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Degree theory in linear complementarity

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Abstract

In this paper, a brief survey of some of the results in linear complementarity theory is presented, using the concept of the degree of a suitably constructed piecewise linear map. Local and global degrees of an LCP map are quite useful in identifying subclasses of Q and Q_0-matrices. Some of the well-known characterizations of these classes are given a newer perspective in terms of degree theory. The class of superfluous matrices defined using global degree is highlighted with relevant examples. Properties of a simplicial polytope relating to an LCP map are of use in global degree analysis. One of the sections here deals with these results. Finally, the use of degree theory in the study of sensitivity and solution stability of linear complementarity problems is brought out.

Keywords: Degree theory, linear map, sensitivity, solution stability, mathematical programming.

1. Introduction

Linear complementarity problem, one of the problems of mathematical programming, has gained immense importance in the last few decades among researchers in various fields. For instance, it arises naturally in many engineering and economic applications like elasto-plastic analysis and portfolio selection problems[1]. Mathematically, the problem can be formulated as follows.

Given an n-vector q and a matrix $M \varepsilon R^{n \times n}$, the linear complementarity problem, denoted by LCP (q, M), is that of finding a nonnegative vector $z \varepsilon R^n$ such that

$$\begin{aligned} w &= Mz + q \geq 0 \\ w^t z &= 0. \end{aligned} \tag{1}$$

A nonnegative vector z satisfying (1) is said to be a solution for the LCP (q, M). The set of all solutions for (1) is denoted by SOL(q, M).

This problem provides an equivalent formulation for many mathematical programming problems like linear programming and convex quadratic programming. The texts by Cottle *et al*[1]. and Murty[2] serve as excellent references on the theory and applications of linear complementarity. The problem of finding a Nash equilibrium point for a bimatrix game was first posed as a linear complementarity problem by Lemke[3]. As a result, he proposed an algorithm for solving the LCP(q, M), which is now well known in the field as Lemke's algorithm.

Besides the above applications, the linear complementarity problem contains two important features that are central to the study of mathematical and equilibrium programming problems. One of them is the concept of complementarity, which plays a key role in nonlinear programming and economic general equilibrium problems. The other factor is linearity, which once the complementarity conditions are chosen, is easier to analyse. These two concepts together help in understanding complex problems. An excellent bibliography on such applications and theory is contained in the books by Cottle *et al.*[1] and Murty[2].

A considerable amount of literature in linear complementarity is devoted to the question of identifying whether the LCP(q, M) has a solution or not, given any specific vector $q \in R^n$ and a square matrix M. Fixing M, for any $q \in R^n$, if the LCP(q, M) has a solution, matrix M is said to belong to class Q. If the set of all q for which LCP(q, M) has a solution is a convex set, then the matrix is known as a Q_0-matrix. Several sufficient conditions were given by Ingleton[4], Karamardian[5] and Lemke[3] for a matrix to belong to class Q; some sufficient conditions were also derived for a matrix to be a Q_0-matrix by several authors, like Garcia[6], Todd[7], Doverspike and Lemke[8], Aganagic and Cottle[9] and Al-Khayyal[10]; the results on Q-matrices were generalized by Murty[11] and Saigal[12]. However, an efficient method of determining membership of any given matrix in these classes is yet to be discovered.

One of the methods of analysing the above-mentioned questions is by reformulating the linear complementarity problem into a problem of finding solutions for a suitably constructed map. The advantage of such a formulation is that one can make use of the concept of degree theory in studying such a system. In this paper, we would like to present a brief survey of the results that are derived in linear complementarity, using the idea of degree of a piecewise linear map. Also, it is possible to present some of the earlier contributions in linear complementarity in terms of local and global degrees of the map.

Before turning to these specific applications of degree theory, we shall briefly recount its history. Degree theory is a branch of differential topology and has been widely used in studying the geometrical aspects of differentiable mappings. The concept of degree of a map dates back to Kronecker[13]. The pertinent idea of using the degree of homotopy maps for solving problems goes back to the works of Poincare[14] and Bohl[15]. It is one of the important tools in the study of generalized equations, optimal control and variational inequality problems.

In the recent years, degree theory has become one of the fundamental ideas in the study of linear complementarity problems. Many authors, the most notable being Garcia and Zangwill[16], Howe[17], Ha[18], Robinson[19], Howe and Stone[20], Kelly and Watson[21], Gowda and Pang[22] and Morris[23], have contributed interesting results to linear complementarity using the concepts of degree theory. The existence of zero-degree Q-matrices[17] speaks in itself how degree theory has furthered both geometric understanding and solution concepts of linear complementarity problems. These works are essentially done by reformulating the problem LCP(q, M) as a piecewise linear map. This technique is also useful in the study of sensitivity and stability analyses of solutions for nonlinear complementarity problems.

Our article is divided into four broader sections, leaving out this introduction. Next section defines some terminology and presents the basic results that are required, both in degree theory and complementarity; the local and global degrees of an LCP map are defined here. Several matrix classes are considered in Section 3, and we analyse their Q-nature using their degree. The fascinating class of superfluous matrices are considered here. Section 4 deals with certain sufficient conditions of Q_0-matrices that are known in the literature, and we present them based on local degrees of the LCP mapping. The structure of a simplicial polytope associated with the linear complementarity problem is also presented. Section 5 considers the stability and sensitivity analysis issues of linear complementarity problems using degree theory. In the end, we conclude with a few remarks and open problems existing in this field.

2. Preliminaries

We follow the notation and terminology as given by Cottle *et al*[1]. Matrices considered here are square, unless otherwise stated explicitly.

$R^{n\times n}$ stands for the class of all real square matrices of order n. Let $M \in R^{n\times n}$. For subsets $J, K \subseteq \{1, ..., n\}$, we denote by M_{JK} the submatrix of M, with rows and columns corresponding to the index sets J and K. For $J = \{1, ..., n\}$, M_{JK} is written for simplicity as $M_{.K}$. The matrix M_{JJ} for $J \subseteq \{1, ..., n\}$, denotes a principal submatrix of M. When $|J| = k$, M_{JJ}, is called a principal submatrix of order k. Then, the determinant of M_{JJ}, denoted by det M_{JJ}, is called a principal minor of order k. For any $J \subseteq \{1, ..., n\}$, $\bar{J}$ denotes the set $\{1, 2, ..., n\}\setminus J$. For any index i, e_i stands for the vector whose ith entry is 1, the rest being zero of appropriate order. By $v(M)$, we denote the minimax value of the two-person zero-sum game, with M as the payoff matrix. By a *signature matrix*, we mean a diagonal matrix, with its diagonal entries being 1 or -1.

A matrix $C(J) \in R^{n\times n}$ for $J \subseteq \{1, ..., n\}$, defined as $C(J)_{.j} = -M_{.j}$ if $j \in J$ and $C(J)_{.j} = I_{.j}$ otherwise, is known as a *complementary matrix*. A *complementary cone* of $[I \vdots -M]$ is denoted by pos$C(J)$, and is defined as the nonnegative linear combination of columns in $C(J)$.

Whenever LCP(q, M) has a solution z and $J = \{i{:}z_i > 0\}$, we have $q \in \text{pos}C(J)$. For more details about complementary cones, we refer to the books of Cottle *et al.*[1] and Murty[2]. We denote by $K(M)$, the union of vectors $q \in R^n$, for which the problem LCP(q, M) has a solution. If for every solution z of LCP(q, M),$(Mz + q) + z > 0$, then we say that the vector q is *nondegenerate* with respect to M. We denote by $\mathcal{K}(M)$ the set of all vectors in R^n which are not nondegenerate with respect to M. It can be seen that this set forms the union of vectors on the facets of complementary cones relative to M. By Sard's theorem[24], it follows that $\mathcal{K}(M)$ has measure zero. Hence, for a matrix to belong to class Q, it suffices to observe that for every $q \in R^n \setminus \mathcal{K}(M)$, LCP($q$, M) has a solution.

A complementary cone pos$C(J)$ relative to M is said to be *degenerate* if the matrix $C(J)$ is singular. It is said to be *strongly degenerate*, if pos$C(J)$ is degenerate and there exists a nonzero, nonnegative vector x such that $C(J)x = 0$. We denote by $\mathcal{C}$, the union of all the strongly degenerate cones relative to M. A *connected component* of a set S containing a point x, is defined as the union of all connected sets C such that $x \in C \subseteq S$. In general, the sets $R^n \setminus \mathcal{C}$ and pos$[I \vdots -M] \setminus \mathcal{C}$ can be written as disjoint unions of their respective connected components.

2.1. *Degree of a map*

The degree of a map can be described by stating its various properties. Let $f : \overline{\Omega} \to R^n$ be a continuous, proper map with $y \notin f(\partial\Omega)$, where Ω is a bounded open subset of R^n, $\partial\Omega$, the boundary of Ω and $\overline{\Omega}$ denotes the closure of Ω. Then, the degree of f at y relative to Ω is defined and is denoted by deg(f, Ω, y). In a sense, the degree gives the number of solutions of the equation $f(x) = y$. It satisfies many properties (see Ha[18] and Lloyd[25]) of which, the following we list are relevant to our study. A solution x of $f(x) = y$ is said to be *isolated* if there exists an open neighbourhood of x which does not contain any other solution of $f(x) = y$.

1. If deg$(f, \Omega, y) \neq 0$, then there is an $x \in \Omega$ such that $f(x) = y$.
2. Suppose that deg(f, Ω, y) is defined. Let ϕ be a continuous function on $\overline{\Omega}$ and let $\varepsilon = \text{dist}(y, \partial\Omega)$ denote the distance of y from $\partial\Omega$. If $\| f - \phi \| < \varepsilon$, where $\| \cdot \|$ stands for the Euclidean norm in R^n, then deg(ϕ, Ω, y) is defined and it equals deg(f, Ω, y).
3. (Homotopy invariance property): If $H(t, x)$ is a continuous function for $(t,x) \in [0,1] \times \overline{\Omega}$ and if $y \notin H(t, \partial\Omega)$ for all $t \in [0,1]$, then
$$\deg(H(0,\cdot), \Omega, y) = \deg(H(1,\cdot), \Omega, y).$$
4. Suppose $f(x) = y$ has an isolated solution x^* in Ω, and at x^*, f is differentiable with a nonsingular Jacobian matrix $J_f(x^*)$, then deg(f, Ω, y) is independent of Ω and is denoted by ind$f(y, x^*)$. It is given by
$$\text{ind}f\,(y, x^*) = sgn\ \det J_f\,(x^*)$$
where *sgn* stands for the sign of any number.

The classical way of defining the degree of a map is using the indices of the map at a point. That is, from property 4, if $x \in \Omega$ is an isolated solution of $f(x) = y$, then the *index* of f at x and y is well defined and is given by the sign of the Jacobian $J_f\,(x)$. Now, the *degree* of f at y could be defined by

$$\deg_f(y) = \sum_{x \in f^{-1}(y)} sgn \det\ J_f\,(x^*). \tag{2}$$

assuming that every solution of $f(x) = y$ is isolated.

Using the homotopy invariance property of degree, it is clear that if $y, y' \varepsilon R^n$ are so chosen that they lie in the same connected component of $R^n \setminus f(\partial\Omega)$ and the solutions of $f(x) = y$ and $f(x) = y'$ are all isolated, then $\deg_f(y) = \deg_f(y')$. Also, if there is only one connected component of $R^n \setminus f(\partial\Omega)$, then the local degree of f is the same for all $y \varepsilon R^n$ and is called the *global degree* of f. It is denoted by $\deg f$.

Degree theory deals with global properties of uniformly continuous maps. The problem LCP(q, M) can be studied using a piecewise linear map (which will be defined in the next section). Hence, the degree of that piecewise linear map is useful in studying the various properties of LCP(q, M).

2.2. *Degree of an LCP map*

We define a piecewise linear map $f_M : R^n \to R^n$, for a given $M \varepsilon R^{n \times n}$, as follows: let $f_M(e_i) = e_i$ and $f_M(-e_i) = -Me_i$, $i = 1, \ldots, n$,. For any $x \varepsilon R^n$, let f_M be linear, *i.e.*,

$$f_M(x) = \sum_{i=1}^{n} f_M(xe_i).$$

Eaves and Scarf[26] have proved that LCP(q, M) is equivalent to finding an $x \varepsilon R^n$ such that $f_M(x) = q$. If x belongs to the interior of some orthant of R^n and $\det M_{II}$ is nonzero where $I = \{i : x_i < 0\}$, then the index of f_M at x is well defined and is given by the sign of $\det M_{II}$, *i.e.*,

$$\operatorname{ind} f_M(q, x) = sgn \det M_{II}.$$

Let $f_M^{-1}(q)$ stand for the set of all vectors $x \varepsilon R^n$, such that $f_M(x) = q$. From the linear complementarity theory, it is clear that the cardinality of $f_M^{-1}(q)$ denotes the number of solutions for the LCP(q, M). In particular, if q is nondegenerate with respect to M, each index of f_M is well defined and we can define the *local degree* of M at q as

$$\deg f_M(q) = \sum_{f_M^{-1}(q)} sgn \det M_{II} \tag{3}$$

where the summation is taken over the index sets $I \subseteq \{1, \ldots, n\}$ such that $q \varepsilon \operatorname{pos} C(I)$.

Now, using the homotopy invariance property, it follows that if $q, q' \varepsilon R^n$ are such that both are nondegenerate with respect to M and lie in the same connected component of $R^n \setminus C$, then $\deg f_M(q') = \deg f_M(q')$. More specifically, when R^n is made up of a single connected component, we have the degree of M defined for every $q \varepsilon R^n$, except possibly for a set of vectors which have measure zero. Such a scalar is called the *global degree* of M and we denote it by $\deg M$. When R^n is made up of more than one connected component, the local degrees in each connected component need not be the same.

We present below two simple examples explaining these facts.

Example 1. Let

$$M = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}.$$

If we draw the complementary cones in R^2, it is easily seen that R^2 is made up of a single connected component; hence, deg$M$ is well defined. For a vector $q > 0$, LCP(q, M) has exactly three solutions. By adding their respective determinantal signs, we get deg$M = -1$.

Example 2. Let

$$M = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}.$$

Here, $R^2 \setminus C$ is made up of two connected components with their local degrees being 0 and -1.

2.3. R_0-matrices

M is said to be an R_0-matrix, if the LCP$(0,M)$ has a unique solution. When $M \varepsilon R_0$, there are no strongly degenerate complementary cones relative to M and C is empty[1]. Hence, $R^n \setminus C$ is made up of a single connected component and it makes sense to talk about the degree of the matrix M.

Combining the concepts of the earlier section, we can state the following result for R_0-matrices.

Theorem 2.1. Let $M \varepsilon R_0$. R^n is a *single connected component and degM is well defined.* If *degM is nonzero, then M is a Q-matrix.*

We present an example of this theorem.

Example 3. Let

$$M = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{bmatrix}.$$

M has got two principal minors zero. However, one can check that LCP$(0, M)$ has a unique solution $z = 0$, implying that M is an R_0-matrix. For a $q > 0$, LCP(q, M) has a unique solution. Therefore, deg$M = 1$ and M is a Q-matrix, degrees being 0 and -1.

In particular, when a matrix has all its principal minors nonzero (called a *nondegenerate* matrix), it belongs to R_0. An interesting property of degree is that it carries over to the principal pivot transforms also. This is stated below. For details on principal pivot transforms, we refer to Cottle *et al.*[1] and Murty[2].

Theorem 2.2. Let $M \varepsilon R_0$. *If* $degM = r$ *and* $\overline{M}$ *is a principal pivot transform of* M, *then* $deg\overline{M} = \pm r$.

For instance, it can be easily verified that the principal pivot transform M given in Example 1, with respect to the principal submatrix m_{11} is given by

$$M = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}.$$

which has got degree 1.

Among the class of nondegenerate matrices, the following are some of the well-known matrices studied in connection with the linear complementarity problem.

Definition 1: A matrix $M \varepsilon R^{n \times n}$ is called

(*i*) a P-matrix (N-matrix), if all its principal minors are positive (negative);

(*ii*) an almost P-matrix (almost N-matrix), if all its proper principal minors are positive (negative) and the determinant of M is negative (positive).

For several properties of these classes, see Murty[2], Cottle *et al.*[1], Mohan and Sridhar[27] and Olech *et al.*[28, 29]. The matrix given in Example 1 is an N-matrix. It can be verified that its inverse belongs to the class of almost P. We give below an example of a P-matrix.

Example 4: Let

$$M = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 3 & -5 \\ -1 & -1 & 2 \end{bmatrix}.$$

This is a P-matrix.

The classes of exact order matrices are defined as generalizations of the classes P and N.

Definition 2: A matrix $M \varepsilon R^{n \times n}$ is called an *N-matrix* (*P-matrix*) *of exact order k*, $k < n$, if every principal submatrix of order $(n - k)$ is an N-matrix (P-matrix), and if every principal minor of order r, $n - k < r \leq n$, is positive (negative). M is called a *matrix of exact order k*, if it is either a P-matrix of exact order k- or an N-matrix of exact order k.

The classes of P and N of exact order k are denoted by E_k^+ and E_k^-, respectively, and the class of exact order k by simply E_k. Mohan *et al.*[30] defined and studied these classes of matrices relating to the LCP(q, M). Constructing examples of these classes are a little difficult. We present below an example of an E_2^--matrix.

Example 5: Consider the matrix

$$M = \begin{bmatrix} -.9 & -2 & -2 & 2 & -2 \\ -1 & -.9 & -3 & 3 & -1 \\ -1 & -3 & -.9 & 3 & -1 \\ 1 & 3 & 3 & -.9 & 1 \\ -2 & -2 & -2 & 2 & -.9 \end{bmatrix}.$$

One can directly verify that every principal minor of order 1, 2 or 3 of M is negative and principal minors of order 4 and the determinant of M are positive. Hence, M is an N-

matrix of exact order 2. It will be understood from later sections that this matrix has zero degree and is not a Q-matrix.

2.4 Stewart's extension formula of degree

In our definition of degree of an LCP map, we made use of $q \varepsilon R^n$ which is nondegenerate with respect to M. The new formula due to Stewart[31] extends these definitions to vectors that are semi-nondegenerate with respect to M. Gowda[32] stated a neater form of this formula and made use of it in proving results on a number of solutions for certain classes of linear complementarity problems.

We will now define what we mean by a vector being semi-nondegenerate. Let z be a solution for the LCP(q ,M). For this z, let us define the following index sets:

$$I = \{i{:}z_i > 0\};\ J = \{i{:}\ w_i = (Mz + q)_i > 0\};\ K = \{i{:}\ z_i = (Mz + q)_i = 0\}. \tag{4}$$

Let us denote the Schur complement of M_{II} in

$$\begin{bmatrix} M_{II} & M_{IK} \\ M_{KI} & M_{KK} \end{bmatrix}$$

by M_{II}^S, which is defined as

$$M_{II}^S = M_{KK} - M_{KI}\, M_{II}^{-1} M_{IK}. \tag{5}$$

When I is empty, $M_{II}^S = M_{KK}$ and when K is empty M_{II}^S is taken as the identity matrix of appropriate order.

The solution z is called *semi-nondegenerate*, if the corresponding principal submatrix M_{II} of M where I is defined as in (4), is nonsingular. A vector $q \varepsilon R^n$ is said to be *semi-nondegenerate* with respect to M, if every solution of the LCP(q, M) is semi-nondegenerate. It follows[11] that for a nondegenerate matrix, every $q \varepsilon R^n$ is semi-nondegenerate with respect to M.

Stewart[31] proved that using the Schur complement, the index of $f_M(q)$ at any semi-nondegenerate solution z can be calculated as

$$\text{ind}\, f_M(q, z) = (sgn\ \det M_{II})\text{ind} f_{M_{II}^S}\ (0{,}0)$$

where the map $f_{M_{II}^S}(0)$ corresponds to the problem LCP($0, M_{II}^S$). The version we stated here is given by Gowda[32].

Now, Stewart's extension formula of degree of M for any $q \varepsilon R^n$ semi-nondegenerate with respect to M is given by the sum of all indices of the map f_M at each semi-nondegenerate solution of the LCP(q, M). It can be stated as

$$\deg M = \sum (sgn \det M_{II}) \deg M_{II}^S \tag{6}$$

where the summation is defined over all the index sets $I \subseteq \{1,...n\}$ such that $q \in \text{pos}C(I)$. Indeed, for M, a nondegenerate matrix, the above definition of degree is valid for any $q \in R^n$.

The following example illustrates the significance of Stewart's formula.

Example 6: Let

$$M = \begin{bmatrix} 1 & 5 & 2 \\ 5 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}.$$

It is easily seen that M has no principal minors zero and that $LCP(I_{.1}, M)$ has the unique solution $(w = I_{.1}, z = 0)$. Now using Stewart's extension formula for $q = I_{.1}$, the degree of M can be found out to be the same as that of the principal submatrix of M, leaving the first row and the first column. From Example 1, it follows that $\deg M = -1$.

In particular, when M is nondegenerate, one can observe that Stewart's formula gives a link between the degree of the matrix M and the degree of its $(n-1)$ principal submatrices. This quite often helps in calculating the number of solutions of the LCP(q, M) for some vectors q which are not nondegenerate with respect to M.

3. Global degree and Q-matrices

In this section, we consider various subclasses of Q-matrices that are known in the literature and analyse their global degree. We consider only R_0-matrices here; this is due to the fact that for matrices that do not belong to R_0, the global degree may not be defined.

Among R_0-matrices, there are several subclasses that are known to be Q, using a result based on the number of solutions for LCP(q,M) for some specified vectors, provided by Ingleton[4], Lemke[3] and Karamardian[5].

Theorem 3.1. *Let $M \in R^{n\times n}$ be an R_0-matrix. If $LCP(d, M)$ has a unique solution for a $d > 0$, nondegenerate with respect to M, then $M \in Q$.*

This result states that if the degree of M is 1, then M is a Q-matrix. The vector d in the above theorem can be made use of in processing these classes of Q-matrices, using Lemke's algorithm[3].

Generalizations of Theorem 3.1, given by Murty[11] and Saigal[12] require that $\deg M$ can be any odd number, not necessarily 1. These results gave insight into more subclasses of Q. But the power of degree theory will be seen in the subsections to follow, when either these techniques become cumbersome, or the problem LCP(q, M) has an even number of solutions for any $q \in R^n \setminus \mathcal{K}(M)$, in which cases the earlier results of linear complementarity fail to give us any clue as to why they must belong to the class Q.

3.1. *Matrices of degree* ± 1

We will at first consider nondegenerate matrices. One of the famous subclasses of Q is the class of P-matrices. Samelson *et al.*[33] gave a complete characterization of this class in terms of the uniqueness of solution for the linear complementarity problem. This is stated below.

Theorem 3.2. *A matrix* $M \in P$ *if and only if for any* $q \in R^n$, *LCP*(q, M) *has a unique solution.*

It then immediately follows that, deg$M$ is 1 whenever M is a P-matrix.

When the matrix M is an N-matrix, Kojima and Saigal[34] proved results on the number of solutions of the LCP(q, M). Several complete characterizations of this class in terms of the number of solutions for the LCP(q, M) for vectors q nondegenerate with respect to M, were proved by Mohan and Sridhar[27] and Parthasarathy and Ravindran[35]. Recently, Gowda[36] brought out a complete picture of results on a number of solutions of LCP(q, M) for some of the classes of N and almost N-matrices, using Stewart's extension formula of degree. Our next theorem gives a characterization of N-matrices. For a more detailed result on the number of solutions, we refer to Gowda[36].

Theorem 3.3. *Let* $M \in R^{n\times n}$ *have no zero entry. Then the following hold good*:

i) *Let* $M < 0$. M *is an* N-*matrix if and only if LCP*(q, M) *has exactly two solutions, for any* $q > 0$.
ii) *Let there exist a signature matrix* $S \neq \pm I$ *such that* $SMS < 0$. *Then,* M *is an* N-*matrix if and only if LCP*(q, M) *has a unique solution for any* $q \ngeq 0$, *and has exactly three solutions for any* $q > 0$.

One can see from these results, that the degree of M, when M is an N-matrix is either 0 or -1.

The classes of almost N- and almost P-matrices were studied by Olech *et al.*[28,29]; they proved that these classes have an intersection with Q if and only if their minimax values are positive. Almost P-matrices are the inverses of N-matrices and hence, the results of this class can easily be stated. A characterization of the number of solutions of LCP(q, M) when M is an almost N-matrix was presented by Mohan *et al.*[30] As cited earlier, Gowda[36] extended this result for any semi-nondegenerate vector for a subclass of almost N. We state these results in the following theorems:

Theorem 3.4. *Let* $M < 0$ *be an almost* N-*matrix. Let* $q \geq 0$. *If* $q \ngtr 0$, *then LCP*(q, M) *has a unique solution. Otherwise, LCP*(q, M) *has exactly*

i) 4 *solutions, if* $M^{-1}q < 0$;
ii) 3 *solutions, if* $M^{-1} q \nless 0$, $q \in pos(-M)$, $(M^S_{II})^{-1} < 0$, *where* M^S_{II} *is as defined in* (5) *with* $z = -M^{-1} q$;
iii) 2 *solutions, if either* $[q \notin pos(-M)]$, *or* $[M^{-1}q \nless 0$, $q \in pos(-M)$ *and* $(M^S_{II})^{-1} \nless 0]$.

Theorem 3.5. *Let $M \varepsilon R^{n\times n}$, for $n \geq 4$, with $S_1 M S_1 < 0$ and $S_2 M^{-1} S_2 < 0$ for signature matrices S_1, $S_2 \neq \pm I$. Then, M is an almost N-matrix if and only if LCP(q, M) has*

a) 3 *solutions, whenever $q > 0$ or $M^{-1}q < 0$;*
b) *a unique solution, if $M^{-1}q \nleq 0$ and $q \ngeq 0$.*

Any matrix M with either $M < 0$ or $M^{-1} < 0$, has deg$M = 0$. Hence, it is clear that M being almost N with all entries of M or M^{-1} being negative has degree zero. On the other hand, for an almost N-matrix M, if both M and M^{-1} have at least one positive entry, from the above theorems it follows that $M \varepsilon Q$ and the degree of M is -1.

Mohan *et al.*[30] observed that E_2-matrices follow the same pattern of N- and P-matrices in their value of degrees. Three different categories of these exact order two matrices were defined and analysed extensively in connection with the linear complementarity problem. Sridhar[37] extended some of these results for E_k, $k \geq 3$, and proved the following result on the degree of an E_k-matrix:

Theorem 3.6. *Let $M \varepsilon R^{n\times n}$ be an E_k^+ (E_k^-)-matrix, for $n \geq k + 1 (n \geq k + 3)$. Then degM is either -1, 0 or 1.*

It was also proved[37] that, for $M \varepsilon E_k \cap R^{n\times n}$, where $n \geq k + 3$, if M is a Q-matrix, then the degree of M is nonzero. We note that this need not in general be true for a nondegenerate Q-matrix. There are Q-matrices for which the degree can be zero. Our next section will provide us an insight into such classes of matrices.

So far in this section, we had considered only nondegenerate Q-matrices. However, most of our results on deg$M$ will still hold good, if we relax the condition of nondegeneracy and define them with R_0-property. For instance, call a matrix M a $P_0(N_0)$-matrix, if all its principal minors are nonnegative (nonpositive). Then the following result is analogous to the observations on N- and P-matrices:

Theorem 3.7. *Let $M \varepsilon R_0$. If*

i) $M \varepsilon P_0$, then $M \varepsilon Q$ and degM = 1;
ii) $M \varepsilon N_0$, $M \nless 0$, then $M \varepsilon Q$ and degM = $-$ 1.

The first part of the above result is due to Aganagic and Cottle[38]; they have also proved the converse, *viz.*, if $M \varepsilon P_0 \cap Q$, then M is an R_0-matrix. The second part of the above theorem has been independently observed by both Eagambaram and Mohan[39] and Pye[40].

3.2. *Superfluous matrices*

The converse of Theorem 2.1, *i.e.*, if $M \varepsilon R_0 \cap Q$, then its degree is nonzero, need not in general be true. The following are the two classical examples where the matrices are in fact, nondegenerate Q-matrices but their degrees are zero.

Example 7 (Kelly and Watson[21]): Let

$$M = \begin{bmatrix} 21 & 25 & -27 & -36 \\ 7 & 3 & -9 & 36 \\ 12 & 12 & -20 & 0 \\ 4 & 4 & -4 & -8 \end{bmatrix}.$$

Kelly and Watson[21] show that M is a nondegenerate Q-matrix. By considering any $q \varepsilon R^n \setminus \mathcal{K}(M)$, one can check that LCP(q, M) has an even number of solutions and the indices of the map f_M add up to zero, *i.e.*, deg$M = 0$. Their aim of considering M, was to show that it lies on the boundary of the class of nondegenerate Q-matrices, implying that this set is not for matrices of order 4.

Example 8 (Howe[17]): Let

$$M = \begin{bmatrix} -4 & 3 & 3 & 6 \\ 3 & -4 & 3 & 6 \\ 3 & 3 & -4 & 6 \\ 6 & 6 & 6 & -4 \end{bmatrix}.$$

It can be easily seen that M is nondegenerate. Howe[17] proved that deg$M = 0$ and that M is a Q-matrix. Though the matrix in Example 8 was known earlier, it was Howe who at first showed with M, that there exist zero-degree Q-matrices.

Incidentally, Howe[17] termed such types of matrices as *superfluous matrices.* The idea of superfluous matrices is found in Stone[41] also. They can be defined as follows:

Definition 3: Let $M \varepsilon R_0$. M is called a *superfluous matrix*, if LCP(q, M) has at least $k + 1$ solutions for every $q \varepsilon R^n \setminus \mathcal{K}(M)$, given that $|\deg M| = k$.

Howe[17] and Stone[20, 41] initiated research on this class of matrices. Q-matrices of degree zero are clearly superfluous. But are there any nonzero degree Q-matrices that are superfluous? Howe's another example (see Theorem 6.7.3 of Cottle *et al.*[1]) puts an end to this query, explaining that there are superfluous matrices of nonzero degree.

Besides Howe and Stone, several others like Garcia *et al.*[42], Morris[23] and Sridhar[43] have contributed results on this class of matrices. One of the striking results[1] on the existence of superfluous matrices of degree k, is presented below.

Theorem 3.8. *If for any integer k, there exists an R_0-matrix of degree k, then there exists a superfluous matrix of degree k.*

However, the above theorem will not be of much help in constructing examples of superfluous matrices of degree k, given any integer k. Indeed, generating examples of superfluous matrices of any degree is a little tedious task as can be seen from Chapter 6 of Cottle *et al.*[1] To circumvent this difficulty, Sridhar[43] introduced a class of matrices denoted by $\bar{Z}$, which have off-diagonal entries positive and diagonal entries negative and studied the solution behaviour of the LCP(q, M) for $M \varepsilon \bar{Z}$. A game-theoretic approach was adapted in identifying examples of superfluous matrices in $\bar{Z}$.

The following is a result[43] which helps in constructing examples of zero-degree Q-matrices of any even order:

Theorem 3.9. *Let $n \geq 4$ be an even integer and $M \varepsilon \bar{Z} \cap R^{n\times n}$ satisfy the following:*

i) $-M_{nn}$ *is an almost P-matrix;*

ii) M has its first two rows identical except for $m_{11} \leq m_{21}$ and $m_{22} \leq m_{12}$, with at least one of them being a strict inequality;

iii) Every 2×2 principal submatrix of M having the index n, has value positive.

Then M is a superfluous matrix of degree zero.

A subclass of superfluous matrices in $\bar{Z}$, of odd order is provided by the following result:

Theorem 3.10. *Let $n \geq 5$ be an odd integer and $M \varepsilon \bar{Z} \cap R^{n\times n}$ satisfy the following:*

i) $-M_{nn}$ *is P-matrix;*

ii) M has its first two rows identical except for $m_{11} \leq m_{21}$ and $m_{22} \leq m_{12}$ with at least one of them being a strict inequality;

iii) Every 2×2 principal submatrix of M having the index n has value positive except for one, for which the value is negative.

Then M is a superfluous matrix of degree zero.

The above theorems provide examples of superfluous matrices of degree zero. The following result from Sridhar[43] asserts with a specific example that there are superfluous matrices having degree k for any integer k:

Theorem 3.11. *Let $M \varepsilon \bar{Z} \cap R^{n\times n}$, for $n \geq 4$ be constructed as follows:*

$$m_{ij} = \begin{cases} -3, \text{ if } i = j; \\ 2, \text{ if } i \neq j, i, j \varepsilon \{1,2,3\} \\ 4, \text{ otherwise.} \end{cases}$$

Let $k \leq 0$ be any given integer. With $n = -k + 4$, M is a superfluous matrix with $\deg M = k$.

For examples of superfluous matrices of degree k, where k is a positive integer, we can consider a principal pivot transform[1] of M in the above theorem, using a principal submatrix of M for which the determinant is negative.

In the example stated in Theorem 3.11, we notice that in order to construct matrices of large degree, the matrix order also needs to be large. We do not know how large degree can be, given a particular order of the matrix. Following is the conjecture due to Morris[23] on the largest possible degree of a matrix of order n by n:

Morris' Conjecture: Let $g(n)$ denote the maximal degree of any R_0-matrix of order $n \times n$. Then the least upper bound of $g(n)$ is

$$\binom{(n-1)}{\left[\frac{(n-1)}{2}\right]},$$

where $[x]$ of any number stands for the largest integer less than or equal to x.

As shown by Morris[23], for a specific example of $\bar{Z}$-matrices, this bound is attained for every order n. In fact, one can show that Morris Conjecture holds good for the class of $\bar{Z}$-matrices. The following result by Sridhar[43] helps us in asserting this.

Theorem 3.12. *Let $M \varepsilon \bar{Z} \cap R^{n\times n}$ satisfy the following, for* $1 \le r < n$:

i) *Every $r \times r$ principal submatrix of M has value negative;*

ii) *Every $(r+1) \times (r+1)$ principal submatrix of M has value positive.*

iii) *Then $M \varepsilon Q$ and degree of M is given by*

$$\deg M = \binom{(n-1)}{r}$$

In the above theorem, by putting $r = n/2$, or $(n-1)/2$, accordingly as n being even or odd, we see that $M \varepsilon \bar{Z} \cap R^{n\times n}$ has its maximal degree, which exactly equals the number quoted by Morris Conjecture.

4. Local degrees and the simplicial polytope

All our results presented earlier on the deg$M$ were concerning matrices which are subclasses of R_0. Naturally, one is tempted to ask the following question: Can degree theory be of any help in studying the linear complementarity problem with matrices which do not belong to R_0? Unfortunately, a unique scalar like deg$M$ cannot be defined in such cases. Nevertheless, the local degrees in different connected components of $R^n \setminus \mathcal{C}$ can throw some light on the number of solutions for the LCP(q, M). These facts are nicely dealt with in several examples in the first two sections of Chapter 6 of Cottle *et al*[1].

We say that LCP(q, M) has an *odd (even) parity* if the cardinality of the set *SOL* (q, M) is odd (even). When M is not an R_0-matrix, the parity of solutions for LCP(q, M) for vectors in two different connected components need not be the same. However, when M is R_0, the results of Murty[11] and Saigal[12] ascertain that LCP(q, M) has the same parity for every q nondegenerate with respect to M. Indeed, Saigal[12] characterized the vectors that have constant parity, based on the number of strongly degenerate facets present in $[I : -M]$, irrespective of whether or not M is in the R_0-class. In order to state this result, we require the following definition.

Definition 4: Let $M \varepsilon R^{n\times n}$. Define a line segment joining two vectors q^0, $q^1 \varepsilon R^n$ as $q^t : [0,1] \to R^n$. We then say that the intersections of this line segment with $\mathcal{K}(M)$ is *non-degenerate*, if any point q^s, $s \varepsilon (0,1)$, of the line segment that intersects $\mathcal{K}(M)$, does not lie in $(n-2)$ or lesser dimensional complementary facets of $[I : -M]$.

For a more detailed study of such paths and their intersections with facets, we refer to Stone[41] and Saigal and Stone[44]. The following is a result that assures that the local degrees have the same parity under certain hypotheses.

Theorem 4.1. *Let $M \in R^{n\times n}$. Define the line segment $q^t:[0,1]\to R^n$, between any two vectors q^0 and q^1 such that it has nondegenerate intersections with $\mathcal{K}(M)$. Suppose q^0, $q^1 \notin \mathcal{K}(M)$, so that $\deg f_M(q^0)$ and $\deg f_M(q^1)$ are well defined as in (3). Then the following are equivalent:*

i) $\deg f_M(q^0)$ and $\deg f_M(q^1)$ have the same parity.

ii) $|SOL(q^0, M)|$ and $|SOL(q^1, M)|$ have the same parity.

iii) There are an even number of pairs (s, C) where $s \in (0,1)$, C is a strongly degenerate complementary cone relative to M, and $q^s \in C$.

Statement (*iii*) implies that the line segment joining q^0 and q^1 must come across an even number of strongly degenerate complementary cones relative to M in order to have the same parity of the local degrees of M at q^0 and q^1. This can be seen in the following example.

Example 9: Let

$$M=\begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix}.$$

The vectors $q^0=(1, 2)^t$ and $q^1=(-2, -1)^t$ lie in two different connected components of $R^2\setminus C$. For these vectors, the problem LCP(q, M) has the same parity of solutions; the line segment joining them passes through exactly two strongly degenerate complementary cones relative to M. Also, local degree at these vectors equals zero.

Coming back to our earlier discussion of the use degree theory in the case of non-R_0-matrices, we can consider the local degrees defined in the various connected components of pos$[I: -M]\setminus C$ and can conclude about the Q_0-nature of M. Before carrying out further this analysis, we would at first interpret some of the classical results known in linear complementarity on Q_0-matrices in terms of degree theory.

As mentioned in our introduction, there are several sufficient conditions under which some specific classes of matrices are found to be Q_0-matrices. Doverspike and Lemke[8] provided a complete characterization of a subclass of Q_0 using the geometrical aspects of the problem LCP(q, M). Aganagic and Cottle[9] gave a constructive characterization of $P_0 \cap Q_0$. Al-Khayyal[10] and Murthy[45] have presented some necessary and sufficient conditions for Q_0-matrices by a linear programming formulation of the LCP(q, M).

Earlier, based on the properties of solutions of the LCP(q, M) for some chosen vectors $q \in R^n$, Garcia[6], Doverspike[46] and Todd[7] introduced subclasses of Q_0 for which the linear complementarity problem can be processed by Lemke's algorithm[3]. A subclass of Q_0-matrices which are not R_0 were also identified by Eagambaram and Mohan[47] and they presented a variant of Lemke's method for computing solutions for the LCP(q, M) when M falls in that class. All these conditions brew up from the assumption that the boundary of pos[I : $-M$] has all the strongly degenerate complementary cones relative to M. This essentially implies that pos[I: $-M$] is made up of a single connected component.

Doverspike's result[46] generalized Garcia's conditions[6]; along with the condition that for some $q > 0$, q nondegenerate with respect to M, the LCP(q, M) has a unique solution; he introduced the condition that the boundary of pos[I: $-M$]$\setminus\mathcal{K}(M)$ has all the strongly degenerate complementary cones relative to M. This result implies that the local degree of M equals 1 in pos[I : $-M$]$\setminus C$ and that pos[I : $-M$]$\setminus C$ is a single connected component.

Todd's condition[7] of Q-matrices replaces the uniqueness of solution for the LCP(q, M) by the following: For some $q > 0$, $q \in R^n\setminus\mathcal{K}(M)$, the indices of the piecewise linear map $f_M(q)$ are positive. This implies that the local degree of M in pos[I: $-M$]$\setminus\mathcal{K}(M)$ calculated using this vector q is positive. Indeed, Todd's condition requires only that the former statement holds good for some principal pivot transform of M. See Cottle *et al.*[1] for details.

In all these results on Q_0-matrices, the condition of pos[I : $-M$]$\setminus C$ being made up of a single connected component can be relaxed. More precisely, Theorem 2.1 can be modified to provide sufficient conditions for Q_0 as follows:

Theorem 4.2. *Let $M \in R^{n\times n}$. If each connected component of pos[I : $-M$]$\setminus C$ has a nonzero local degree, then M is a Q_0-matrix.*

The above theorem does not require the matrix to belong to R_0; it directly follows from the properties of local degree. Also, the local degrees in each connected component of pos[I : $-M$]$\setminus C$ may vary.

4.1. *Geometric approach to LCP*

There are several methods of studying the problem LCP(q, M). One of them, introduced by Morris[23], is by defining a polytope using the matrix M. It is given by

$$\mathcal{P}_M = \{x \in R^n : x \geq 0, Mx \geq 0, e'x = 1\}.$$

We note that this $\mathcal{P}_M$ is a subset of the $(n-1)$-dimensional unit simplex of R^n. Let us call a vertex v of $\mathcal{P}_M$ an $\bar{i}$ -complementary vertex, if $v_j(Mv)_j = 0$ for all indices $j \neq i$. The following result gives the connection between the polytope $\mathcal{P}_M$ and the linear complementarity problem.

Lemma 4.1. *Let $M \in R^{n\times n}$ be an R_0-matrix. If, for some $i \in \{1,...,n\}$, we have $M_{.i} \notin \mathcal{K}(M)$, then there is a one to one correspondence between the solutions to the LCP($M_{.i}$, M) and the $\bar{i}$ -complementary vertices of $\mathcal{P}_M$.*

The condition stated in the above lemma is not necessarily true for any R_0-matrix. However, if the matrix M is totally nondegenerate, *i.e.*, every submatrix of M is nonsingular, then we have $M_{.i} \notin \mathcal{K}(M)$ for every index i. Besides this, for totally nondegenerate matrices, Morris observes the following connection between the complementary vertices of $\mathcal{P}_M$ and the linear complementarity problem for certain specified vectors.

Lemma 4.2. *Let $M \varepsilon R^{n\times n}$ be totally nondegenerate. Let $cv(M)$ denote the number of complementary vertices of $\mathcal{P}_M$. If $G(n)$ is defined as*

$$G(n) = max\{cv(M): M \varepsilon R^{n\times n} \text{ is totally nondegenerate}\},$$

then for $n \geq 2$, we have $(n-1)G(n) \leq 2nG(n-1)$.

Morris[23] used this lemma to derive an upper bound on the degree of totally nondegenerate matrices.

Theorem 4.3. *Let $M \varepsilon R^{n\times n}$ be a totally nondegenerate matrix. If $n \geq 4$, then $|\, degM \,| \leq 3 \times 2^{n-4}$.*

If $M \varepsilon R^{n\times n}$ is an R_0-matrix, but not totally nondegenerate, then Lemma 4.1 may not hold good, in which case, the maximum of $cv(M)$ will be less than that of $G(n)$. Hence, the above-mentioned upper bound holds good for M being R_0 also. But in view of Morris Conjecture stated in the earlier section, this bound is far greater than the actual value of $g(n)$ for large n, for R_0-matrices.

4.2. *Connected components of R_0*

Consider the set of all n by n, R_0-matrices. A *connected component* of this set of R_0-matrices is a subset for which the structure of the polytope $\mathcal{P}_M$ remains unchanged for any type of perturbation within R_0. As the degree of any R_0-matrix is well defined, Garcia et *al.*[43] and Howe and Stone[20] showed that these connected components of R_0 could be identified by their respective degree. For instance, the classes E_k^+ of the first category[37] are associated with maps of degree one. The advantage of finding the various connected components of R_0 is that homotopy algorithms[20], when constructed for solving these problems, would benefit from having all matrices along a continuous path in R_0.

One of the questions raised by Howe and Stone[20] is that whether the set of all matrices of degree one is connected. For $n = 3$, Morris[48] proved this result; he went on further and identified that R_0 is made up of seven connected components, three of which have degree 2 and the rest of them have degrees, -2, -1, 0 and 1, respectively. He presented representatives of each of these components; interestingly, the following 3 by 3 matrix due to Murty[11], and its certain principal pivot transforms represent the connected components of degrees -2 and 2.

Example 10 (Murty[11]): Let

$$M = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}.$$

By considering the solutions for the $LCP(e, M)$, one can notice that the degree of M is -2. It is clear that the principal pivot transforms of M with respect to the diagonal entries have degree 2.

Let us call a vertex v of the polytope $\mathcal{P}_M$ as *complementary*, if $\{i : v_i = 0$ or $(Mv)_i = 0\} = \{1,...,n\}$. Morris[48] gives a nice characterization of R_0-matrices in terms of the vertices of $\mathcal{P}_M$ as follows:

Theorem 4.4. $M \in R^{n\times n}$ is an R_0-matrix if and only if there is no vertex of $\mathcal{P}_M$ which is complementary.

We call a facet of $\mathcal{P}_M$ complementary if all the vertices of the facet are complementary. In order to analyse the Morris Conjecture on the maximal degree, from Lemma 4.2, one finds that the maximal degree could be calculated just by considering the incidence relationships of facets of $\mathcal{P}_M$. Hence it is worth determining the maximal number of complementary facets of $\mathcal{P}_M$, given that we consider the polytope for R_0-matrices. It is known[48] that n times $g(n)$ is less than or equal to the maximal number of complementary vertices of $\mathcal{P}_M$. Relating this to Morris Conjecture, one can say that, if the conjecture is true, then $\mathcal{P}_M$ can have at most

$$n\binom{(n-1)}{\left[\frac{(n-1)}{2}\right]}$$

complementary vertices. This is proved affirmatively by Morris[49] for two special classes of polytopes: (*i*) polytopes for which every face is complementary, and (*ii*) polytopes that have exactly one with vertices of the same subscript.

5. Stability results using degree of M

The sensitivity and stability studies of any mathematical programming problem deal with what changes the solutions of the problem are subjected to due to disturbances in the input data. In the case of linear complemetarity problems, the degree of the LCP map comes to hand in proving most of these results. Robinson[50] set the stage for this work, by formulating the LCP map as a generalized equation and contributed interesting results; later, Ha[51], Gowda and Pang[22] and many others took up these issues and brought out the relevance of degree theory in such studies.

Let $\mathcal{M} \subseteq R^{n\times n}$ be a subset of matrices, and $Q \subseteq R^n$ be a set of vectors; for each pair (q, M), $q \in Q$ and $M \in \mathcal{M}$ we consider the solution set SOL(q, M). Sensitivity analysis is concerned with the investigation of this solution set as q and M vary in Q and $\mathcal{M}$ respectively. The special case of Q and $\mathcal{M}$ being neighbourhoods around a fixed pair $(\bar{q}, \bar{M})$ is of more interest and is called the *stability analysis* of the linear complementarity problem. In the customary way, before getting into the results using degree theory, we would like to present some results on the properties of the solution map SOL(q, M). Though these results do not involve degree theoretic approach, they would throw better insight into the studies of solution stability.

5.1. *Properties of SOL(q, M)*

Let us be concerned with the effect of changes in vector q fixing a matrix M, on the solution set of the linear complementarity problem (1). Following Cottle *et al.*[1], we would like to call SOL(q, M) as $S(q)$, as the changes are made only in the vector q.

The solution map $S(.)$ possesses two elementary properties: (a) S is a closed mapping, *i.e.*, for any sequence of vectors $\{q^k\}$ converging to $\bar{q}$ with $z^k \varepsilon S(q^k)$, if $\{z^k\}$ also converges to some $\bar{z}$, then $\bar{z} \varepsilon S(\bar{q})$. (b) The map $S(\cdot)$ is polyhedral; that is, the graph,

$$\{(q, z) \varepsilon R^n \times R^n : z \varepsilon S(q)\}$$

is a finite union of convex polyhedra. The following theorem, proved by Robinson[19] for polyhedral multifunctions, is presented here for the solution map of (1).

Theorem 5.1. *Consider the LCP* $(\bar{q}, M)$ *for a given matrix* $M \varepsilon R^{n\times n}$ *and a vector* $\bar{q}$. *There exist a constant* $c > 0$ *and a neighbourhood* $V \subseteq R^n$ of $\bar{q}$ *such that, for all vectors* $q \varepsilon V$,

$$S(q) \subseteq S(\bar{q}) + \| q - \bar{q} \| \mathcal{B}, \tag{7}$$

where $\| \cdot \|$ *denotes the standard Euclidean norm of* R^n *and* $\mathcal{B}$ *is the associated unit ball around* $\bar{q}$.

The above theorem implies that, if $S(\bar{q})$ is empty, then for vector q in a sufficiently small neighbourhood around $\bar{q}$, $S(q)$ is also empty; when $S(\bar{q})$ has at least one element, it ensures that for all vectors q sufficiently close to $\bar{q}$, if $z(q)$ is any solution of the LCP(q, M), then there must exist a $\bar{z} \varepsilon S(\bar{q})$ such that

$$\| z(q) - \bar{z} \| \leq c \| q - \bar{q} \|. \tag{8}$$

This inequality shows that the solutions of the perturbed LCP(q, M) lie at a distance proportional to the magnitude of change in $\bar{q}$, from some of the solutions of the LCP($\bar{q}$, M). Such a mapping is called *locally upper Lipschitzian* with modulus c.

We illustrate Theorem 5.1, with an example below.

Example 11: Let M be the identity matrix of order 2 and $\bar{q} = (1,0)'$. We have $\bar{z} = (0,0)'$, as the only solution for the LCP($\bar{q}$, M). Now, for any $q = (1 \pm\varepsilon_1 \pm \varepsilon_2)'$, where ε_1 and ε_2 are so chosen that q sufficiently close to $\bar{q}$, we see that $z(q)$ equals $\bar{z}$, if $\varepsilon_2 \geq 0$ and $z(q) = (0, -\varepsilon_2)'$, otherwise. In any case, the inequality (8) holds good with $c = 1$.

In the above example, we note that M is a P-matrix and $SOL(q, M)$ is a singleton set for any $q \varepsilon R^n$. Hence, it is easier to check indeed that the inequality in (8) holds good for any pair of arbitrary vectors q, $\bar{q} \varepsilon R^n$. In particular, fixing a matrix M, if the expression in (7) holds good for all q, $\bar{q} \varepsilon R^n$, then the mapping $S(\cdot)$ is called a *Lipschitzian mapping*.

The problem of characterizing the class of all matrices M for which the corresponding solution map $S(\cdot)$ is Lipschitzian is still open. There are many sufficiency results proved in the last few years. Gowda[36] has proved that solution mappings corresponding to P-matrices, and N-matrices with no positive entry are all Lipschitzian. He has also noted that almost N-matrices without a positive entry cannot have this property. The following are two interesting characterizations of some subclasses of Lipschitzian maps due to Gowda[36,52]:

Theorem 5.2. Let $M \in R^{n \times n}$. Then the following statements hold good:

i) If $M < 0$, then the solution mapping $S(\cdot)$ is Lipschitzian if and only if M is an N-matrix.

ii) If LCP(q, M) has a unique solution for a $q > 0$, then M is a P-matrix if and only if the solution mapping $S(\cdot)$ is Lipschitzian.

There are also results on Lipschitzian maps available in the literature, based on certain assumptions like copositivity of the matrix M. For further details we refer to the last chapter in Cottle *et al.*[1]

Theorem 5.1 does not assure that for vectors q sufficiently close to $\bar{q}$, the problem LCP(q, M) has a solution, even when the solution set $S(\bar{q})$ is nonempty. Mangasarian[53] and Robinson[54] noted that this could be achieved by assuming that certain Schur complement is in the class Q.

Theorem 5.3. Let $M \in R^{n \times n}$ and $q \in R^n$ be given. Suppose z is a unique solution of the LCP(q, M) and M_{II} is nonsingular, where the index I is as defined in (4). If the Schur complement $M_{KK} - M_{KI} M_{II}^{-1} M_{IK}$, where K is as defined in (4), is a Q-matrix, then there exists a neighbourhood V of q such that for all $q' \in V$, $S(q) \neq \phi$.

The converse of the above theorem holds good with a stronger assumption of global uniqueness of solution for the LCP(q, M).

5.2. *Stability issues using degree theory*

As mentioned in the previous section, the general idea of studying stability of LCP(q, M) at a solution point can be transformed into a problem of finding a solution of a system $F(z) = 0$, where F is derived from the solution map SOL(q, M). For instance, we can define the problem (1) as

$$F(z) = z \wedge (Mz + q)$$

where $\wedge$ stands for the vector minimum. There are other ways of formulation, see for example, Kojima and Saigal[55], Howe[17], Howe and Stone[20], Garcia *et al.*[42] and Ha[51].

Suppose it is shown that the degree of such a constructed map F is nonzero; then, by a property of degree, it follows that for all maps G sufficiently close to F, $G(z) = 0$ also has a solution, *i.e.*, any smaller perturbation of the LCP(q, M) will have its solution set nonempty.

We give the notions of stability and strong stability of LCP(q, M) at a solution point, as defined by Ha[51].

Definition 5: A solution $z \varepsilon SOL(q, M)$ is said to be *stable*, if there exist neighbourhoods V of z and U of LCP(q, M) such that

(i) for all $(\bar{q}, \bar{M}) \varepsilon U$, the set $S_v(\bar{q}, \bar{M}) = \text{SOL}(q, M) \cap V$ is nonempty, and

(ii) $\sup \{\| y - z \|: y \varepsilon S_V(\bar{q}, \bar{M})\} \to 0$, as $(\bar{q}, \bar{M})$ approaches (q, M).

If in addition to the above conditions, we require that $S_V(\bar{q}, M)$ is a singleton, then LCP(q, M) is said to be *strongly stable* at z.

A solution z of LCP(q, M) is said to be isolated (or locally unique) if there exists a neighbourhood V of z such that $S_V(q, M) = \{z\}$.

Jansen and Tijs[56] defined the robustness of a solution for the LCP(q, M) and compared it with the stability definitions of Ha. A solution z of LCP(q, M) is said to be *robust*, if condition (i) in the above definition of stability holds good for some neighbourhoods V of z and U of LCP(q, M). They proved that a solution of a linear complementarity problem is stable if and only if it is isolated and robust. It is also observed[56] that if $z = 0$ is a nondegenerate solution for the LCP(q, M) for a $q > 0$, then 0 is a stable solution for the problem.

Let us define for a solution z of LCP(q, M), the index sets I, J, K, as in (4). If M_{II} is nonsingular, then for the map $f_M(q)$, $\text{ind} f_M(q, z) \neq 0$. Following is a result on stability of the LCP(q, M), based on the index of $f_M(q)$ at a solution point. See Gowda[36] and Ha[51].

Theorem 5.4. *Suppose z is a semi-nondegenerate solution of the LCP(q, M), such that $\text{ind} f_M(q, z) \neq 0$. Then z is stable.*

We note that this result includes the observations of Jansen and Tijs[56].

The next result we state is established by Ha[51] on the strong stability of solutions of linear complementarity problems:

Theorem 5.5. *LCP(q, M) is strongly stable at z if and only if the following conditions hold:*

(i) M_{II} is nonsingular, and

(ii) the Schur complement M_{II}^S is a P-matrix.

This result is quite remarkable and gives a complete characterization of strong stability. The Schur complement M_{II}^S plays a major role in the stability results.

Ha failed to provide a complete characterization of stability, although there were several sufficient conditions provided by him. It was Gowda and Pang[22] who came up with such a characterization, which could be stated as follows:

Theorem 5.6. *Let z be a solution of the LCP(q, M) with $\det M_{II} \neq 0$, where the index I is as defined in (4). Then the following statements are equivalent:*

(i) *z is stable for LCP(q, M),*

(ii) *the zero vector is stable for LCP(0, M^S_{II}), where M^S_{II} is as defined in (5),*

(iii) M^S_{II} ε *int(Q)* $\cap$ R_0.

The above theorem asserts that z is an isolated solution of LCP(q, M) if and only if zero is isolated for the problem LCP($0, M^S_{II}$). This characterization of isolated solutions for LCP(q, M) has been observed by Mangasarian[53].

A matrix M is said to be *semimonotone*, if LCP(q, M) has a unique solution for any $q > 0$. The degree of an R_0 semimonotone matrix is 1. A matrix M is said to be *fully semimonotone*, if every principal pivot transform of M is semimonotone. The following result gives the index of $f_M(q)$ for a fully semimonotone matrix M without the assumption of R_0:

Theorem 5.7. *Let $M \ \varepsilon\ R^{n\times n}$ be a fully semimonotone matrix. If z is a semi-nondegenerate solution of LCP(q, M), then indf$_M$(q,z) = ± 1.*

This result is observed by Gowda and Pang[22]; see also Gowda[36].

6. Concluding remarks

In this article, we tried to bring out the current research that makes use of the degree of an LCP map in studying the problem LCP(q, M). We have not covered fully the results on stability or sensitivity analyses of linear complementarity problems; the last chapter of Cottle *et al.*[1] presents a complete picture on this. Degree theory as a tool in linear complementarity has given a better understanding of the geometrical aspects of the problem. Nevertheless, the problems stated in the introduction, about complete characterizations of Q- and Q_0-matrices, still remain unsolved. Degree theory can probably help in settling problems of characterizing the class of Lipschitzian maps. One of such problems is due to Prof. Jong-Shi Pang: If the solution map SOL(q, M) corresponding to a Q-matrix is Lipschitzian, is it true that the matrix M is a P-matrix? Gowda[52] (part (*ii*) of Theorem 5.2) has provided a partial answer to this question.

The number of connected components of R_0-matrices is shown to be seven by Morris[23], for matrices of order 3 by 3, as mentioned in Section 4; each of these connected components is identified with respect to its global degree. In connection to this, one does not know an answer to the following, *viz*., how many connected components the R_0-class is made up of, for higher order matrices; and if an answer is found, it will be interesting to know their associated degrees. This problem is very much related to Morris Conjecture on the maximal degree.

We are sure that further work on local and global degree analyses can guide us better into our future research, on these and other unsolved mysteries in linear complementarity.

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