v}')$.

(vi) $(\underline{v'})$ defined by $(\underline{v'})(S) = \max_{1 \le k \le l} v'_k(S)$ for all $S \varepsilon \mathcal{N}$.

(i)–(vi) alone will be used in our later discussion of various solution approaches.

(*a*) *Parametrization of multicriteria characteristic function*: To reduce the multicriteria characteristic function to a single-criterion function, the players may cooperatively determine a real-valued utility function on $v(\mathcal{N})$. One way of doing this would be for the players to agree on a weight vector, λ, or a set of such weight vectors, $\Omega \subseteq L = \{\lambda \varepsilon R^l \mid \lambda \ge 0 \text{ and } \Sigma_{k=1}^{l} \lambda_k = 1\}$, which would be used to parametrize the characteristic function and obtain a single-criterion characteristic function. We will discuss two different methods of parametrizing a characteristic function $v = (v_1, v_2, \ldots, v_l)$. One method of parametrizing a characteristic function and the other is through a parametrization of the underlying normal-form game (as in Section 3.2(c)).

Given $\lambda \varepsilon L$, we can define the parametrized game v_λ^c (the superscript c denotes parametrization directly on the characteristic function) by $v_\lambda^c(S) = \lambda \cdot v(S) \; \Sigma_{k=1}^{l} \lambda_k v_k(S)$ for all $S \varepsilon \mathcal{N}$. We will denote the (0, 1)-normalization of v_λ^c by $\left(v_\lambda^c\right)'$. On the other hand, we might first (0, 1)-normalize v to obtain $v' = (v'_1, v'_2, \ldots, v'_l)$ and parametrize as above to obtain $(v')_\lambda^c$. Clearly, $(v')_\lambda^c$ is in (0, 1)-normalized form. In general, $\left(v_\lambda^c\right)' \neq \left(v_\lambda^c\right)'$.

In the parametrization approach through the underlying normal form, we first use a given weight vector λ to parametrize the underlying normal form (as in Section 3.2) and then induce a single-criterion characteristic function. Specifically, given $\lambda \varepsilon L$, we parametrize Pi's normal-form payoff function p^i to obtain a real-valued function $\Sigma_{k=1}^{l} \lambda_k p_k^i$ as his payoff. We denote the characteristic function derived from this parametrized normal form by v_λ^N (the superscript N denotes parametrization through the underlying normal form).

The fact that, for each $S \varepsilon \mathcal{N}$, $v_\lambda^N(S)$ is defined as the maximum of an aggregated parametrized payoff function, where the maximum and minimum are taken over compact sets of mixed strategies, allows us to show that v_λ^N is a continuous function of λ.

With each multicriteria characteristic-function-form game $v = (v_1, v_2, \ldots, v_l)$ derived from a normal-form game, using the parametrization process just described, we define the single-criterion characteristic function v^* by $v^*(S) \triangleq \max_{\lambda \varepsilon L} v_\lambda^N(S)$ for each coalition $S \varepsilon \mathcal{N}$. It can be seen that v^* will be somewhat simplistic if we replace v_λ^N by v_λ^c since in that case we would have $v^* = \overline{v}$. It is possible to have $v^*(S) > \overline{v}(S)$ for some $S \varepsilon \mathcal{N}$. This can be shown through a simple example. We denote the (0, 1)-normalization of v^* by $(v^*)'$.

For each $\lambda \varepsilon L$, we could also (0, 1)-normalize v_λ^N to obtain $\left(v_\lambda^N\right)$ if $\Sigma_{i=1}^{n} v_\lambda^N(i) < v_\lambda^N(N)$. Then we similarly define v^{**} by $v^{**}(S) \triangleq \max_{\lambda \varepsilon L} v_\lambda^N(S)$. Note that v^{**} is in (0, 1)-normalized form. It is evident that $(v^*)'$ may not equal v^{**} .The relationships between these games and between v and $(v^N)'$ are worth investigating.

(*b*) *Solution concepts yielding a set or sets of many solution points*: In this section we focus primarily on an extension of the core concepts to multicriteria games, which we call the multicriteria core.

Suppose that each criteria v_k' of a multicriteria (0, 1)-normalized characteristic-function-form game, $v' = (v_1', v_2' \ldots, v_l')$, represents the coalition values for a different outcome of some future uncertain event. Given any outcome of this future event, no coalition will object to an imputation in the core of every characteristic function. Consequently, we define the multicriteria core of the game v', denoted by MC(v'), to be the intersections of the cores of the coordinate function, that is $\mathrm{MC}(v') = \bigcap_{k=1}^{l} C(v_k')$. In the next lemma, we shall see that v' has a close relationship with multicriteria core.

Lemma 3.1. *For a* (0, 1)-*normalized characteristic-function-form game* $v' = (v'_1, v'_2, \ldots, v'_l)$, $\mathrm{MC}(v') = C((v'))$.

Proof. $x \varepsilon \mathrm{MC}(v') = \bigcap_{k=1}^{l} C(v_k')$ if and only if $x(N) = 1$ and $x(S) \geq v_k'(S)$ for $k = 1, 2, \ldots, l$ and for all $S \varepsilon \mathcal{N}$. This is equivalent to $x(N) = 1$ and $x(S) \geq \max_{1<k<l} v_k'(S) = (v')(S)$ for all $S \varepsilon \mathcal{N}$, which is equivalent to $x \varepsilon C(\overline{(v')})$.

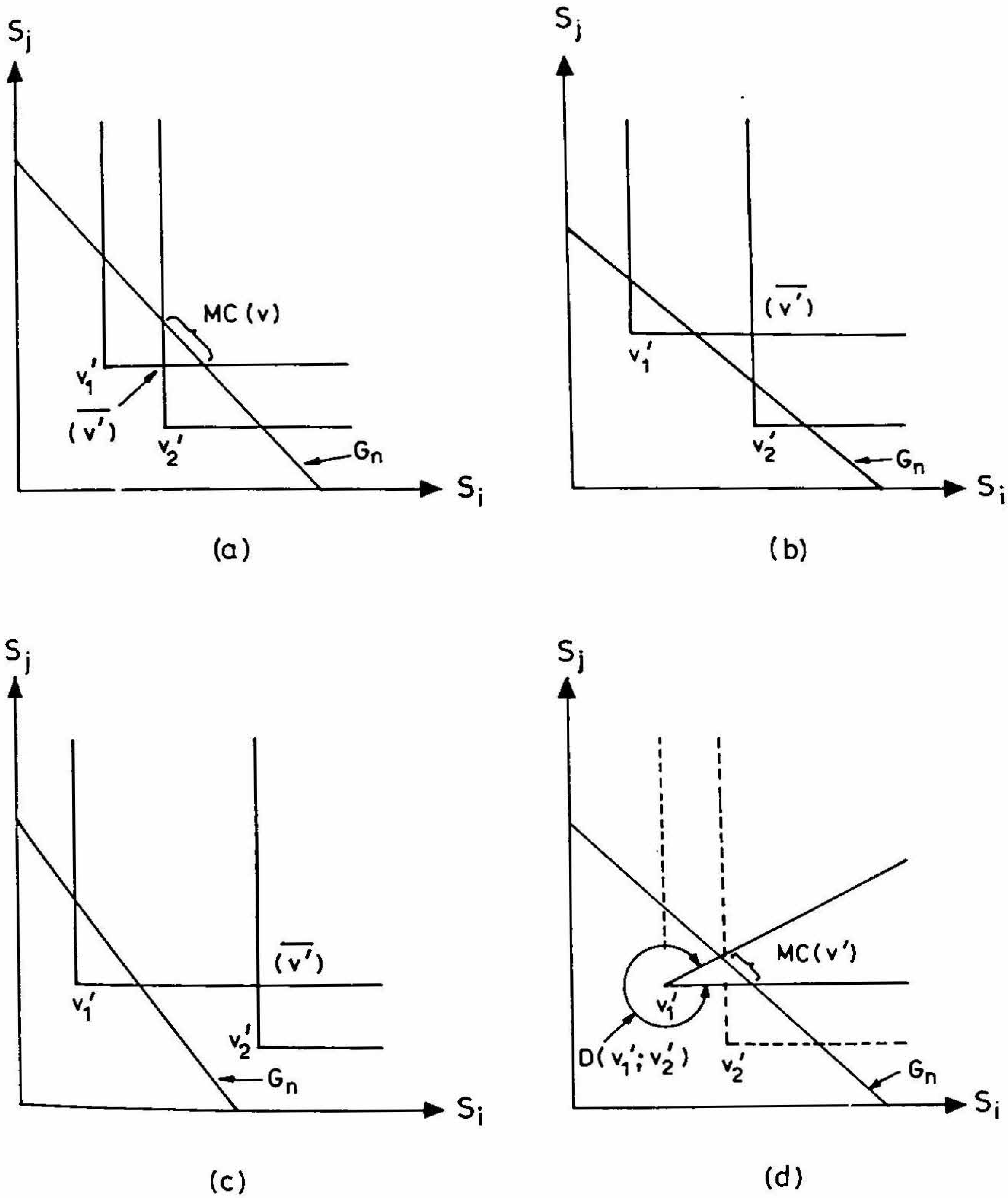

FIG. 3.(a) $C(v_1') \neq \emptyset$, $C(v_2') \neq \emptyset$, $\mathrm{MC}(v') \neq \emptyset$; (b) $C(v_1') \neq \emptyset$, $C(v'_2) \neq \emptyset$, $\mathrm{MC}(v') \neq \emptyset$; (c) $C(v_1') \neq \emptyset$, $C(v_2') \neq \emptyset$, $\mathrm{MC}(v') \neq \emptyset$; (d) the domination structure $D(v_1'; v_2')$.

In Figs 3(a)-(c), for the case of two criteria, we illustrate three possible locations of the game points v_1', v_2' in Y with respect to G_n, where G_n is the n-dimensional simplex in the coalition space Y. We represent Y as a two-dimensional space. In case $\mathrm{MC}(v') = \varnothing$ (as in B and C) another solution approach would be required. Lemma 3.1 provides a way to define a domination structure on I, the imputation space, a (0, 1)-normalized game $v' = (v_1', v_2', \ldots, v_l')$which yields the multicriteria core. Using the notation of the domination structure[13], and given a coalition $S \varepsilon \mathcal{N}$, we define the following core of each $x \varepsilon I$:

$$D_S(x) = \begin{Bmatrix} \Lambda^{<} \text{ if } x(S) \leq \max\limits_{1 \leq k \leq l} \{v_k'(S)\} \\ \{0\} \text{ otherwise} \end{Bmatrix}$$

Then we can obtain $\mathrm{MC}(v') = \bigcap\limits_{j=1}^{m} N(D_{S_j})$.

We will show that the multicriteria core induces general domination structures in the coalition space Y. Given the (0, 1)-normalized multicriteria game $v' = (v_1', v_2')$ we treat v_1' as a reference game point and enlarge the core $D(v_1') = (Y \backslash \Lambda^{\geq}) U\{0\}$ to form the core $D(v_1'; v_2')$ as follows: To $D(v_1')$ we add rays induced by imputations which are objectionable to any coalition at the other criteria (game point), v_2'. More specifically, given an imputation x, if $x(S) < v_2'(S)$ for some coalition S, then we adjoin the ray $\{a(\phi(x) - v'_1) \mid \alpha \geq 0\}$ to $D(v')$. This can be seen in Fig. 3 (d). It is apparent from the figure that $\mathrm{MC}(v')$ is not dominated by v'_1 under $D(v'_1;\ v'_2) = \{x \,\varepsilon\, I \mid \phi(x) - \varepsilon\, v'_1 + D(v'_1;\ v'_2)^c\}$. $D(v'_1;\ v'_2)^c$ is the complementary one of $D(v'_1;\ v'_2)$). Clearly, if we reduced the multicriteria game to a single-criterion game v'_1 and used the classical domination structure, we would lose some crucial information from the game point of v'_2. Consequently, to avoid this loss of information we must use the more general core $D(v'_{1,};\ v'_2)$.

In general, given a (0, 1)-normalized multicriteria game $v' = (v'_1, v'_2, \ldots, v'_l)$ for each $k = 1, 2, \ldots, l$ we define the following core: $D(v'_k;\ v'_1, v'_2, \ldots, v'_{k-1}, v'_{k+1}, \ldots, v'_l) = (Y \backslash \Lambda^{\geq})\ U\ \{0\} U\ \{a(\phi(x) - v'_k) \mid \alpha > 0,\ x \,\varepsilon\, I$ such that $x(S) < v_j(S)$ for some $S \,\varepsilon\, \mathcal{N}$ and some $j = 1, 2, \ldots, l,\ j \neq k\}$. As above, this domination core can be used to generate $\mathrm{MC}(v')$. Similarly, for each weight vector λ, we can define the domination structure $D((v_\lambda^N)';\ v'_1, v'_2, \ldots, v'_l)$, which generates $\mathrm{MC}(v')$ using $(v_\lambda^N)'$ as the reference point.

If all the players agree on a probability distribution λ over the possible outcomes, it would be reasonable to apply the core and other solution concepts to $(v_\lambda^N)'$. By the continuity of v_λ^N and of the (0, 1)-normalization process, one can show that if $C((v_\lambda^N)') \neq \varnothing$, then $c((v_\lambda^N)')$ is a continuously varying set function of λ. In fact, this continuity property holds over the parameter space for v_λ^c.

Instead of agreeing on a unique distribution, it is more likely that the players would agree on an interval estimate of the probabilities for each possible outcome. Thus, the players might agree to use a set Ω, of probability weight vectors. Then each imputation in $\bigcap_{\lambda \varepsilon \Omega} C((v_\lambda^N)')$ has the property that has no coalition will object no matter which $\lambda \varepsilon \Omega$ is the actual distribution for the future event. We might also use $\bigcap_{\lambda \varepsilon \Omega} C((v')_\lambda^c$ or $\bigcap_{\lambda \varepsilon \Omega} C((v_\lambda^c)')$ depending upon the particular application. On the other hand, the players might want to consider imputations in $\bigcup_{\lambda \varepsilon \Omega} C((v_\lambda^N)')$, which gives all nondominated imputations under each possible probability distribution in the estimate set Ω.

Further, the core and other solutions concepts could be applied to the associated single-criterion games $(\bar{v})'$, $(v*)'$ or $v**$. These games indicate the best that a coalition can do under particular circumstances. Thus, the coalitions may agree to use one of these game points in determining the final solution.

An additional solution approach can be obtained by using the full payoff space of the underlying normal-form game. Assuming that each player has the same l criteria, we can view the normal-form game as a single-criterion game with ln players. In other words, each criterion for each player is associated with a different player in a game with an enlarged player set. Then the characteristic function for the game with ln players could be derived. A reasonable solution would be relaxed core where the only permitted coalitions are those which do not split up each player's criteria set in the original game. This approach implicitly assumes that for each imputation each player in the original game receives the sum of the coordinates associated with each criterion under his control. Other solution concepts would be applied to this characteristic function derived from the full payoff space.

(c) *Solution concepts yielding a unique solution point*: Arbitration schemes such as the Shapely value, nucleolus and convex nuclei can be applied to all of the single-criterion game points associated with a multicriteria game $v = v_1,, v_2, \ldots , v_l)$. For example, as in the preceding section, $(\bar{v})'$, $(v*)$ or $v**$ would be reasonable game points that the players might agree to use.

If the players can agree to a set of probability weight vectors Ω, then each arbitration scheme generates the set of solution points for the games parametrized by all $\lambda \varepsilon \Omega$. In view of the continuity of v_λ^N *and* v_λ^c and continuity property of the arbitration schemes, one can show that the Shapley value, nucleolus and convex nuclei are continuous functions of the parameter λ.. As in the preceding section, we may consider the characteristic function induced by the full payoff space of the underlying normal form. Then arbitration schemes can be applied to this single-criterion characteristic function. Again, the assumption is that at the final solution point, each player receives the sum of the coordinates associated with the criteria under his control.

We can also derive a number of new arbitration schemes using the parametrization process. For example, if $C((\overline{v'}))) = \emptyset$ or $C(v**) = \emptyset$, the players may treat $(\overline{v'})$ or $v**$ as a kind of 'utopia' game point. Both $(\overline{v'})$ and $v**$ represent the best value a coalition can have under certain conditions. Therefore, the players may agree to use the imputation x whose game point image, $\phi(x)$, best approximates $(\overline{v'})$ or $v**$ in the sense of some distance measure, such as an l_p-norm.

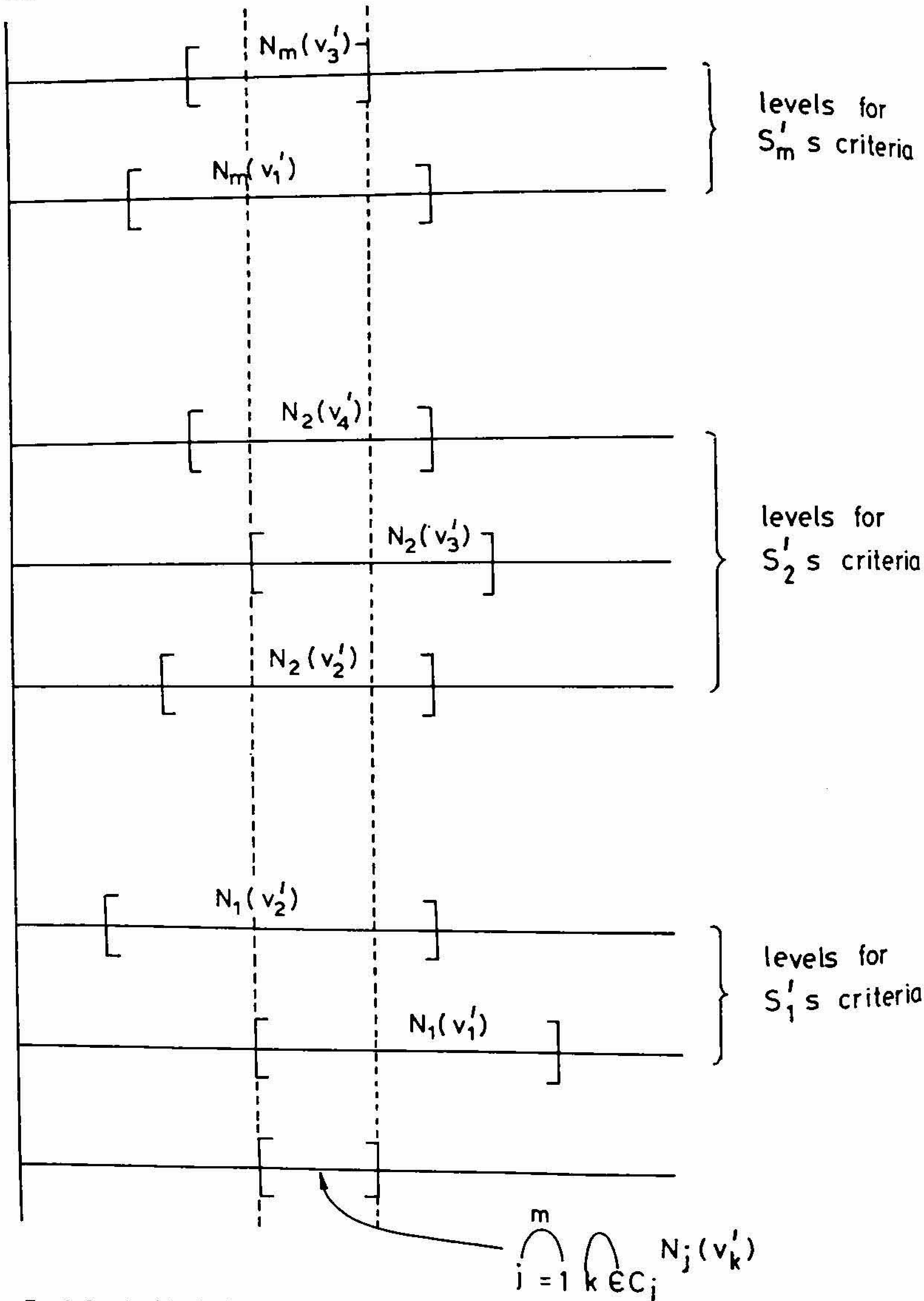

FIG. 4. Levels of domination structures yielding a relaxed multicriteria core.

Another arbitration scheme using the parametrization process involves using the parametrized games $(v_\lambda^N)'$ and $(v')_\lambda^c$ to approximate v^{**} and $(\overline{v'})$, respectively. In other words, the players would agree to use the parameter λ_0 for which the distance $d((v_\lambda^N)'$, v^{**} or the distance $d((v')_\lambda^c, (\overline{v'}))$ is minimal. Then after λ_0 has been located, solution concepts from the previous section could be applied to $(v_{\lambda_0}^N)'$ or $(v')_{\lambda_0}^c$.

(*d*) *The general case*: Here we shall discuss the various formulations in which the players may have different criteria. For the first approach we consider a (0, 1)-normalized multicriteria game $v = (v'_1, v'_2, \ldots, v'_l)$ where each proper coalition of more than one player, $S_j, j = 1, 2, \ldots, m$, determines an index set $C_j \subseteq \{1, 2, \ldots, l\}$ containing the indices of the criteria which are of concern to S_j as a coalition. In this case we would expect that each coalition S_j will object to potential distribution laws only on the basis of the criteria indexed by C_j.

A natural solution approach would then be to use a relaxed multicriteria core, namely,

$$\bigcap_{j=1}^{m} \bigcap_{k \varepsilon C_j} N_j(v'_k),$$

the imputations which are nondominated under the classical domination structure for any coalition S_j with respect to an criterion indexed iny C_j. We use $N_j(v'_k)$ to denote the imputations which are nondominated *via* coalition S_j on criteria v'_k.

In Fig. 4, there is a set of 'levels' (copies of I) for each coalition. Given a coalition S_j, there is one level for each criterion indexed in C_j. We project all the levels on to a single copy of I and take the intersection to obtain the relaxed multicriteria core.

Now, given a normal-form multicriteria game where not all of the players necessarily have the same criteria, we consider the methods for inducing a characteristic-function-form game. Each coalition will have as its criteria set, the criteria which are of concern to at least one of the members of the coalition. Intuitively, an individual player would not join a coalition unless the coalition pays some attention to all of that player's criteria. For each criterion for a given coalition, we could compute the maximum value by ignoring all of the other criteria. Clearly, a coalition would not necessarily obtain the maximum values thus derived on all of its criteria simultaneously, but these values could serve as a basis for bargaining or arbitration. If a given coalition is not concerned about a particular criterion, one may assign that coalition a value of zero on that criterion. In this way we construct a characteristic function where each coalition has a value for every criterion in the game. We then take the (0, 1)-normalization and proceed to use the appropriate solution approach.

A similar, but intuitively less appealing, approach is to assume that each coalition considers only those criteria which concern all of its members. In this case, we could assume that there is at least one criterion common to all the players.

Yet another approach would be to induce a characteristic function from the full payoff space by viewing that space as the payoff space for a single-criterion game with an enlarged player set (for a given Pi, there is one 'fictitious' player controlling each of player i's criteria). As in Section 3.3, we would place a restriction on the coalitions that are permitted to form. No group of fictitious players associated with the criteria set of one player in the original game could be split among more than one coalition in the fictitious game.

3.4. *Discussion*

A variety of problems remain unsolved. One of them is the extension of domination structures and multicriteria concept to various categories of games which are not discussed here. These could be games in partition function form[14], games without side-payments[15–17], constrained games[18] and differential games[19].

4. Equilibrium solution in multicriteria games

4.1. *Introduction*

In game theory, the word 'equilibrium' connotes a situation where a player cannot improve his outcome by acting unilaterally. If the game has N players then an N-tuple of player strategies (belonging to the product space of the individual strategy spaces) is said to constitute an equilibrium solution if it results in an equilibrium in the above sense. The component strategies of the N-tuple are called the equilibrium strategies of the respective players. It has to be emphasized that an equilibrium strategy for a player cannot be specified in conjunction with the equilibrium strategies of the other players. In single-criterion games common solution concepts are the Nash solution (non-cooperative games) and the Pareto solution (cooperative games). For multicriteria cooperative games the Pareto solution remains a viable solution. For multicriteria non-cooperative games an equilibrium solution was first proposed by Shapley[20]. His definition was motivated by a problem from a combat situation. Since then there have appeared a number of papers[21–27] on, and related to, this topic. In this section we survey some of these articles. We begin with the definition and characterization of equilibrium solutions in multicriteria games, then we look at the relationship between equilibrium and minimax in the multicriteria case, and finally we view some applications.

4.2. *Equilibrium solutions—Definition and characterization*

As mentioned in the introduction, Shapley[20] was the first to provide a definition of equilibrium in multicriteria games. His definition was for zero-sum games. Here, following Ghose and Prasad[26], we present a generalization of Shapley's definition to nonzero-sum games. Before doing that we shall formulate the game and establish the terminology.

We consider a two-person nonzero-sum multicriteria game (referred to as the M-game) with strategy spaces X and Y and cost $J^i: X \times Y \rightarrow R^p$, $i = 1, 2$, with component functions $J^i_j: X \times Y \rightarrow R$, $j = 1, 2, \ldots, p$. We also define, for future use, a scalarized game $S(\alpha^1, \alpha^2)$ derived from the M-game. The game $S(\alpha^1, \alpha^2)$ is a two-person nonzero-sum single-criterion game with strategy spaces X and Y and cost functions $\hat{J}^i: X \times Y \to R$, given by $\hat{J}^i(x, y) \underline{\underline{\Delta}} \Sigma_{j=1}^{p} \alpha_j^i J_j^i(x, y)$, $\alpha^i \varepsilon R^p$, $i = 1, 2$. We assume the players to be minimizers.

The following notation will be used in this and the following section. We use the symbol $\leqq$ to denote the natural order on R^p, that is, for $x, y \varepsilon R^p$, $x \leqq y$ if and only if $x_i \leq y_i$ for all $i = 1, 2, \ldots, p$; also $x \lneqq y$ if and only if $x \leqq y$ and $x \neq y$; and we define $R_+^p = \{z \varepsilon R^p \mid z \geqq 0\}$.

A pair of strategies (x^*, y^*), $x^* \varepsilon X$, $y^* \varepsilon Y$ is the Nash equilibrium solution of the nonzero-sum M-game if and only if

(i) $J^1(x^*, y^*) \ngeq J^1(x, y^*)$ for all $x \varepsilon X$, and
(ii) $J^2(x^*, y^*) \ngeq J^2(x^* y)$ for all $y \varepsilon Y$.

If the M-game is zero-sum, then (x^*, y^*) is an equilibrium solution if and only if $J^1(x^*, y) \ngeq J^1(x^*, y^*) \ngeq J^1(x, y^*)$ for all $x \varepsilon X$ and for all $y \varepsilon Y$. Such solutions are also known as generalized saddle points[22] or vector saddle points[28]. We now consider the existence and characterization of equilibrium points in the M-game; as before, we follow the notations of Ghose and Prasad[26]. But first we make the following assumptions:

(i) $X \subseteq R^n$ and $Y \subseteq R^m$ are non-empty, convex and compact.
(ii) The functions J_j^1 are strictly convex in x for a fixed y and the functions J_j^2 are strictly convex in y for a fixed x; $j = 1, 2, \ldots, p$.
(iii) The functions J_j^i, $i = 1, 2, \ldots, p$ are jointly continuous in (x, y).

The following theorems (4.1–4.4) are stated with the above assumptions.

Theorem 4.1. *If $(x^*, y^*) \varepsilon X \times Y$ is a Nash equilibrium solution for the $S(\alpha^1, \alpha^2)$ game with some $\alpha^1 > 0$, $\alpha^2 > 0$, then (x^*, y^*) is also a Nash equilibrium solution for the M-game.*

Theorem 4.2. *For every $\alpha^i \geqq 0$, $\Sigma_{j=1}^{p} \alpha_j^i = 1$, $i = 1, 2$, the $S(\alpha^1, \alpha^2)$ game has a Nash equilibrium solution.*

Theorem 4.3. *There exists a Nash equilibrium solution for every M-game.*

Theorem 4.4 (*Necessary condition*). *If $(x^*, y^*) \varepsilon X \times Y$ is a Nash equilibrium solution for the M-game then there exist vectors $\alpha^i \geqq 0$, $\Sigma_{j=1}^{p} \alpha_j^i = 1$, $i = 1, 2$, such that (x^*, y^*) is also a Nash equilibrium solution for the $S(\alpha^1, \alpha^2)$ game.*

For the zero-sum case, Chan and Lau[28] have proved general results on the existence and characterization of equilibrium solutions (vector saddle points). They consider the strategy and criterion spaces to be subsets of locally convex topological spaces with the criterion space being partially ordered by a closed convex cone. Their results involve sub-differentials or Gateaux derivatives of the cost function. Some other results on this topic are found in Corley[23] and Tanaka[29]. Hannan[21] and Zeleny[24] have given some results on equilibrium solutions in matrix games: they also describe the computation of equilibrium solutions by conversion to a linear programming problem.

4.3. *Equilibrium and minimax*

Consider a two-person zero-sum single-criterion game with strategy spaces U and V and cost function $L : U \times V \rightarrow R$. Then it is a well-known fact that this game has an equilibrium solution (saddle point) if and only if

$$\min_{u \varepsilon U} \max_{v \varepsilon V} L(u, v) = \min_{v \varepsilon V} \max_{u \varepsilon U} L(u, v).$$

There have been several attempts to extend the definition of minimax and maximum to the multicriteria zero-sum case and to investigate the connections between minimax and equilibrium. Nieuwenhuis[27] has generalized the definitions of minimax and maximum solutions to static multicriteria games and has also proved some minimax theorems. Corley[23] has addressed the issues of minimax, maximum and equilibrium solutions in multicriteria matrix games. Tanaka[29] has defined weak saddle points for static multicriteria games and has investigated the relation of inclusion among the set of minimax, maximin and weak saddle points. Ferro[30] provides some minimax theorems for vector-valued function. We refer the reader to these sources for detailed expositions. An interesting fact is that in the multicriteria case a solution which is both a minimax and a maximin is also an equilibrium solution.

4.4. *Applications*

As mentioned in Section 4.1, Shapley[20] was motivated to define equilibrium strategies in multicriteria games by a real-world problems. This was to analyse a combat situation in which movement of forces and the inhibition of such movements played a critical part. Chan and Lau[28] have discussed the application of equilibrium solutions (vector saddle points) in distributed-parameter differential games. Undaneta and Chankong[31] have employed a solution concept similar to the equilibrium solution to obtain controller settings for systems running under disturbances. They mention tuning power system stabilizers and the coordination of directional overcurrent relays taken from the power system field as typical applications for each solution concepts.

5. Security strategies in multicriteria games

5.1. *Introduction*

The concept of security in a game is based on a worst-case scenario in which a player assumes that all the other players choose strategies, in response to his chosen strategy, so as to yield the worst possible value of his criterion. The player's criterion value in such a situation is called the security level corresponding to the player's strategy. A security strategy for a player is that strategy which yields him the 'best' security level. There are thus two fundamental notions involved in the definition of a security strategy for a player.

(i) guaranteed criterion value, or security level, of a strategy, and
(ii) selection of the strategy (or strategies) which yield the best security level.

Security strategies, therefore, have the desirable property of yielding a guaranteed criterion value irrespective of the other player's strategies. They are intended to cope with situations where a player is uncertain as to the strategies employed by the other players and wants to adopt a 'safe' mode of play.

In single-criterion games security strategies are defined as min sup or max inf strategies[32]. However, in multicriteria games, the situation is not quite so straightforward. This is due to the fact that different order relations can be specified on the criterion space such as the natural order and the lexicographic order. The concept of optimality commonly used with the natural order is Pareto optimality. Haurie[33] has provided a definition of security strategies in mulcriteria games based on Pareto optimality. This definition is consistent with the two fundamental notions underlying security concepts mentioned earlier. Before considering the definition, we introduce some terminology and notation which we will follow throughout the present section unless otherwise stated.

We shall consider, without loss of generality, a two-person ($P1$ and $P2$) zero-sum multicriteria game M, with strategy spaces X (for $P1$) and Y (for $P2$). The cost function is $f: X \times Y \rightarrow R^p$; $P1$ is the minimizer and $P2$ the maximizer. We define an index set $I = \{1, 2, \ldots, p\}$. We shall, without loss of generality, consider security strategies for $P1$ only. We define the auxiliary cost function for $P1$, $v : X \rightarrow R^p$ as

$$v(x) = \left(\sup_{y \varepsilon Y} f_i(x, y))\right)_{i \varepsilon I}$$

and assume that v is well defined. Also, for situations where the supremum is attained in the above, we define $Y_i(x) = \{\bar{y} \ \varepsilon \ Y \mid f_i\ (x, \bar{y}) = v_i(x)\}$. We now proceed to define, following Haurie[33], a security for $P1$ in game M.

A strategy $x^* \varepsilon X$ is a *security strategy* for $P1$ if and only if, for all $x \varepsilon X$, $v(x) \leqq v(x^*) \Rightarrow v(x) = v(x^*)$. Ghose[34] has called such strategies *Pareto optimal security strategies* (POSS).

In this section, we first survey characterization of security strategies based on scalarization and directional derivative techniques. We then briefly survey some results on dynamic games, and finally, we discuss some applications of security in the multicriteria game context.

5.2. *Characterization of security strategies*

(a) *Results based on scalarization*: The scalarization process is a commonly used technique in vector optimization problems. Goffin and Haurie[35] have obtained results for security strategies based on this technique. As before, we consider the game M; we first state the following sufficiency conditions.

Theorem 5.1. *Let* $\alpha_i > 0$, $i \in I$, $x^* \in X$ *be such that for all* $x^* \in X$ $\sum_{i=1}^{p} \alpha_i v_i(x^*) \leq \sum_{i=1}^{p} \alpha_i v_i(x)$. *Then* x^* *is a security strategy for* $P1$.

Corollary 5.1. *Let* $\alpha_i > 0$, $i \in I$, *and* $x^* \in X$ *be such that for all* $x \in X$,

$$\sup_{y_1, y_2, \ldots, y_p \in Y} \sum_{i=1}^{p} \alpha_i f_i(x^*, y_i) \leq \sup_{y_1, y_2, \ldots, y_p \in Y} \sum_{i=1}^{p} \alpha_i f_i(x, y_i).$$

Then x^* *is a security strategy for* $P1$.

For a necessary scalarization condition to hold, a convexity assumption is needed.

Theorem 5.2. *If* $v(X)+R_+^p$ *is convex and* x^* *is a security strategy for* $P1$, *then there exist* $\alpha_i \geq 0$, *for all* i *and* $\alpha_k > 0$, *for some* k *such that, for all* $x \in X$, $\sum_{i=1}^{p} \alpha_i v_i(x^*) \leq \sum_{i=1}^{p} \alpha_i v_i(x)$.

Schmitendorf[36] has provided scalarization results similar to those of Goffin and Haurie[35]. Ghose[37] has obtained results on security strategies in multicriteria matrix games. He shows their existence and also provides necessary and sufficient scalarization conditions. He also proves that a finite number of scalarizations are sufficient to obtain all security strategies. Results similar to those of Goffin and Haurie[35] for a static continuous kernel multicriteria game are proved in Ghose and Prasad[26].

(*b*) *Results based on directional derivatives*: The first results in this area were reported by Goffin and Haurie[35]. Using Danskin's[38] and Bram's[39] results they obtained a necessary and sufficient Lagrange multiplier rule in the form of an inequality. Schmitendorf[36] obtained a multiplier rule in the form of an equality using a generalized Motzkin's lemma[40]. We now state his main result. For this the following assumptions are made:

(*i*) We consider the game M defined previously.
(*ii*) $X \subseteq R^n = \{x \mid g(x) \leqq 0)$, where g: $R^n \rightarrow R^q$ is C^1 on R^n.
(*iii*) $Y \subseteq R^m$ and is compact.
(*iv*) $f: X \times Y \rightarrow R^p$ is C^1.

Also, we denote the partial derivative of f_i, $i \in I$, with respect to x by $f_{ix}(x, y)$ and similarly, for g_i, $i = 1, 2, \ldots, q$, by $g_{ix}(x)$.

Theorem 5.3. *Let* $x^* \in X$ *be a security strategy for* $P1$. *Then there exist*

(*i*) *non-negative integers* α_i, $i \in I$,
(*ii*) *scalars* $\lambda_1^i \geq 0$, $i = 1, 2, \ldots, \alpha_1$, $\lambda_2^i \geq 0$, $i = 1, 2, \ldots, \alpha_2, \ldots, \lambda_p^i \geq 0$, $i = 1, 2, \ldots, \alpha_p$,
(*iii*) *scalars* $\mu_i \geq 0$, $i \in I$
(*iv*) *vectors* $y_1^i \in Y_1(x^*)$, $i = 1, 2, \ldots, \alpha_1$, $y_2^i \in Y_2(x^*)$, $i = 1, 2, \ldots, \alpha_2, \ldots, y_p^i \in Y_p(x^*)$, $i = 1, 2, \ldots, \alpha_p$, *such that*

$$\sum_{i=1}^{\alpha_1} \lambda_1^i f_{1x}(x^*, y_1^i) + \sum_{i=1}^{\alpha_2} \lambda_2^i f_{2x}(x^*, y_2^i) + \ldots +$$

$$\sum_{i=1}^{\alpha_p} \lambda_p^i f_{px}(x^*, y_p^i) + \sum_{i=1}^{p} \mu_i g_{ix}(x^*) = 0, \tag{9}$$

$$\mu_i \; g_i\,(x^*) = 0,\; i \in I, \tag{10}$$

$$\sum_{i=1}^{\alpha_1} \lambda_1^i + \sum_{i=1}^{\alpha_2} \lambda_2^i + \ldots + \sum_{i=1}^{\alpha_p} \lambda_p + \sum_{i=1}^{p} \mu_i \neq 0. \tag{11}$$

This theorem provides necessary conditions for weak Pareto optimality and is of the Fritz–John type; Kuhn–Tucker-type conditions may be obtained by imposing certain constraint qualifications. Schmitendorf[40] has also proved sufficiency condition, with the convexity assumptions, which we now state.

Theorem 5.4. *Let g be a convex function and, for every $y \in Y$, $f_i(., y)$, $i \in I$, be a convex function. Let $x^* \in X$. If conditions* (*i*)–(*iv*) *and equations* (9) *and* (10) *of Theorem* 5.3 *are satisfied with $\alpha_i > 0$, $i \in I$ and $\Sigma_{i=1}^{\alpha_k} \lambda_k^i \neq 0$, $k = 1, 2, \ldots, p$, then x^* is a security strategy for P1.*

Schmitendorf[40] has also provided a sufficiency condition assuming the functions f and g to be C^2. Ishizuka and Shimizu[41] have obtained results similar to those of Schmitendorf[40]. Ishizuka[42] has derived very general necessary conditions using a generalized Tucker's theorem. His results are based on directional derivative estimates for the optimal value function in mathematical programming[43, 44]. Most of the earlier results can be obtained as special cases of Ishizuka's theorems.

The first results for security strategies in dynamic multicriteria games were obtained by Haurie[33]. He provides necessary conditions for games with open-loop information structure using results from the theory of reachability of perturbed systems[45, 46]. Schmitendorf[47] has obtained some sufficiency results for games with open-loop information structure.

5.3. *Applications*

One of the main applications of security strategies is in worst-case designs. Examples of this in the single-criterion case are the min-max controller design[48,49], min-max filter design[50] and stabilization of vibrations in buildings during earthquakes[51]. In the multicriterion case, Ishizuka and Shimizu[41] and Shimizu and Hirata[50] have reported applications of security strategies to the design of two-dimensional recursive digital filters. Prasad and Ghose[52], Grimm *et al.*[53] and Ghose and Prasad[54] have reported the applications of security strategies in bicriterion games to two-target or combat differential games. In Ghose and Prasad[54, 55] the delineation of qualitative outcome regions using security strategies of the players has been discussed.

6. Conclusions

In this paper we have attempted to survey the major results in the area of multicriteria game theory. Key solution concepts and their applications to many real-world problems have been discussed. Blackwell's approachability–excludability theory was shown to have potential applications in the area of microeconomics, in general, and repeated games, in particular. In the area of cooperative games several solution concepts were shown to be possible in the multicriteria framework. A number of unsolved problems were also identified. In a purely non-cooperative mode of play, equilibrium and security concepts were shown to play a crucial role in defining acceptable strategies for players. Here too, several practical applications of the theory were discussed.

Acknowledgements

The authors thank one of the referees of the constructive comments provided and for bringing to their notice two important references[7,21].

References

1. VON NEUMANN, J. AND MORGENSTERN, O. *Theory of games and economic behaviour*, 1944, Princeton University Press.
2. EATWELL, J., MILGATE, M. AND NEWMAN, P. (eds) *The new palgrave: Game theory*, 1989, Macmillan.
3. BLACKWELL, D. An analog of the minimax theorem for vector payoffs, *Pacif. J. Math.*, 1956, **6**, 1–8.
4. HOU, T. F. Approachability in a two-person game, *Ann. Math. Stat.*, 1971, **42**, 735–744.
5. SACKROWITZ, H. A note on approachability in a two-person game, *Ann. Math. Stat.*, 1972, **43**, 1017–1019.
6. HOU, T. F. Weak approachability in a two-person game, *Ann. Math. Stat.*, 1969, **40**, 780–813.
7. VIEILLE, N. Weak approachability, *Math. Op. Res.*, 1992, **17**, 781–791.
8. SHIMKIN, N. AND SHWARTZ, A. Guaranteed performance regions in Markovian system with competing decision makers, *IEEE Trans.*, 1993, **AC-38**, 84–95.
9. KATZ, M. Infinitely repeatable games, *Pacif. J. Math.*, 1960, **10**, 879–885.
10. BENOIT, J. P. AND KRISHNA, V. Finitely repeated games, *Econometrica*, 1985, **53**, 905–922.
11. BERGSTRESSER, K. AND YU, P. L. Domination structures and multicriteria problems in *N*-person games, *Theory Decision*, 1972, **8**, 5–48.
12. YU, P. L. Cone convexity, cone extreme points and non-dominated solutions in decision problems with multiobjectives, *J. Optimiz. Theory Applic.*, 1974, **14**, 319–377.
13. YU, P. L. *Multiple-criteria decision making: Concepts, techniques and extensions*, 1985, Plenum Press.
14. THRALL, R. M. AND LUCAS, W. F. *N*-person games in partition function form, *Nav. Res. Logistics Q.*, 1963, **10**, 281–298.

15. BILLERA, L. J. Some recent results in n-person game theory, *Math. Progmg*, 1971, **1**, 58–67.

16. SHAPLEY, L. S. AND SCARF, H. On cores and indivisibility, *J. Math. Econ.*, 1974, **1**, 23–27.

17. JENTZSCH, G. Some thoughts on the theory of cooperative games, *Ann. Math. Stud.*, 1964, **52**, 407–442.

18. CHARNES, A. AND SORENSEN, S. Constrained n-person games, *Int. J. Game Theory*, 1974, **3**, 141–158.

19. ISAACS, R. *Differential games*, 1965, Wiley.

20. SHAPLEY, L. S. Equilibrium points in games with vector payoffs, *Nav. Res. Logistics Q.*, 1959, **6**, 57–61.

21. HANNAN, E. L. Reformulating zero-sum games with multiple goals, *Nav. Res. Logistics Q.*, 1982, **29**, 113–118.

22. RODDER, W. A generalized saddle point theory: Its application to duality theory for linear vector optimum problem, *Eur. J. Op. Res.*, 1977, **1**, 55–59.

23. CORLEY, H. W. Games with vector payoffs, *J. Optimiz. Theory Applic.*, 1985, **47**, 491–498.

24. ZELENY, M. Games with multiple payoffs, *Int. J. Game Theory*, 1975, **4**, 179–191.

25. GHOSE, D. AND PRASAD, U. R. Solution concepts in two-person multicriteria games, *J. Optimiz. Theory Applic.*, 1989, **63**, 167–189.

26. GHOSE, D. AND PRASAD, U. R. Solution concepts in continuous-kernel multicriteria games, *J. Optimiz. Theory Applic.*, 1991, **69**, 543–553.

27. NIEUWENHUIS, J. W. Some minimax theorems in vector-valued functions, *J. Optimiz. Theory Applic.*, 1983, **40**, 463–475.

28. CHAN, W. L. AND LAU, W. T. Vector saddle point and distributed parameter differential games, *Comput. Math. Applic.*, 1989, **18**, 195–208.

29. TANAKA, T. Some minimax problems of vector-valued functions, *J. Optimiz. Theory Applic.*, 1988, **59**, 505–524.

30. FERRO, F. Minimax theorems for vector-valued functions, *J. Optimiz. Theory Applic.*, 1991, **68**, 35–48.

31. UNDANETA, A. J. AND CHANKONG, V. A multiobjective minimax approach to controller settings for systems running under disturbances, *Contr. Theory Advd Technol.*, 1989, **5**, 391–411.

32. BAŠAR, T. AND OLSDER, G. J. *Dynamic non-cooperative game theory*, 1982, Academic Press.

33. HAURIE, A. On Pareto optimal decisions for a coalition of a subset of players, *IEEE Trans.*, 1973, **AC-18**, 144–148.

34. GHOSE, D. *Multicriteria games with applications to two-target game problems*, Ph.D. Thesis, Deptt of Electrical Engng, Indian Institute of Science, Bangalore, India, May 1989.

35. GOFFIN, J. AND HAURIE, A. Necessary conditions and sufficient conditions for Pareto optimality in a multicriterion perturbed system, *Proc. 5th IFIP Conf. on Optimiz.*, 1973, pp. 184–193.

36. SCHMITENDORF, W. E. Static multicriteria problems: necessary conditions and sufficient conditions, *Proc. IFAC Symp. on Large-Scale Systems*, Udine, Italy, 1976, pp. 441–447.

37. GHOSE, D. A necessary and sufficient condition for Pareto optimal security strategies in multicriteria matrix games, *J. Optimiz. Theory Applic.*, 1991, **68**, 463–481.

38. DANSKIN, J. The theory of max-min with applications, *SIAM J. Appl. Math.* 1966, **14**, 641–644.

39. BRAM, J. The Lagrange multiplier theorem for max-min with several cor traints, *SIAM J. Appl. Math.*, 1966, **14**, 665–667.

40. SCHMITENDORF, W. E. Necessary conditions and sufficient conditions for static min-max problems, *J. Math. Anal. Applic.*, 1977, **57**, 683–693.

41. ISHIZUKA, Y. AND SHIMIZU, K. Necessary and sufficient conditions for the efficient solutions of nondifferentiable multiobjective problems, *IEEE Trans.*, 1984, **SMC-14**, 625–629.

42. ISHIZUKA, Y. Optimality conditions for directionally differentiable multiobjective programming problems, *J. Optimiz. Theory Applic*, 1992, **72**, 91–111.

43. ROCKAFELLAR, R. T. Directional differentiability of the optimal value function in a nonlinear programming problem, *Math. Progmg Study*, 1984, **21**, 213–226.

44. FIACCO, A. V. Optimal value continuity and differential stability bounds under Mangasarian–Fromovitz constraint qualifications, *Mathematical programming with data perturbations*, Vol. 2 (A. V. Fiacco, ed.), 1983, pp. 65–90, Marcel Dekker.

45. DELFOUR, M. C. AND MITTER, S. K. Reachability of perturbed linear systems and min sup problems, *SIAM J. Contr.*, 1969, **7**, 521–533.

46. BERTSEKAS, D. P. AND RHODES, I. B. On the minimax reachability of targets and target tubes, *Automatica*, 1971, **7**, 233–247.

47. SCHMITENDORF, W. E. Optical control of systems with multiple criteria when disturbances are present, *J. Optimiz. Theory Applic.*, 1979, **27**, 135–146.

48. WITSENHAUSEN, H. S. A minimax control problem for sampled linear systems, *IEEE Trans.*. 1968, **AC-13**, 5–21.

49. GLOVER, J. D. AND SCHWEPPE, F. C. Control of linear dynamic systems with set constrained disturbances, *IEEE Trans.*, 1971, **AC-16**, 411–423.

50. SHIMIZU, K. AND HIRATA, T. Optimal design using min-max criteria for two-dimensional recursive digital filters, *IEEE Trans.*, 1986, **CAS-33**, 491–501.

51. GUTMAN, S. AND LEITMANN, G. Stabilizing feedback control for dynamical systems with bounded uncertainty, *Proc. IEEE Conf. on Decision and Control*, Miami, 1976, pp. 94–99.

52. PRASAD, U. R. AND GHOSE, D. Formulation and analysis of combat problems as zero-sum bicriterion differential games, *J. Optimiz. Theory Applic.*, 1988, **59**, 1–24.

53. GRIMM, W., PRASAD, U. R., WELL, K. H. AND BERGER, E. Open loop guidance for prelaunch maneuvering in medium-range air combat, *Proc. IEEE Conf. on Decision and Control*, 1988, vol. 2, pp. 1442–1447.

54. GHOSE, D. AND PRASAD, U. R. Multicriterion differential games with applications to combat problems, *Comput. Math. Applic.*, 1989, **18**, 117–126.

55. GHOSE, D. AND PRASAD, U. R. Determination of rational strategies for players in two target games. *Comput. Math. Applic.*, 1993, **26**, 1–11.