J. Indian Inst. Sci., May–June 1995, 75, 289–296

Symplectic techniques in mechanics and optics*

N. MUKUNDA
Jawaharlal Nehru Centre for Advanced Scientific Research, Bangalore 560 064, and Centre for Theoretical Studies and Department of Physics, Indian Institute of Science, Bangalore 560 012, India.

Abstract

We give a brief introduction to the real symplectic groups and the geometry of canonical variables that they describe. The role of the metaplectic groups in the context of quantum mechanics is highlighted. Several applications of symplectic techniques to classical ray and wave optics, noise matrices and the uncertainty relations, squeezing, and Gaussian states are outlined.

Keywords: Symplectic groups, symplectic geometry, metaplectic groups, uncertainty principles, squeezed states.

1. Introduction

The most familiar kinds of geometry are real Euclidean and complex unitary geometries. These arise in a variey of physical contexts and in various dimensions. There is a third and somewhat unfamiliar kind of geometry called 'symplectic geometry', which is relevant in classical and quantum mechanics and optics, and can be exploited in many interesting ways in these fields. However, it does not lend itself to easy and intuitive visualization or picturization of the kind one is used to. The purpose of this account is to introduce this kind of geometry and indicate some of its uses.

To begin, let us recall the main features of real Euclidean geometry which govern the properties of ordinary space. Here we have the notions of vectors, length of a vector and unit vectors, the forming of real linear combinations of vectors, the angle between two vectors and their scalar product, the property of perpendicularity, and Pythagoras' theorem. All these form the basis of intuitive geometrical pictures in many situations. Euclidean three-dimensional geometry can be generalized to any number of real dimensions, as well as to the geometry of space–time in special relativity.

Real Euclidean geometry also generalizes to a complex form called 'unitary geometry', which is basic to the structure of quantum mechanics[1]. One again deals with vectors but they can now be multiplied by complex numbers, and complex linear combinations can be formed. The notions of lengths of vectors, scalar products, perpendicularity, Pythagoras' theorem all generalize to the complex domain. In this context, it is very interesting to see that these geometric notions get closely linked to the laws of probability theory in the physical interpretation of quantum mechanics. Thus, the fact that one can normalize a

*Text of invited lecture delivered at the Annual Faculty Meeting of the Jawaharlal Nehru Centre for Advanced Scientific Research at Bangalore, on November 11, 1994.

vector to unit length, and Pythagoras' theorem, reflect the laws that the total probability of something happening, and the sum of all partial mutually exclusive probabilities in any given situation, are both unity. Two vectors being perpendicular in Hilbert space corresponds to a certain probability being zero. That the scalar product of two unit vectors cannot exceed unity in magnitude corresponds to the fact that no probability can exceed unity. All this works in any number of dimensions.

2. Symplectic geometry and the symplectic groups

The answer to the question—in what sense is symplectic geometry the third kind of geometry?—is an elegant one. When one attempts to classify all possible compact simple Lie groups, one finds that there are three great infinite families of such groups, and five isolated and so-called exceptional cases[2–6]. The latter are of dimensions 14, 52, 78, 133 and 248. Leaving these aside, the three infinite families are the real orthogonal groups $SO(n)$, the complex unimodular unitary groups $SU(n)$, and the symplectic unitary groups $USp(2n)$. The first two, supplemented by translations, are the bases for real Euclidean and complex unitary geometries, respectively, and each exists in any number of dimensions. The third family involves groups defined in even-dimensional spaces alone, and they lead to symplectic geometry.

For our purpose we are concerned with the real noncompact forms of the groups $USp(2n)$, namely, the real symplectic groups $Sp(2n, R)$ defined in 2,4,6..., real dimensions. They are arrived at as follows. Consider a quantum system with n Cartesian degrees of freedom and $2n$ Hermitian canonical operators $\hat{q}_j, \hat{p}_j, j = 1,2,..., n$ obeying the Heisenberg commutation relations

$$\left[\hat{q}_j, \hat{p}_k\right] = i\delta_{jk}, \left[\hat{q}_j, \hat{q}_k\right] = \left[\hat{p}_j, \hat{p}_k\right] = 0 \tag{1}$$

(Here, we have set $\hbar = 1$). Arrange them into a $2n$-component column vector $\hat{\xi} = (\hat{\xi}_a)$ with Hermitian operator entries defined by and obeying:

$$\begin{gathered}\hat{\xi} = \left(\hat{\xi}_a\right) = \left(\hat{q}_1 \cdots \hat{q}_n \; \hat{p}_1 \cdots \hat{p}_n\right)^T, \; a = 1, 2, \ldots, 2n; \\ \left[\hat{\xi}_a, \hat{\xi}_{a'}\right] = i\beta_{ab}, \\ \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{gathered} \tag{2}$$

The $2n$-dimensional real antisymmetric nonsingular matrix β captures the structure of the commutation relations (1). Then a real $2n \times 2n$ symplectic matrix S is any matrix such that

$$\hat{\xi}'_a = S_{ab}\hat{\xi}_b \tag{3}$$

obeys the same commutation relations (2) as $\hat{\xi}$. This leads to the matrix condition

$$S\beta S^{\mathrm{T}} = \beta, \tag{4}$$

and the real symplectic group $Sp(2n, R)$ consists of all such matrices[7–9].

For comparison with $SO(n)$ and $SU(n)$, let us list the definitions of all three groups of matrices:

$$SO(n) = \{R = real,\ n \times n \mid RR^T = 1,\ detR = 1\},$$
$$SU(n) = \{U = complex,\ n \times n \mid UU^\dagger = 1,\ detU = 1\},$$
$$Sp(2n, R) = \{S = real,\ 2n \times 2n \mid S\beta S^T = 1\}. \tag{5}$$

One can sense that a new kind of geometry is involved here: the unit matrix is replaced by the even-dimensional antisymmetric β. This leads to all the unusual features of symplectic geometry of which we mention a few[10–12].

Let $x_a, y_a, \ldots$ be real $2n$-component vectors. The symplectic scalar product of x with y is defined as

$$x.y = x^T \beta y = x_a \beta_{ab} y_b$$
$$= x_1 y_{n+1} - x_{n+1} y_1 + x_2 y_{n+2} - x_{n+2} y_2 + \ldots + x_n y_{2n} - x_{2n} y_n. \tag{6}$$

From the structure of this expression we see the appropriateness of the name for this geometry. The word 'symplectic' comes from the Greek "συμπλεκτικοζ" meaning "twining or plaiting together, to twine, to plait, to weave", and we see just this feature in the series of terms in $x.y$.. This scalar product is antisymmetric:

$$x \cdot y = -y \cdot x,$$
$$x \cdot x = 0. \tag{7}$$

Because of this we lose the intuitive ideas of length of a vector, unit vectors, the angle between two vectors, perpendicularity, Pythagoras' theorem, etc.! This is why this geometry is unfamiliar and difficult to visualize. Even with the idea of a subspace something new comes in. With both Euclidean and unitary geometries, any two subspaces of the same dimension are basically similar and can be transformed into one another. But in symplectic geometry this is no longer the case. Apart from dimension, an invariant and significant characteristic is the symplectic rank of a subspace—basically this is a measure of the number of complete canonical pairs contained in the subspace. So two subspaces of the same dimension but different symplectic ranks are intrinsically different and cannot be transformed into one another.

3. The metaplectic representation of $Sp(2n, R)$ in quantum mechanics

Since the hermiticity of $\hat{q}'s$ and $\hat{p}'s$ and their commutation relations are maintained by the linear transformations (3), these changes are unitarily produced. Thus we have unitary operators $\mathcal{U}(S)$, $\mathcal{U}(S')$,...such that

$$S', S \in Sp(2n, R): S_{ab}\hat{\xi}_b = \mathcal{U}(S)\hat{\xi}_a \mathcal{U}(S)^{-1},$$
$$\mathcal{U}(S)^\dagger \mathcal{U}(S) = 1,$$
$$\mathcal{U}(S')\, \mathcal{U}(S) = (phase\ factor)\, \mathcal{U}\,(S'S). \tag{8}$$

These operators are fixed up to phases; however, these cannot be adjusted so as to lead to a true unitary representation of $Sp(2n, R)$. The best that can be achieved is

$$\mathcal{U}(S')\,\mathcal{U}(S) = \pm\,\mathcal{U}(S'S). \tag{9}$$

We actually have here a faithful unitary representation of the metaplectic group $Mp(2n)$, which is a double cover of $Sp(2n, R)$[13]. The situation is similar to the so-called double-valued representation of the group of rotations in space, provided by the spin1/2 solution to the angular momentum commutation relations. The generators of $\mathcal{U}(S)$ are all Hermitian quadratic expressions in the $\hat{q}'s$ and $\hat{p}'s$.

Usually when we deal with the symmetry group of some physical system in quantum mechanics, various inequivalent unitary irreducible representations of it may appear; and generally different energy eigenstates belong to different irreducible representations. But here in the case of $Mp(2n)$, we have a particular and fixed unitary representation, which happens to be the sum of two irreducible representations. No other representations come in, and the ones occurring are completely determined by the number of degrees of freedom. So it is more like a symmetry of the basic language of canonical variables rather than of any specific system.

4. Some uses of the symplectic groups

We now describe briefly various uses of the family of groups $Sp(2n, R)$, covering classical ray and wave optics, quantum mechanics and optics.

4.1. *Classical ray and wave optics*[14]

In linear Gaussian optics, each axially symmetric optical system is representable by a matrix of $Sp(2, R)$; this acts directly on ray variables to transform input rays into output rays. For nonaxially symmetric systems, we need to use the group $Sp(4, R)$. All this is due basically to Fermat's principle, which governs ray propagation, being a variational principle just as in Lagrangian and Hamiltonian mechanics. When we go over to the wave description, then the metaplectic groups $Mp(2)$ and $Mp(4)$ come in[15]; their actions on the wave amplitude are *via* the generalised Huyghens kernel.

4.2. *The Wigner distribution*

In the phase space description of quantum states one uses the so-called Wigner function or distribution[16]. To a general state described in operator form by a density matrix $\hat{\rho}$ one associates a real classical phase space function $W(\xi)$, which is a partial Fourier transform of the configuration space matrix elements of $\hat{\rho}$:

$$W(\xi) = (2\pi)^{-n} \int d^n q' < q - \frac{1}{2}q' | \,\hat{\rho}\, | q + \frac{1}{2}q') e^{ip \cdot q'}. \tag{10}$$

Then it turns out that this function has a very simple transformation law under the symplectic group[17, 18]:

$$\hat{\rho}' = \mathcal{U}(S)\,\hat{\rho}\,\mathcal{U}(S)^{-1} \Leftrightarrow W'(\xi) = W(S^{-1}\xi). \tag{11}$$

This can be and has been exploited in many ways[17, 18].

4.3. *Variance matrix and uncertainty relations*

The variance or noise or uncertainty matrix of a quantum state $\hat{\rho}$ is defined by

$$V = (V_{ab}),$$
$$V_{ab} = \tfrac{1}{2} Tr\left(\hat{\rho}\left\{\hat{\xi}_a, \hat{\xi}_b\right\}\right) - <\hat{\xi}_a><\hat{\xi}_b>,$$
$$<\hat{\xi}_a> = Tr\left(\hat{\rho}\hat{\xi}_a\right). \tag{12}$$

This is a real symmetric positive definite $2n \times 2n$ matrix, and these are the only conditions on it at the classical level. Under the symplectic group it has a nice transformation law:

$$\hat{\rho}' = \mathcal{U}(S)\hat{\rho}\,\mathcal{U}(S)^{-1} \Rightarrow V' = SVS^T. \tag{13}$$

In passing, we mention a very useful theorem due to Williamson[19, 11]. Since V is positive definite, it turns out that we can find suitable $S \in Sp(2n, R)$ such that V' is diagonal. These diagonal elements are not in general the eigenvalues of V. The point is that a general symmetric V, not possessing the property of positive (or negative) definiteness, cannot generally be brought to diagonal form by a symplectic $S \in Sp(2n, R)$, but only by a matrix of $SO(2n)$.

We mentioned above that the only conditions on a noise matrix arising from a classical phase space probability distribution are reality, symmetry and positive definiteness. Any such matrix is classically physically admissible or realizable. In quantum mechanics, however, there are further conditions, namely, the Heisenberg uncertainty relations. Using symplectic group methods these can be handled completely for any number of degrees of freedom[20]. The necessary and sufficient conditions in quantum mechanics for a matrix V to be a physically acceptable noise matrix are (apart from reality and symmetry)

$$V + \frac{i}{2}\beta \geq 0. \tag{14}$$

We have here a positive semidefiniteness condition for the Hermitian matrix on the left, in the complex domain. Even in one dimension, this goes beyond the usually stated Heisenberg uncertainty relation.

4.4. *Squeezing and squeezed states*

Squeezed states of radiation are nonclassical states of the quantized radiation field with unusual properties. Most studies have dealt with a single mode of the field. The methods described above allow us to set up a satisfactory squeezing criterion with desirable invariance properties for any number of modes of the radiation field[20]. It reads:

$$\hat{\rho} \text{ is a squeezed state} \Leftrightarrow \ell(V) = \text{least eigenvalue of } V < \frac{1}{2}. \quad (15)$$

Useful classifications of squeezed states, squeezing transformations and their intrinsic properties can be given using the $Sp(2n, R)$ machinery.

4.5. *Geometric phase for squeezed states*

Even with one degree of freedom, in the context of geometric phases, there are interesting results connected with $Sp(2, R)$ and $Mp(2)$. In the case of polarization optics, there is the familiar result of Pancharatnam[21]: after a closed circuit on the Poincare sphere, the resulting geometric phase is the negative of the solid angle enclosed. For squeezed coherent states, the Poincare sphere gets replaced by the unit timelike hyperboloid in a 2+1-dimensional Minkowski space; each squeezed state corresponds to a point on this two-dimensional surface[22]. And after a closed cycle of squeezing transformations, the accumulated geometric phase is one quarter of the enclosed hyperbolic solid angle on this surface.

4.6. *Gaussian pure states*[18]

Lastly, we mention the description and transformation properties of Gaussian wave functions for quantum systems with n degrees of freedom. Any such normalized and centred wavefunction can be written, apart from a phase, as

$$\psi_{u,v}(q) = \pi^{-n/4}(\det u)^{1/4} \exp[-\frac{1}{2}q^T(u+iv)q],$$
$$u,v = \text{real symmetric } n \times n \text{ matrices}, \; u > 0. \quad (16)$$

Thus this is an $n(n + 1)$-parameter family of pure states. Now it turns out that *any two* such states can be connected to one another by (many) suitable elements of $Sp(2n, R)$ *via* the metaplectic unitary representation. In fact, these states are in one–one correspondence with points of the coset space $Sp(2n, R)/U(n)$, where $U(n)$ is a maximal compact subgroup of $Sp(2n, R)$. The symplectic transformation law of the labels u, v is also very interesting. If we write $S \in Sp(2n, R)$ in terms of $n \times n$ blocks as

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (17)$$

then we find:

$$\mathcal{U}(S)\psi_{(u,v)}(q) = (\textit{phase factor})\psi_{(u',v')}(q), \quad (18)$$

where

$$\Lambda = (iu - v)^{-1} \to \Lambda' = (iu' - v')^{-1} = (A\Lambda + B)(C\Lambda + D)^{-1}. \quad (19)$$

This is a multidimensional matrix form of the familiar Mobius transformation law, and the complex symmetric matrix Λ is the generalization of the lower half plane for a single complex variable.

5. Conclusions

We have given a brief introduction to the structure and properties of the real symplectic groups and their associated geometry, and outlined a few applications in mechanics and optics. There are other applications too such as for instance: the prediction of a new class of optical beams which have been recently experimentally produced; geometrical generalization of Hamilton–Jacobi theory; the semiclassical limit of quantum mechanics; universality classes of random matrix ensembles; to mention a few.This brief account should hopefully convey the flavour of symplectic geometry.

Acknowledgements

It is a pleasure to acknowledge the collaboration and cooperation of R. Simon, B. Dutta and Arvind in the work described herein.

References

1. WEYL, H. *The theory of groups and quantum mechanics*, 1959, Dover.
2. RACAH, G. Group theory and spectroscopy, *Ergebn. Exakten Naturw.*, 1965, **37**, 28–84.
3. WEYL, H. *The classical groups*, 1946, Princeton University Press.
4. PONTRJAGIN, L. *Topological groups*, 1958, Princeton University Press.
5. GILMORE, R. *Lie Groups, Lie Algebras and some of their applications*, 1974, Wiley.
6. WYBOURNE, B. G. *Classical groups for physicists*, 1974, Wiley.
7. GUILLEMIN, V. AND STERNBERG, S. *Symplectic techniques in physics*, 1984, Cambridge University Press.
8. WOLF, K. B. *Integral transforms in science and engineering*, 1979, Plenum Press.
9. ARVIND, DUTTA, A. B., MUKUNDA, N. AND SIMON, R. *The real symplectic groups in quantum mechanics and optics*, IISc preprint, 1995.
10. ABRAHAM, R. AND MARSDEN, J. E. *Foundations of mechanics*, 1978, Benjamin-Cummings.
11. ARNOLD, V. I. *Mathematical methods of classical mechanics*, 1978, Springer.
12. MARMO, G., MORANDI, G. AND MUKUNDA, N. A geometrical approach to the Hamilton–Jacobi form of dynamics and its generalizations, *La Rivista del Nuovo Cimento*, 1990, **13**(8), 1–74.
13. WEIL, A. On certain groups of unitary operators, *Acta Math.*, 1963, **11**, 143–211.
14. STAVROUDIS, O. N. *The optics of rays, wave fronts and caustics*, 1972, Academic Press.
15. BACRY, H. AND CADILHAC, M. The metaplectic group and Fourier optics, *Phys. Rev. A*, 1981, **23**, 2533–2536.
16. WIGNER, E. P. On the quantum correction for thermodynamic equilibrium, *Phys. Rev.*, 1932, **40**, 749.
17. SIMON, R., SUDARSHAN, E. C. G. AND MUKUNDA, N. Gaussian Wigner distributions in quantum mechanics and optics, *Phys. Rev. A*, 1987, **36**, 3868–3880.

18. SIMON, R., SUDARSHAN, E. C. G. AND MUKUNDA, N. — Gaussian pure states in quantum mechanics and the symplectic group, *Phys. Rev. A*, 1988, **37**, 3028–3038.

19. WILLIAMSON, J. — On the algebraic problem concerning the normal forms of linear dynamical systems, *Am. J. Math.*, 1936, **58**, 141–163.

20. SIMON, R., MUKUNDA, N. AND DUTTA, B. — Quantum noise matrix for multimode systems: $U(n)$-invariance, squeezing and normal forms, *Phys. Rev. A*, 1994, **49**, 1567–1583.

21. PANCHARATNAM, S. — Generalized theory of interference, and its applications. Part I. Coherent pencils, *Proc. Indian Acad. Sci. A*, 1956, **44**, 247–262.

22. MUKUNDA, N. AND SIMON, R. — Quantum kinematic approach to the geometric phase–I. General formalism; II—The case of unitary group representations, *Ann. Phys.*, 1993, **228**, 205–268 and 269–340.