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The theory of proportional weirs

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Abstract

This paper reviews the work of Keshava Murthy and collaborators in the area of proportional weirs. It brings forth the lacuna that existed in the theory of weirs and focusses on why there was need a for a generalized theory of weirs. It outlines the theorem of slope discharge continuity developed, emphasizing the importance of a datum or reference plane for every weir. It explains briefly the mathematical theory developed and its application to the design of several important weirs having wide application. The theory and application of geometrically simple weirs is briefly outlined.

Keywords: Weirs, notches, proportional weirs, flow measurement, hydrometry, linear weir, quadratic weir, geometrically simple weir.

1. Introduction

The study of weirs has been a subject of long-standing interest in hydraulics as is evidenced by the continuous flow of literature on this subject. The importance of weirs as a discharge-measuring device has been very well recognized. Sharp-crested weirs or notches are among the oldest, simplest and most accurate measuring devices used to measure the rate of flow in natural and artificial streams. It is well known that given any defined geometrical shape of a weir, the discharge through it can be found out. This will be a function of h, the head causing flow. The conventional sharp-crested weirs of standard geometrical shapes like rectangular, triangular, trapezoidal and parabolic have been extensively investigated and their performance and characteristics well understood. But the reverse problem of finding the shape of a weir to produce a discharge which is a given function of h, called the 'problem of the design of proportional weir', is of considerable interest in many fields like hydraulic, environmental and chemical engineering, and is, in general, not as simple and involves the solution of integral equations. The study of proportional weirs (P-weirs), besides having considerable practical application, is of fundamental and academic interest in hydraulics. The linear proportional weir with its linear discharge head characteristics has been a subject of considerable interest, with applications in diverse fields. Such weirs are used as control outlets for float-regulated dosing devices in chemical engineering, as a simple measuring device in hydraulics and irrigation and as an outlet for grit chambers in environmental engineering to maintain constant velocity in sedimentation tanks irrespective of fluctuations in discharge.

The first attempt to design a linear proportional weir was made by Oscar van Pelt Stout in 1898[1], while he was a professor of civil engineering at the University of Nebraska. He found out that the equation of such a weir is given by $y \propto x^{-1/2}$, where y and x are coordinates measured along the horizontal and vertical axes, respectively. This weir, although theoretically exact, suffers from the practical difficulty of having an infinite crest width ($y \rightarrow \infty$ as $x \rightarrow 0$), which is physically unrealizable. Cowgill and Banks[2,3] showed that the equation of the curve describing a weir producing a discharge $Q = bH^m$ ($m > 1/2$), H being the depth of flow, is proportional to $x^{m-3/2}$. Stout's case can be obtained as a special case of this ($m = 1$).

In 1908, Sutro overcame the deficiency in Stout's model by providing a rectangular base of depth a and width $2W$ and fitting above this weir a designed complementary weir. For all flows through this weir (Fig. 1) above the rectangular base, the discharge is proportional to the head measured above a 'reference plane' or 'datum' located at $a/3$ above the weir crest. Though this worked out satisfactorily in this case, the rational basis for the selection of the datum was never explained. It was wrongly believed for over 50 years that the reference plane of the weir could be arbitrarily chosen. This erroneous notion was largely responsible for a lot of empiricism that crept into this important branch of hydraulics. The status of the subject was well summarized in 1966 by Singer and Lewis[4] in the following words: "... In spite of its merits, the primary device is only known to a small number of specialists. There are several reasons for its relative obscurity, the most important one being the lack of up-to-date technical information. Technical literature that exists is very old. Few textbooks on hydraulics have a chapter on proportional weirs, and the ones which do present the data in such a compressed form that one is rarely tempted to make further inquiries...." Though all these are true, the main reasons for its relative obscurity and empiricism are more complex. It is in this regard that the work on proportional weirs was taken up at the Department of Civil Engineering at the Indian Institute of Science in the late sixties to develop a theoretical understanding of the theory of proportional weirs. The aim of the project was two fold: (i) to develop mathematical theory of P-weirs, and (ii) to apply the same to design important weirs unsolved till then to prove its effectiveness.

As a part of the mathematical theory, the theorem on 'slope discharge continuity' was recognized and proved. It states: "In any physically realizable weir having a finite number of finite discontinuities, the rate of charge of discharge is continuous at all points of discontinuity. The theorem has been proved rigorously using the theory of Laplace Transforms[5,6] and experimentally verified. Accordingly, every weir is associated with a unique reference plane or datum above which only are all heads reckoned. A new parameter λ, called the 'datum constant', which fixes the datum is introduced. The choice of the datum at $a/3$ above the crest in the case of Sutro weir is precisely meant to satisfy the slope discharge continuity theorem although it was unrecognized.

1.1. *Logarithmic weirs*

The above mathematical theory was applied to design a logarithmic weir. From the works of Cowgill[2] and Banks[3] it is clear that any attempt to design a weir which has a

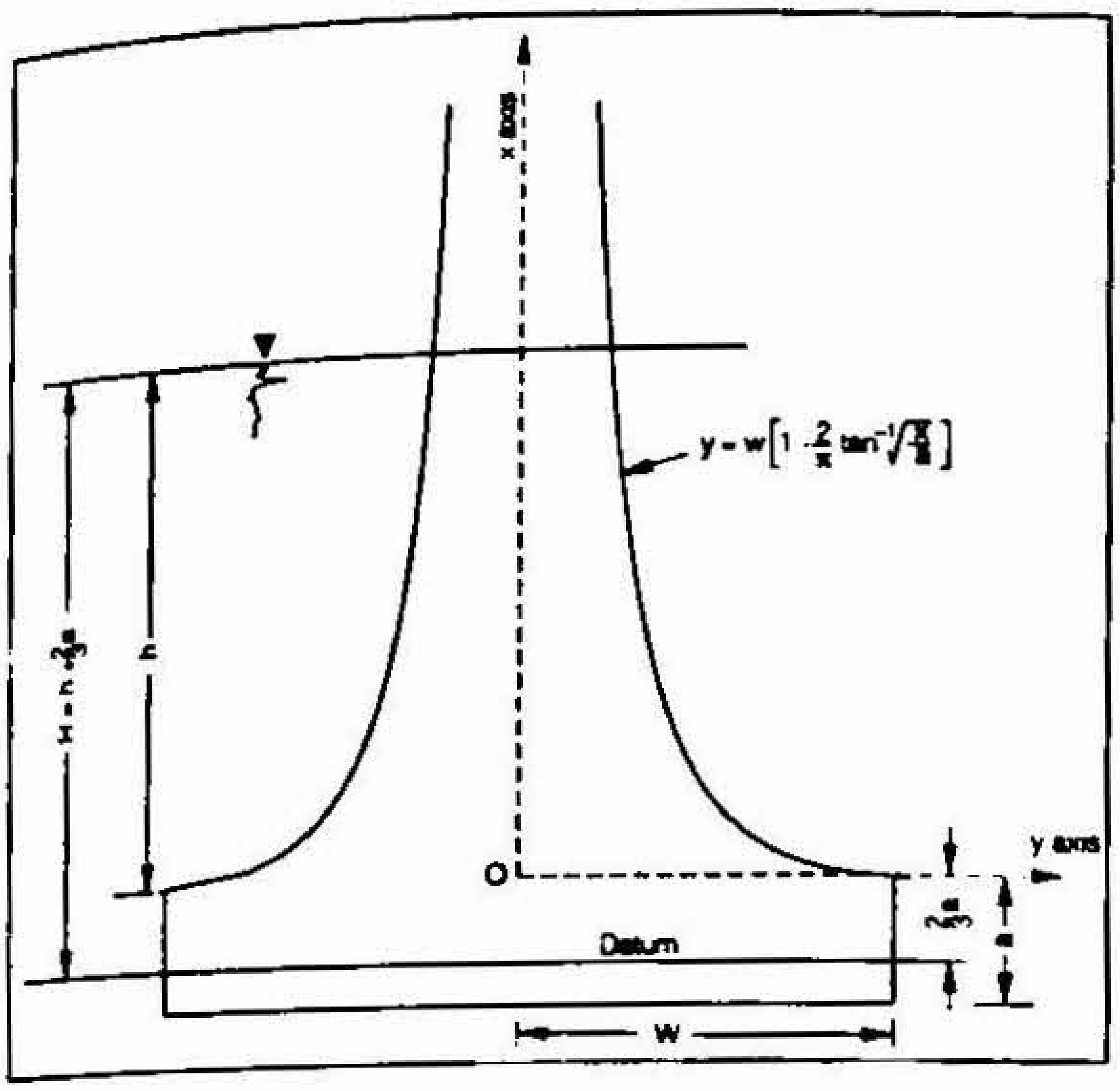

FIG. 1. Linear proportional weir (Sutro weir).

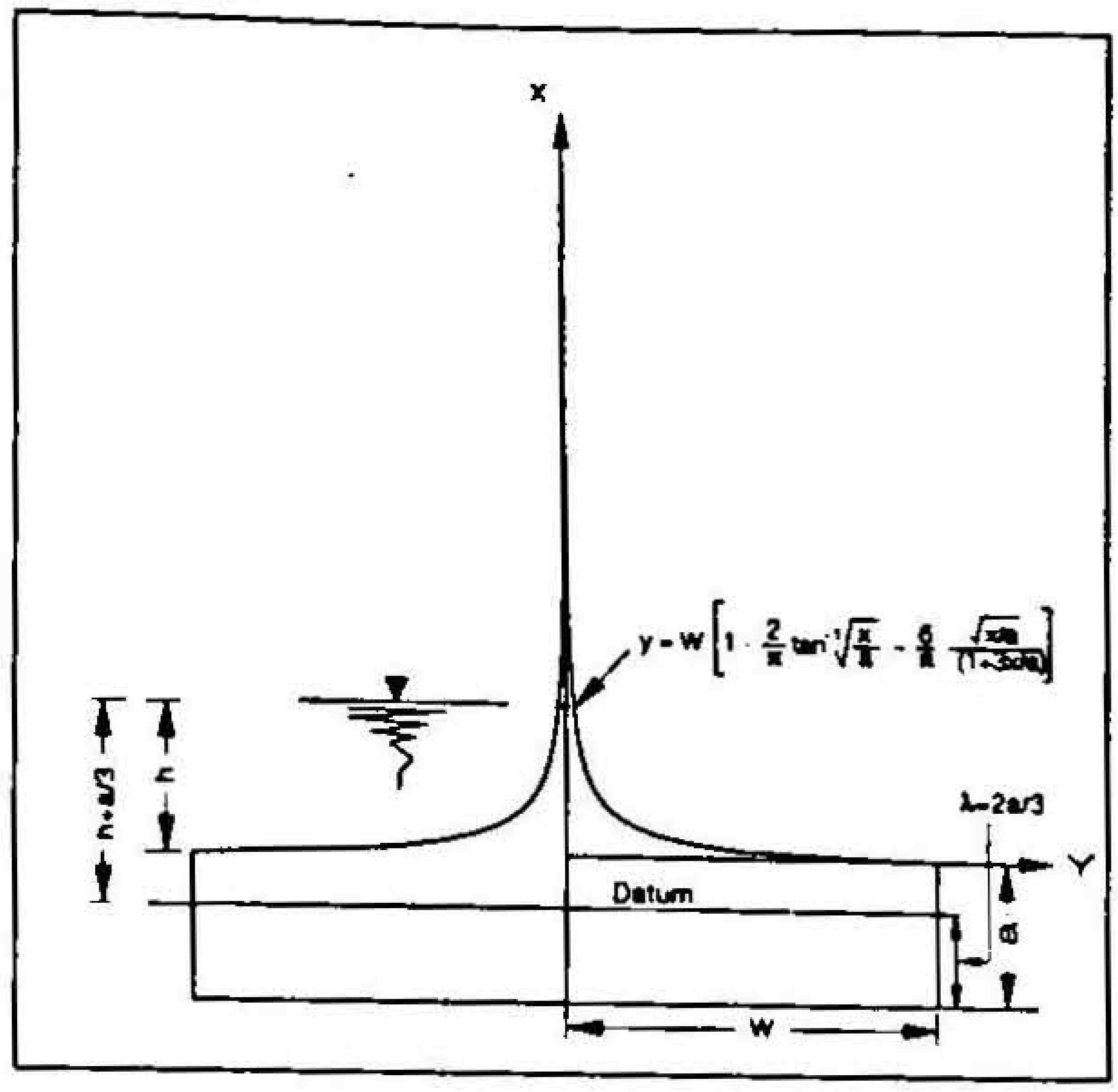

FIG. 2. Rectangular-based quadratic weir.

term h^m in the discharge equation $Q = f(h)$, where $m < 3/2$ invariably leads to a weir having infinite crest width. These are classified as 'compensating base weirs' requiring a base for their design. Logarithmic weir is one such weir. Using the general theory developed, the weir was designed with a rectangular base. Incidentally, logarithmic weirs give larger changes of head for a given change of flow, compared to conventional weirs and hence are useful in irrigation canals as sensitive measuring devices[6, 7].

1.2. *Quadratic weirs*

The quadratic weir which gives a discharge proportional to the square root of the head-causing flow, has applications in bypass flow measurement, was first unsuccessfully tried by Haszpra[8] in 1965 in Hungary. An exact solution to the problem was given by Keshava Murthy[9]. The function defining the weir has a very significant property of fast convergence leading a weir of zero width after a certain height, rendering it into a proportional orifice (Fig. 2). For all flows through this weir above the rectangular base of width $2W$ and depth a, the discharge is proportional to the square root of the head measured above a datum located at $2a/3$ above the crest, both while acting as a 'notch' as well as when it is acting as an 'orifice'. A new concept of notch–orifice was introduced for the first time. Several exact designs of quadratic weirs have been studied by Keshava Murthy *et al.*[6,9–11]

1.3. *Orifice-notch*

Though the Sutro weir was used as an outlet weir for grit chambers or sedimentation tanks to maintain constant average velocity necessary for the collection of the grit, it suffered from the main drawback in that it had to be fixed with its crest at the bed of the channel without leaving a clear gap of about 8–12 inches for the collection of the grit. This was recognized as early as in 1936[12] and had remained unsolved.

This was taken up with the formulation and theory and a generalized synthesis procedure for the design of weirs having their base in any given shape to a depth a so that the discharge through it is proportional to any singular monotonically increasing function of the depth of flow measured above a certain datum. The problem is reduced to finding out an exact solution of a Volterra integral equation in Abel's form. The maximization of the datum below the crest of the notch was investigated. It was proved that for a weir notch made out of one continuous curve and for a flow proportional to the mth power of the head, it is impossible to bring the datum lower than $(2m-1)a$ below the crest of the notch.

A new concept of an 'orifice-notch' having discontinuity in the curve and having division of flow into two distinct portions was developed. The division of flow was shown to have a beneficial effect on the lowering of the datum below the crest and still maintaining the proportionality of flow[5]. This could be used effectively as grit chamber outlet weir, hence solving the long-standing problem in this regard.

1.4. *Proportional* v-*notches*

Although weirs for which the discharge $Q \propto bH^m$, $m \geq 3/2$, do not require a base (like conventional rectangular, v-notch, parabolic weirs), it was shown that they can also be designed with bases with advantage. These are classified as 'noncompensating' base weirs'. Keshava Murthy and Pillai[13,14] designed a modified proportional v-notch weir which produces the same head–discharge relationship, *i.e.*, $Q \propto h^{5/2}$, as that of the conventional v-notch. They used a rectangular base which not only eliminates the difficulty of fixing the weir to plumb, but also increases the indication accuracy. Interestingly, it is seen that the proportional weir regains the geometrical simplicity of the conventional v-notch weir, as the profile approaches fast a straight line. Other designs of noncompensating base weirs include proportional three-halves weir[15] (to replace rectangular weir) and the proportional parabolic weir[16].

2. Geometrically simple weirs

2.1. *Introduction*

Linear proportional weirs have recently attracted considerable interest because of their wide application in varied fields. Further, a linear proportional weir has greater indication accuracy in that a ± 1% error causes an equal percentage error in discharge compared to ± 1.5% error in rectangular weir and ± 2.5% error in v-notches. Although exact solution for the linear proportional weirs has been given by Sutro, Keshava Murthy and others[1,17–19] often these weirs are difficult to fabricate as they require sophisticated equipment and skilled labour.

Recently, a few practical proportional linear weirs have been proposed by Ramamurthy[20], Venkataraman and Subramanya[21] with the main objective of simplifying the weir geometry while incurring negligibly small errors in the discharge computation.

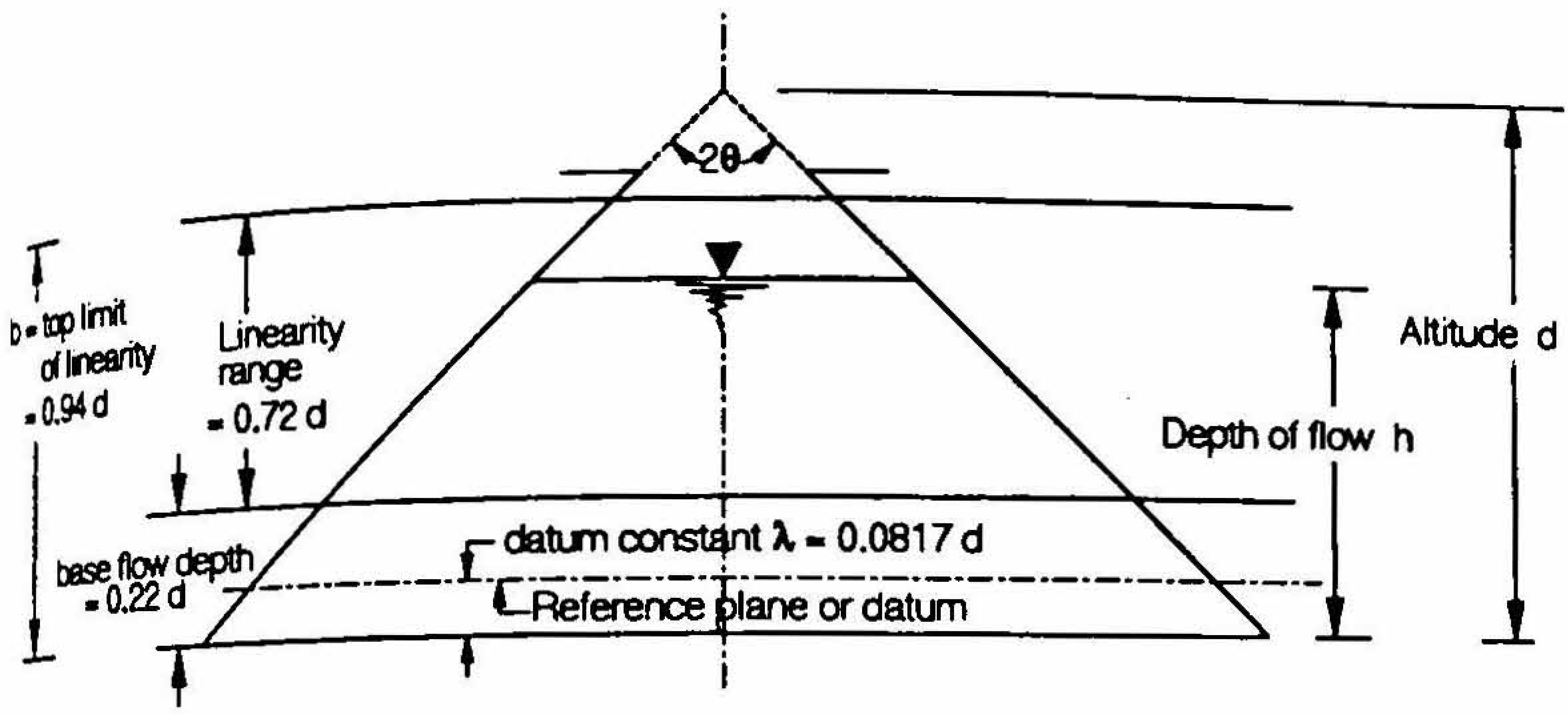

FIG. 3. Characteristics of the inverted v-notch as a linear proportional weir.

2.2. *Inverted* v-*notch*

The work of inverted v-notch was motivated by a casual remark by Troskolansky in his well-known book *Hydrometry*[22], in which he states that a closed trapezoidal weir with a vertex angle of approximately 50° has a near-linear head-discharge relationship. Surprisingly this had not been investigated. Neither its discharge–head relationship was known nor its range of validity. The inverted v-notch was analysed (Fig. 3) in the background of the general theory of proportional weirs. It was shown through an optimization procedure developed that the flow through this weir of half crest width W and depth d for depths above 0.22 d is proportional to the depth of flow measured from a reference plane situated at 0.08 d for all heads in the range $0.22\,d \leq h \leq 0.94\,d$ within a maximum percentage deviation of ± 1.5 from the theoretical discharge. Nearly 75% of inverted v-notch can be used effectively as the measuring range. Experiments are in very good agreement with the theory, giving a constant coefficient of discharge of 0.62[23,24].

2.3. *Chimney weir*

The inverted v-notch was improved in the 'chimney weir' (Fig. 4) with respect to its range of applicability. It was shown that the range of linearity can be considerably enhanced by more than 200% by the addition of a rectangular weir of width 0.265 W (W is half crest width of inverted v-notch) at a depth of 0.735 d (d is the altitude of the inverted v-notch) above the crest of the weir[24, 25]. The design parameters of the weirs, *viz.*, linearity range, base depth, reference plane, are estimated by solving the nonlinear programming problem using a numerical optimization procedure. It is shown that for flows above a threshold depth of 0.22 d the discharges are proportional to the depth of flow measured above a reference plane situated at 0.08 d above the weir crest for all heads in the range $0.22\,d \leq h \leq 2.43\,d$, with a maximum percentage deviation of ± 1.5 from the theoretical discharge. A significant result of the analysis is that the same linear head–discharge relationship governing the flow through the inverted v-notch is also valid for the

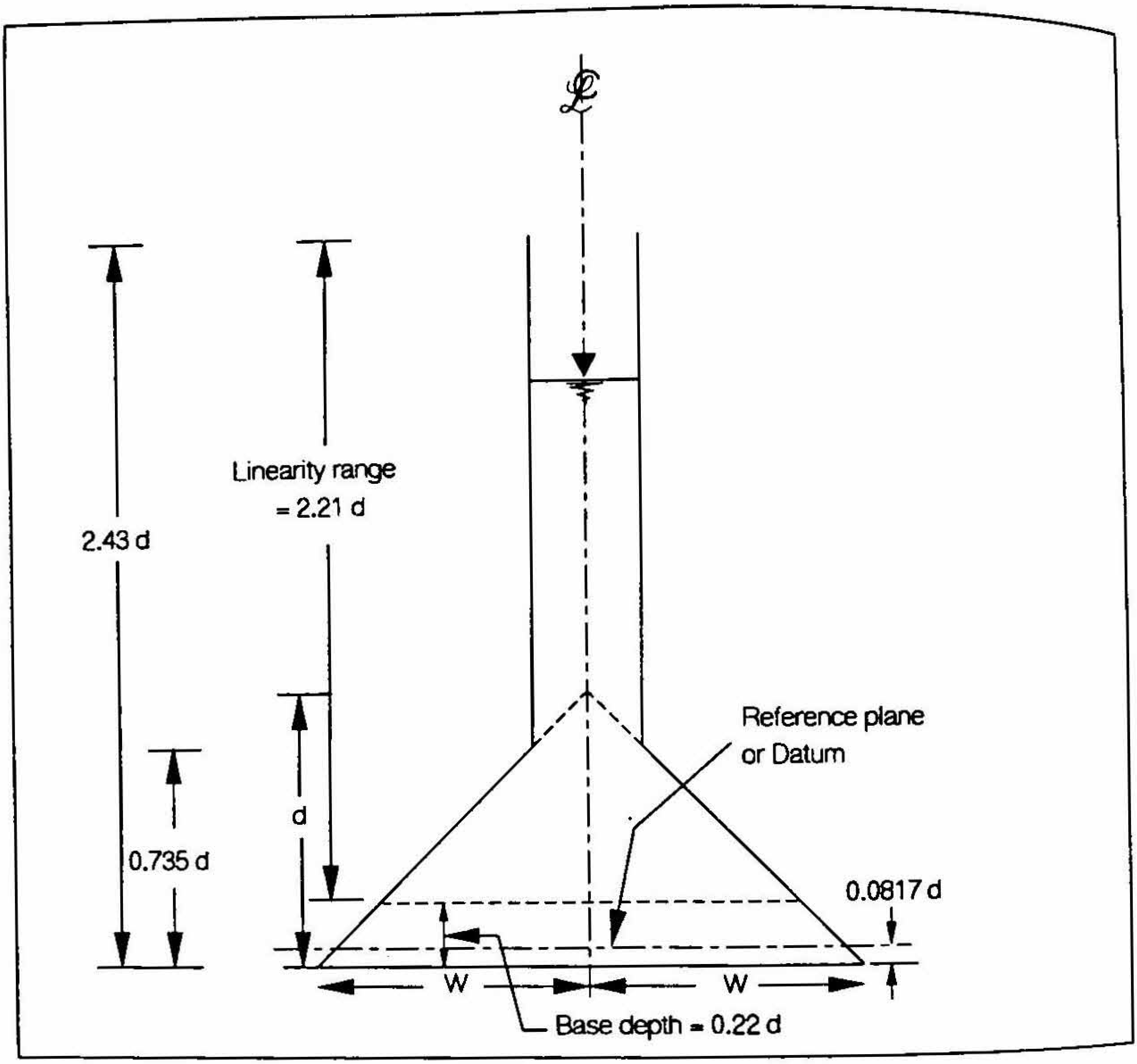

FIG. 4. Salient features of a chimney plate weir.

extended chimney weir. Experiments with three different chimney weirs show excellent agreement with the theory, giving a constant coefficient of discharge for each weir.

The inverted v-notch weir and chimney weir are being tested in some minor irrigation canals in Karnataka.

2.4. *Bell-mouth weirs*

Troskolansky[22] while referring to approximate weirs mentions that flow through the intervening space obtained by keeping a semicircular disc in a rectangular channel produces an approximate linear head–discharge relationship. Unaware of this, Venkataraman and Subramanya[21] have experimentally observed the linear discharge characteristics of weir called 'quadrant plate weir'. However, no analytical investigation has been

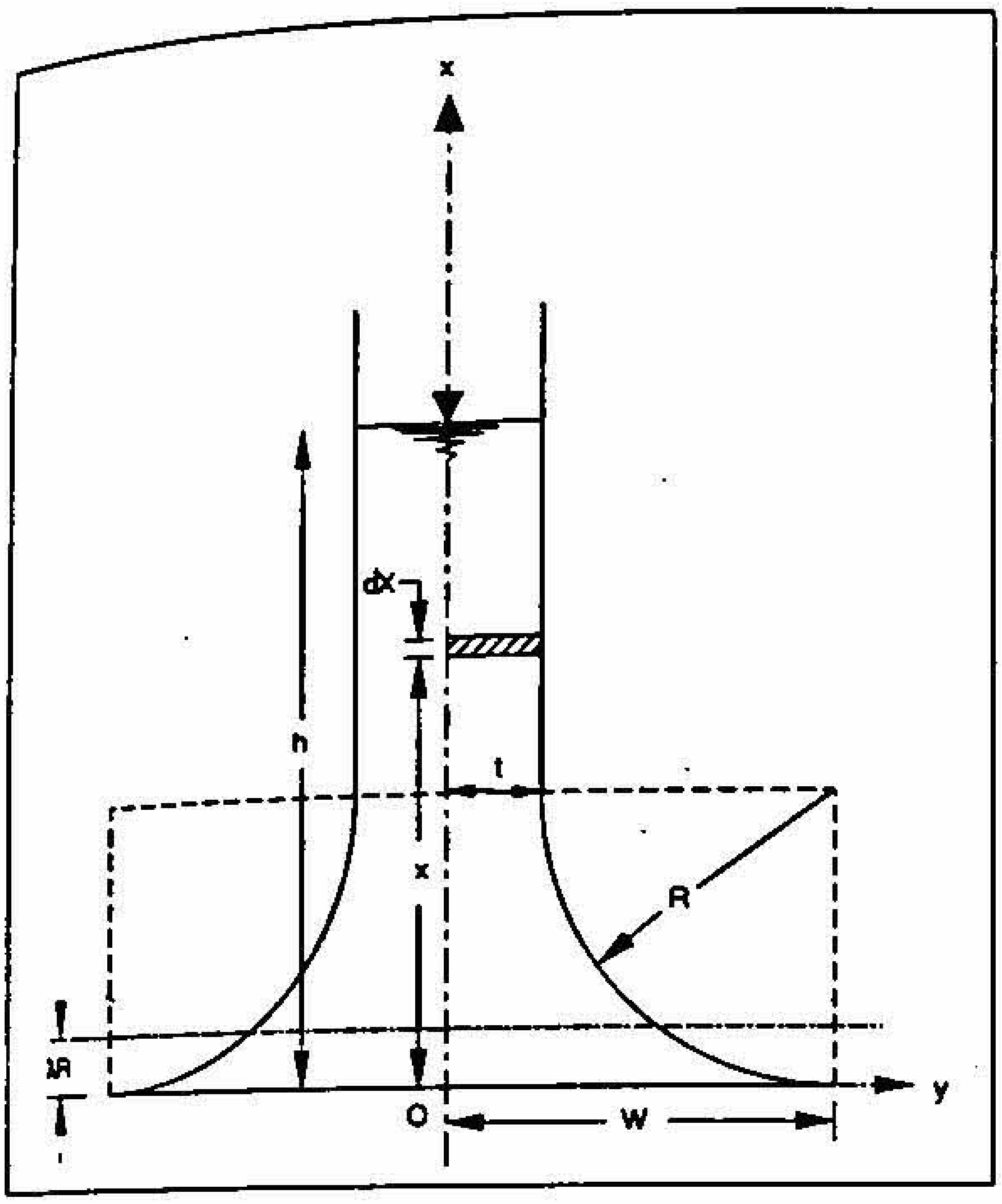

FIG. 5. Typical EBM weir.

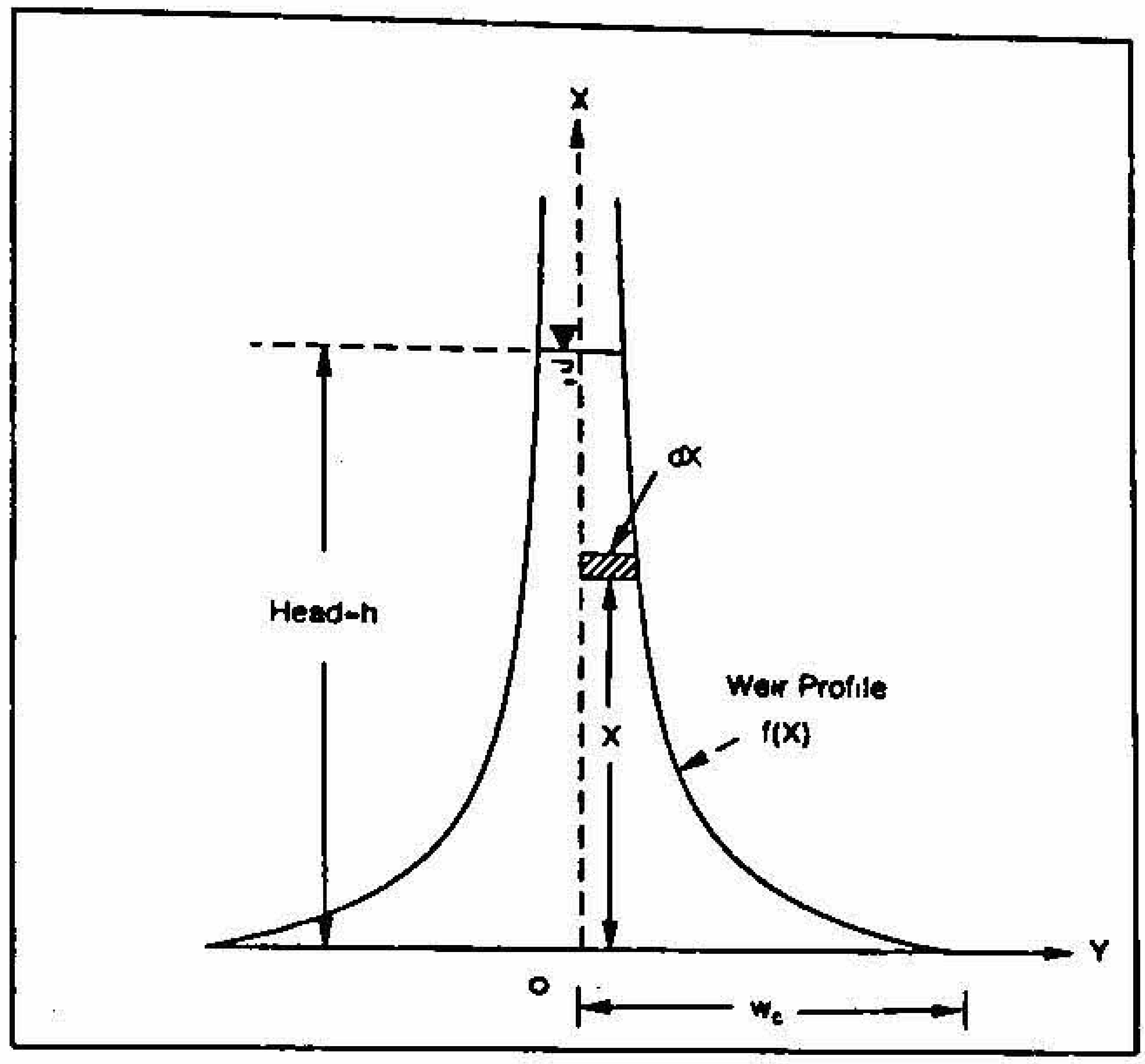

FIG. 6. Definition sketch of the proposed weir.

made. A detailed theoretical analysis of the flow through the quadrant plate weir, referred to as bell-mouth weir (Fig. 5), is made in the light of the generalized theory of proportional weirs using the 'range of point method' of numerical optimization procedure. It is shown that the flow through the quadrant plate weir has a linear discharge–head relationship valid for certain ranges of head within a maximum deviation of ± 1% from the theoretical discharge. Further, it is shown through the optimization procedure that the measuring range of quadrant plate weir can be considerably enhanced by extending the tangents to the quadrants at the terminals of the quadrant plate weirs. These are discussed in the paper on 'bell-mouth weir' by Keshava Murthy and Giridhar[24,26].

3. Self-basing linear weirs

3.1. *Introduction*

It is clear from the works of Banks[3] and Cowgill[2] that the linear weir belongs to the class of 'compensating weirs' invariably requiring a base for its design, rendering it into a compound weir defined by two separate profiles. Further, it has been shown by Keshava Murthy[5,6] that these weirs have a unique reference plane or datum above which all heads are reckoned. These weirs pass discharge, for all flows above the base weir, proportional to the head above the datum.

Of late, there has been some special interest in the design and development of approximate linear weirs which produce near-linear head–discharge relationship (henceforth referred to as h–q function). The prime motivation for this interest is geometrical simplicity and consequent ease in the fabrication, which is necessary in field conditions, where it is hard to find sophisticated equipment and skilled labour. Ramamurthy *et al.*[20] have designed a quadrant plate weir by replacing the curved profile of the Sutro weir[1] by the quadrant of a circle. Optimum dimensions of the weir have been found by minimizing the percentage deviation of discharge produced by the quadrant plate weir from the corresponding Sutro weir.

The first significant mention of a geometrically simple linear weir appears to be by Troskolansky[22], where he refers to two approximate linear weirs: one an inward trapezium (with an apex angle of 50°) and the second a quadrant weir (where the curved profiles are quadrants of a circle). It is said that these weirs can pass near-linear discharges in certain ranges of head. Keshava Murthy and Giridhar have analytically investigated in depth these two weirs, *viz*., the inverted v-notch[23] and bell-mouth weir[24,26]. In their study, they have used a numerical optimization procedure to fix the weir parameters governing the threshold depth (base depth) and the datum of the weir. The experiments fully confirm the results.

One of the main outcomes of the above investigation is the emergence of the linear weir defined by a single profile unlike the exact linear weirs like Sutro weir, etc.[1,17-19], which are essentially compound weirs defined by two profiles: one for the base and the other for the complementary weir above. It has to be underlined here that a portion of the profile of the weir above the crest itself acts as base for the weir. In other words, the base weir becomes an integral part of the whole weir itself. Hence, these weirs are appropriately called 'self-basing linear weirs'. One of the main drawbacks of the above designed self-basing weirs is that they have a limited range of head which makes their choice difficult in practice. The exact solution for the self-basing linear weirs is the one obtained by Stout[1], which, however, is physically unrealizable. In what follows, we are concerned with the finding of a very good approximate solution to this problem. Succinctly, we seek an answer to the question 'Is it possible to find an approximate solution for a self-basing linear weir defined by one single profile with infinite measuring range?' It is shown that the significant property of rapid convergence (hence an approximate solution of Fredholm's integral equations of a particular kind) of a quadratic weir can be eminently exploited to arrive at a practical self-basing linear weir (henceforth referred to as SBL weirs) of very high degree of accuracy.

3.2. *Preliminary consideration*

Referring to Fig. 6, the discharge through the sharp-created weir (symmetrical about the x-axis), defined by $y = f(x)$ for a head h is given by

$$q = 2C_d\sqrt{2g}\int_0^h f(x)\sqrt{h-x}\,dx, \tag{1}$$

where q is the discharge or rate of flow, h, the head above the weir crest, g, the acceleration due to gravity, and C_d, the coefficients of discharge.

The coefficient of discharge is assumed to be constant (approximately equal to 0.6), which is true for streamlined flows through sharp-crested weirs (this is confirmed later by experiments).

Nondimensionalizing the above equation we have, for example,

$$Q = \int_0^H f(H)\sqrt{H - X}\,\mathrm{d}X = \phi(H), \tag{2}$$

where

$Q = \dfrac{q}{2C_d\sqrt{2g}W_s^{5/2}}$,

W_s = half crest width,

$H = \dfrac{h}{W_s}$,

$X = \dfrac{x}{W_s}$.

For a proportional weir the h–q function $\phi(H)$ is known a priori, in which case eqn (2) is the standard Volterra integral equation and can be reduced to Abel's form by differentiating with respect to H(using Leibnitz's rule), so that

$$\int_0^H \frac{f(X)}{\sqrt{H - X}}\,\mathrm{d}x = 2\phi'(H). \tag{3}$$

Solving eqn (3)[13], which is in Abel's form, we get the weir profile

$$f(X) = \frac{2}{\pi}\int_0^X \frac{\phi''(H)}{\sqrt{X - H}}\,\mathrm{d}H. \tag{4}$$

The above solution is realizable only if $\phi(H)$ and $\phi'(H)$ are continuous in the range $0 \le X \le \infty$ and $\phi(0) = \phi'(0) = 0$, where $\phi(H)$ is a continuous and monotonically increasing function of head.

3.3. *Characteristics of the discharge function of the SBL weir*

The discharge function of the SBL weir, in addition to satisfying the conditions specified in the previous section, should satisfy another property, *viz.*, it should tend to become linear very rapidly after a certain small threshold depth, the variation from linearity becoming smaller as the head increases. In other words, the error in replacing this h–q relation (henceforth referred to as the primary h–q function) by a linear one, should rapidly and continuously decrease as H increases. It has to be emphasized here that an exact linear relationship used as the primary h–q function results in a physically unrealizable form $y \propto x^{-1/2}$ (Stout profile[1], for which $Y(0) = \infty$).

3.4. *Forms of primary h–q function*

The above-mentioned near-linear property of the primary h–q function of the SBL weir can be realized in three following forms (Fig. 7):

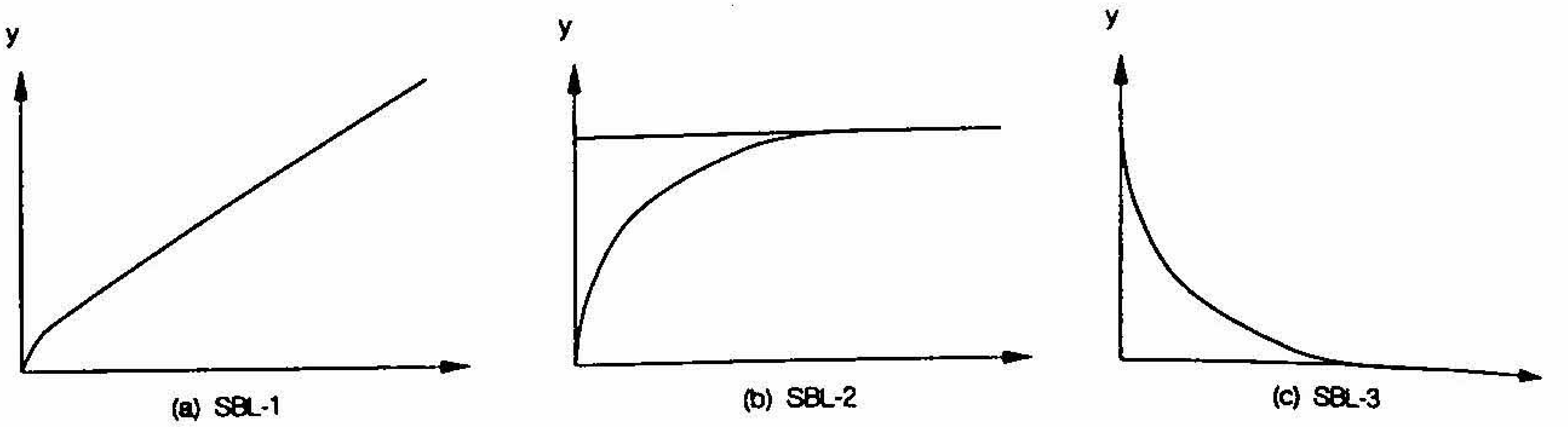

FIG. 7. Forms of primary h–q function.

(*i*) A function although nonlinear in $0 \leq X \leq a$ rapidly becomes linear and retains this till infinity such that $f(X)$ can be replaced by a form $mX \pm C$ in the range of $a \leq X \leq \infty$ with increasing accuracy, so that this function can be used directly as the primary h–q function $\phi(H)$ (Fig. 7a).

(*ii*) An increasing function which rapidly approaches its constant value at infinity such that this could be treated as the first differential of the primary h–q function, integrating which one can get the primary h–q function $\phi(H)$ (Fig. 7b).

(*iii*) A continuously decreasing function which rapidly approaches zero, so that this could be treated as $\phi''(H)$ *vs* H curve, from which $\phi(H)$ can be obtained (Fig. 7c).

The above three forms of functions from which the primary h–q function of an SBL weir can be generated are termed for convenience as, SBL-1, SBL-2 and SBL-3-type generating functions, respectively.

3.5. *Generating functions through exact solutions of some proportional weirs*

In practice, it is very difficult to find a function which possesses all the essential properties of a primary h–q function discussed above. However, some of the existing proportional weir functions (equations defining profile shape of the weirs) do posses certain special characteristics which can be exploited to generate the primary h–q function of SBL weir.

It has been observed that the profile of a proportional v-notch weir[13] designed using any base weir (Fig. 8a) is nearly a straight line after small values of x and the linearity rapidly improves with increasing x. Hence, by shifting the origin from O to O' (Fig. 8a), the weir function considered with reference to the axes $O'X'$ and $O'Y$ (Fig. 8a) can be treated as SBL-1-type generating function

The quadratic weir[9-11], shown in Fig. 8b, has been found to become a 'proportional orifice' or a weir of almost zero width for $X \geq X_0$ (X_0 is a small initial value), so that the function defining this weir considered with reference to the axes $O'X'$ (Fig. 8b) can be treated as SBL-2-type generating function. Similarly, the function defining the quadratic weir profile with reference to the axes OX and OY (Fig. 8b) can be considered to represent the SBL-3-type generating function.

3.6. *Choice of generating function*

An important aspect to be considered in the choice of a particular type of generating function is whether or not it results in a finite nonzero crest width for the designed linear weir. An infinite crest width is physically unrealizable and a zero crest width renders practical difficulties in the fixing of the weir to plumb and symmetry and in the accurate measurement of the initial head.

Cowgill[2] and Banks[3] have shown that an h–q relationship of the form $Q \propto H^m$ can be obtained for a flow in a weir having its profile in the form of $y \propto x^{m-3/2}$, from which it is evident that the weir will have a finite crest width if and only if the least power of the head term in the discharge function $\phi(H)$ is 3/2.

It is found that out of the three types of the generating functions developed using the proportional v-notch and quadratic weirs, only for the SBL-2-type generating functions developed using the quadratic weirs the least power of head term is 3/2. For the rest it is greater than 3/2. In addition, the SBL-1-type generating function developed using the proportional v-notch is not amenable to exact integration and hence a closed-form solution of eqn (4) is not possible.

In the light of the above discussion, only SBL-2-type generating functions, developed using the exact solutions of quadratic weir, are considered for the design of the SBL weir.

4. Development of self-basing linear weir utilizing the exact solution of a quadratic weir

4.1. *Generating the head discharge function*

The function defining the profile of the parabolic-based quadratic weir[11] is (Fig. 9), for example,

$$y = W\left[\sqrt{1+\frac{x}{a}} - \sqrt{\frac{x}{a}} - \frac{2\sqrt{\frac{x}{a}}}{1+\frac{x}{a}}\right] = f(x). \tag{12}$$

It has been mathematically proved[14] that $f(x)$ is positive, single-valued and a continuously decreasing function. Further, the proportional weir transforms itself into a proportional orifice or a weir of almost zero width after a small finite height. Hence, by shifting the origin to O' (Fig. 6)

$$y' = W\left[1 - \sqrt{1+\frac{x'}{a}} + \sqrt{\frac{x'}{a}} + \frac{2\sqrt{\frac{x'}{a}}}{1+\frac{x'}{a}}\right]. \tag{13}$$

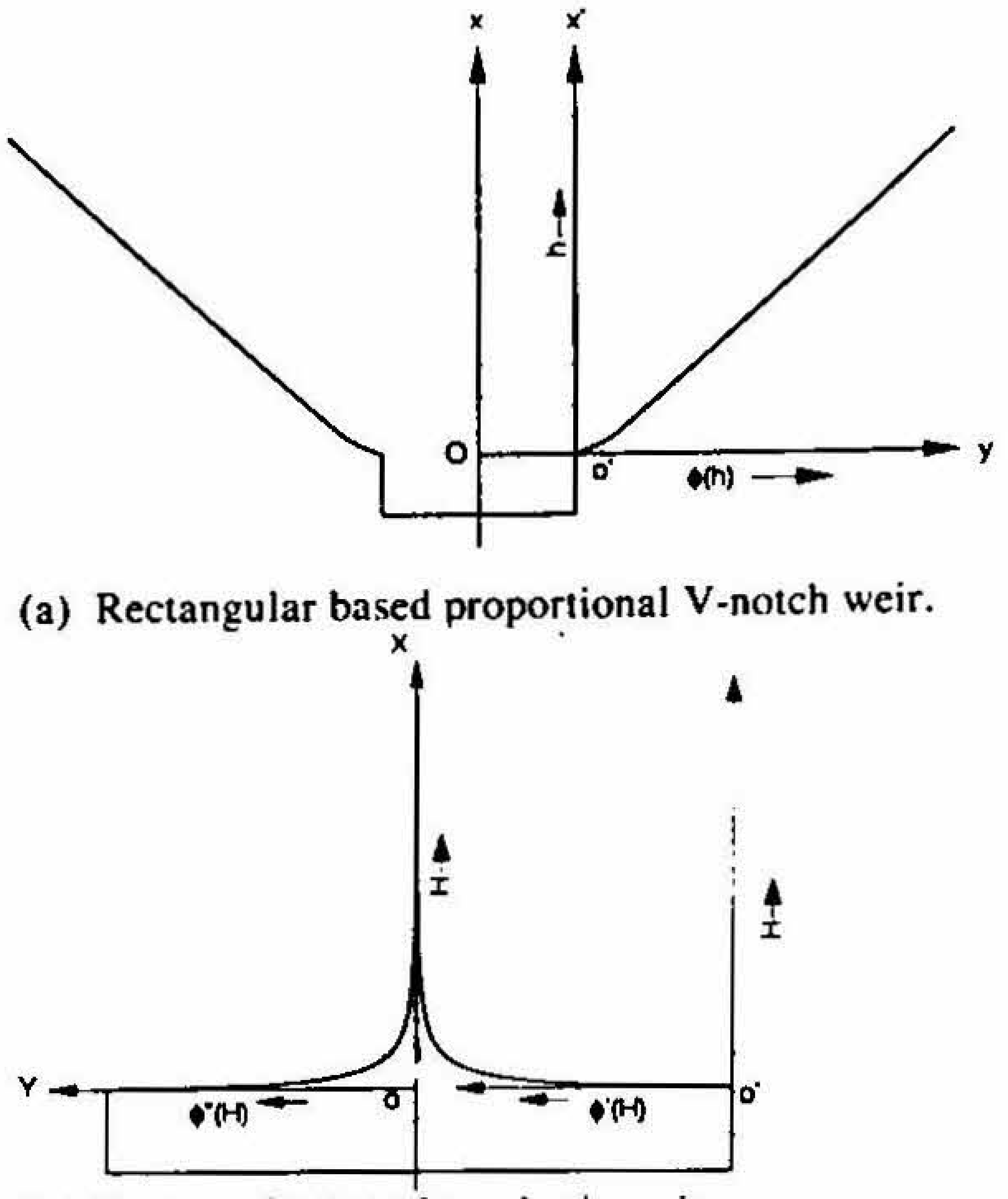

FIG. 8. Generating functions using profiles of P-weirs. (a) Rectangular-based proportional v-notch weir, (b) Rectangular-based quadratic weir.

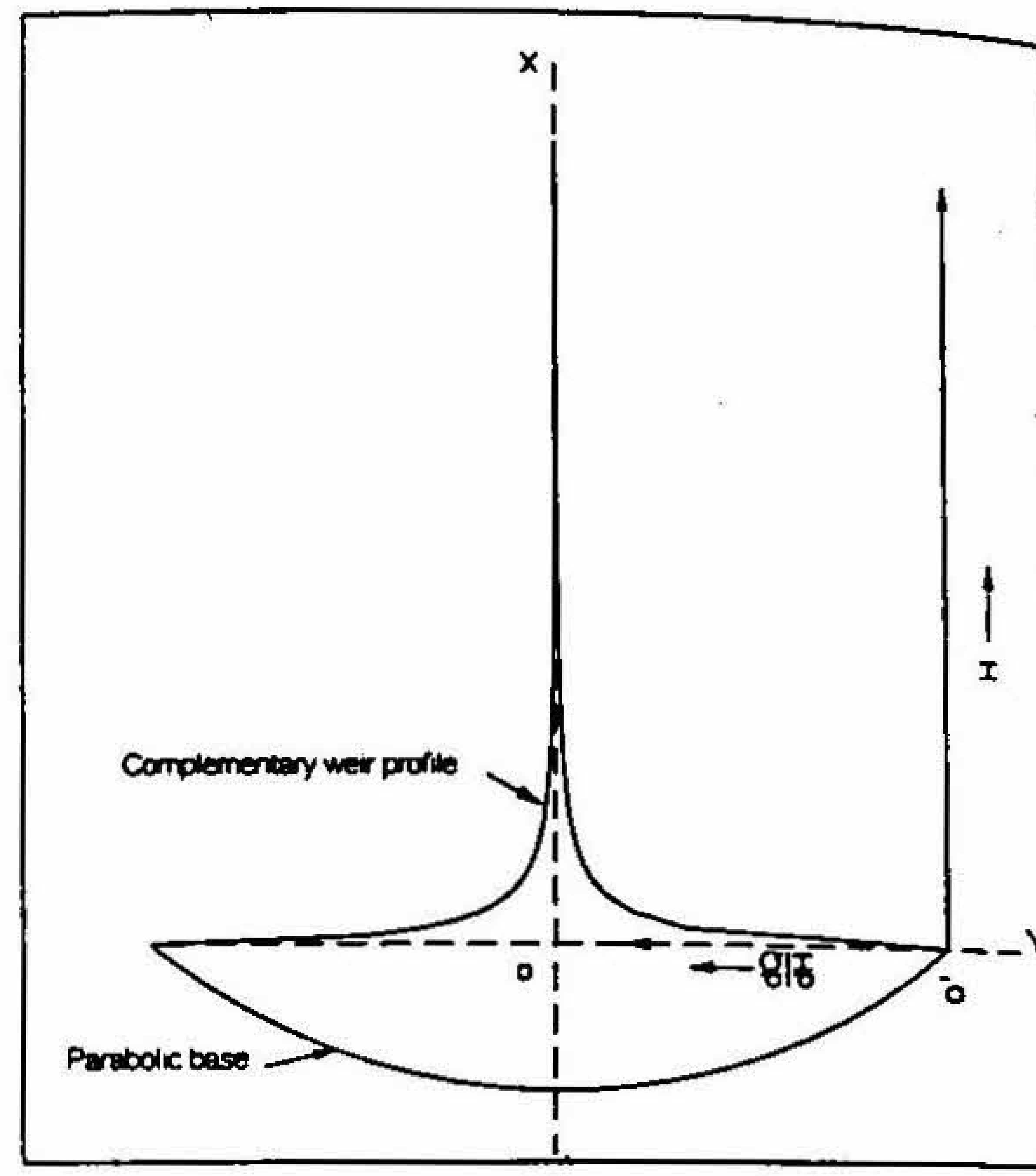

FIG. 9. Parabolic-based quadratic weir.

Nondimensionalizing, we get

$$Y = 1 - \sqrt{1+X} + \sqrt{X} + \frac{2\sqrt{X}}{1+4X}; \; Y = \frac{y'}{W} \text{ and } X = \frac{x'}{W}. \tag{14}$$

Choosing the above function as SBL-2-type generating function to develop the self-basing weir, we have (Fig. 9),

$$\phi'(H) = 1 - \sqrt{1+H} + \sqrt{H} + \frac{2\sqrt{H}}{1+4H}. \tag{15}$$

Integrating eqn (15) with respect to H,

$$\phi(H) = H - \frac{2}{3}\left[(1+H)^{3/2} - H^{3/2} + H^{1/2}\right] - \frac{1}{2}\tan^{-1}(2\sqrt{H}) + C. \tag{16}$$

The constant of integration C is evaluated using the initial condition $\phi(0) = 0$. It is found that $C = 2/3$. Therefore, the required primary h–q function is

$$\phi(H) = \left(H + \frac{2}{3}\right) - \frac{2}{3}\left[(1+H)^{3/2} - H^{3/2} + H^{1/2}\right] - \frac{1}{2}\tan^{-1}(2\sqrt{H}). \tag{17}$$

Expanding the terms on the right-hand side of the above equation and simplifying, we get

$$\phi(H) = 2H^{3/2} - \frac{1}{4}H^2 - \frac{16}{5}H^{5/2} - + \ldots \tag{18}$$

From eqn (18) it can be observed that the least power of the head term in the primary h–q function is 3/2 and hence it is confirmed that the designed SBL weir will have a finite, nonzero crest width.

4.2. *To derive the function $f_s(X)$ defining the self-basing weir*

Referring to Fig. 6, the discharge equation for flow through the proposed SBL weir in the nondimensional form is

$$\int_0^H \sqrt{H-X}\, f_s(X)\,\mathrm{d}X = \phi(H). \tag{19}$$

Substituting for $\phi(H)$ from eqn (17),

$$\int_0^H \sqrt{H-X}\, f_s(X)\,\mathrm{d}X = \left(H+\frac{2}{3}\right) - \frac{2}{3}\left[(1+H)^{3/2} - H^{3/2} + H^{1/2}\right] - \frac{1}{2}\tan^{-1}(2\sqrt{H}). \tag{20}$$

Differentiating with respect to H using Leibnitz's rule,

$$\int_0^H \frac{f_s(X)}{\sqrt{H-X}}\,\mathrm{d}X = 2\left[1 - \sqrt{1+H} + \sqrt{H} + \frac{2\sqrt{H}}{1+4H}\right] = \phi'(H). \tag{21}$$

Equation (21) is recognized as the Volterra integral equation in Abel's form, whose solution is[13]

$$f_s(X) = \frac{2}{\pi}\int_0^X \frac{\phi''(H)}{\sqrt{X-H}}\,\mathrm{d}H. \tag{22}$$

Differentiating the RHS of eqn (21) and evaluating the integral in eqn (2) we get

$$f_s(X) = 1 - \frac{2}{\pi}\tan^{-1}\sqrt{X} + \frac{2}{(1+4X)^{3/2}}. \tag{23}$$

Equation (23) gives the profile of the SBL weir for flow through which the h–q relationship $\phi(H)$ is almost linear after a small base depth. The accuracy to which it is linear can be best explained by comparing this theoretical h–q relationship with an exact linear h–q relationship.

4.3. *Linear discharge characteristics of the designed weir*

It is found that, beyond a small base depth, as the head increases, the h–q graph approximates the asymptote (tangent at infinity) and the deviation of the value given by the tangent at infinity from that given by the h–q graph becomes increasingly negligible.

Let

$$Q_L = mH \pm C, \tag{24}$$

be the equation of the asymptote. Constants m and C and be found as

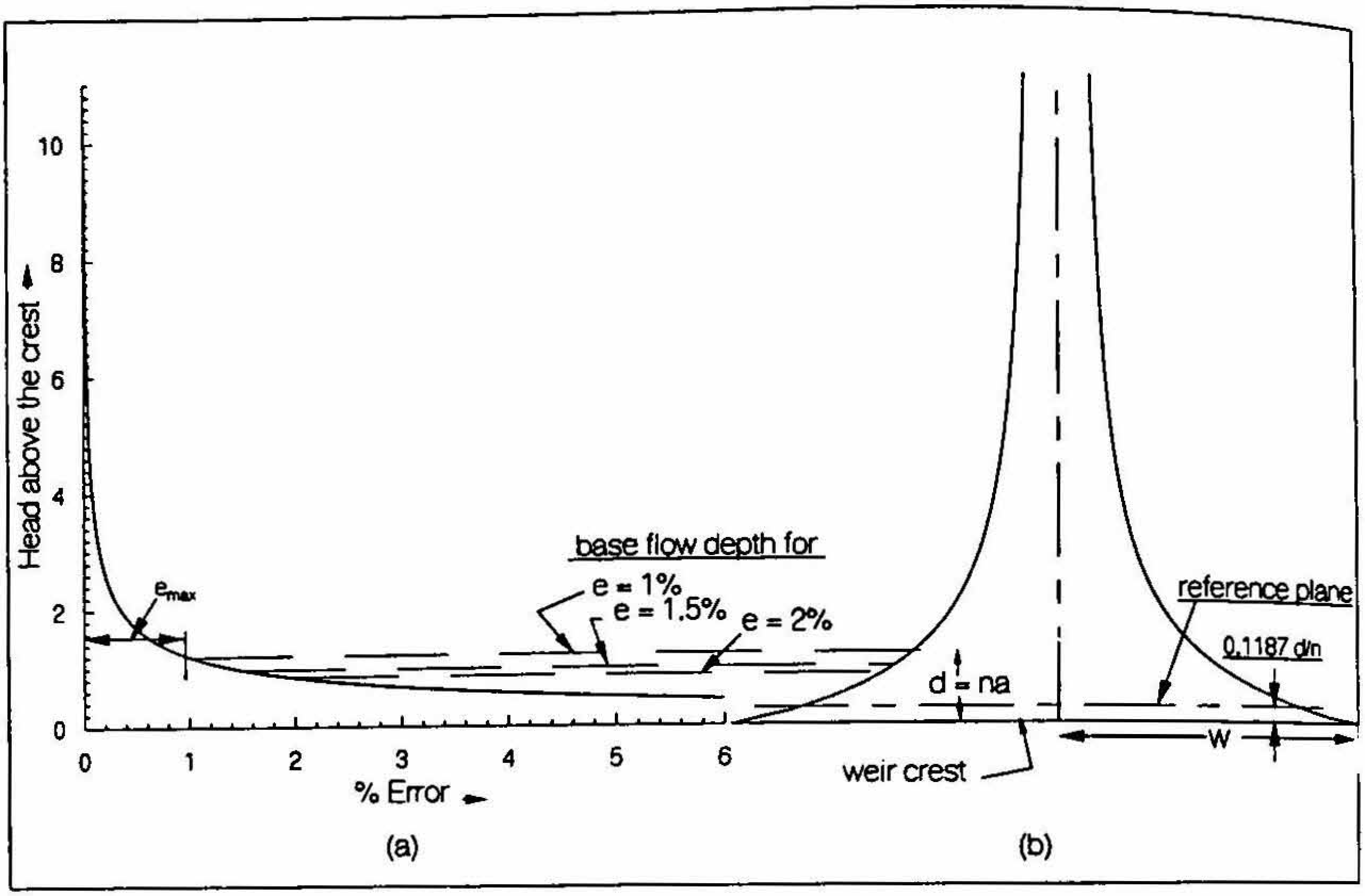

FIG. 10. Parameters of the designed SBL weir.

$$m = \lim_{H\to\infty} \frac{dQ}{dH} = \lim_{H\to\infty}\left[1-\sqrt{1+H}+\sqrt{H}+\frac{2\sqrt{H}}{1+4H}\right]=1 \tag{25}$$

and

$$C = \lim_{H\to\infty}(H-Q) = \lim_{H\to\infty}\left[-\frac{2}{3}+\frac{2}{3}\left\{(1+H)^{3/2}-H^{3/2}\right\}-H^{1/2}+\frac{1}{2}\tan^{-1}(2\sqrt{H})\right]$$

$$C = \frac{\pi}{4}-\frac{2}{3} = 0.1187. \tag{26}$$

Hence, the equation of the asymptote, which is the linear relationship of the SBL weir, is

$$Q_L = H - 0.1187. \tag{27}$$

The absolute value of the deviation of this discharge from the one obtained by the theoretical h–q relationship given by eqn (17) is

$$E_r(H) = Q - Q_L. \tag{28}$$

It is seen that $E(0)$ = finite, $E(\infty) = 0$ and

$$\frac{dE_r}{dH} = \frac{dQ}{dH} - m = \frac{dQ}{dH} - 1. \tag{29}$$

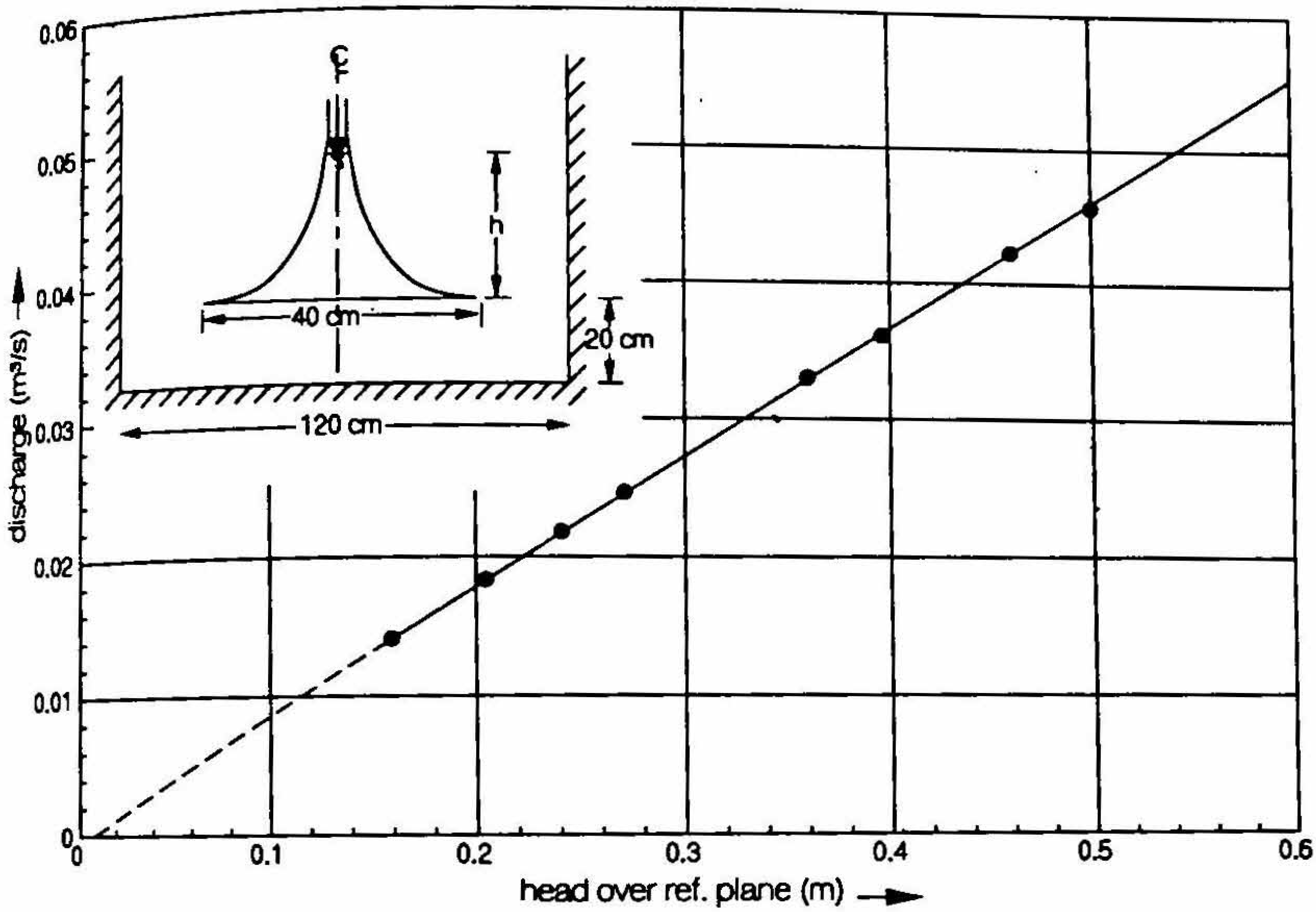

FIG. 11a. Discharge-head variation (experimental).

From eqn (14) it is found that the above expression is negative throughput the range $0 \leq H \leq \infty$. Hence, it is concluded that the relative error e given by

$$e = \frac{Q - Q_L}{Q} \tag{30}$$

continuously decreases as H increases in the range $0 \leq H \leq \infty$, which is also evident from Fig. 5a. Further from Fig. 10a, it can also be observed that the error, which is nearly 1% at $H = 1.2$, decreases very rapidly and is very nearly zero (accurate to third decimal place) at $H = 4$, beyond which for all practical purposes, the designed weir is as accurate as an exact linear weir. If a limit for the maximum permissible value of the error

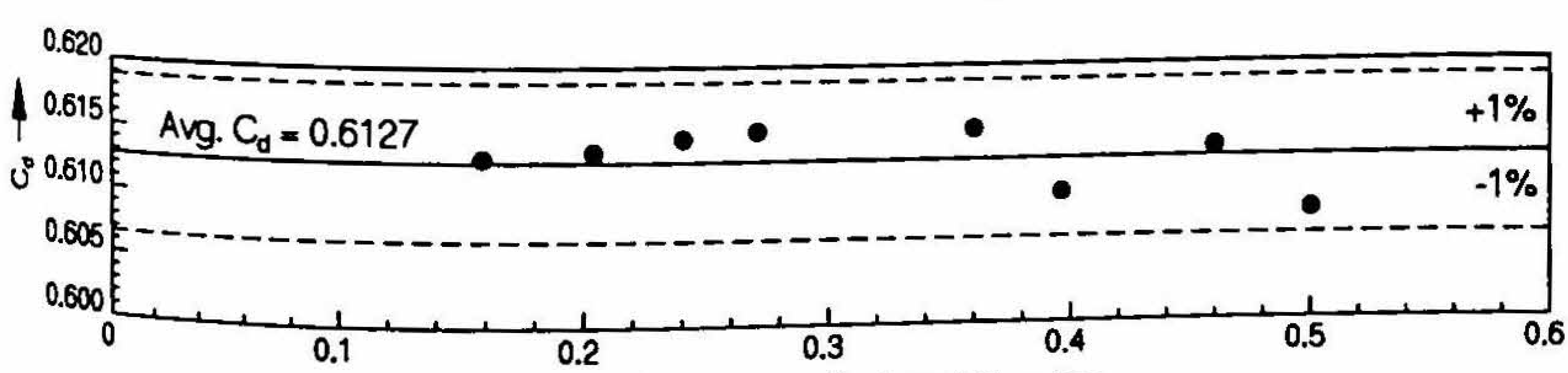

FIG. 11b. Variation of C_d with head.

FIG. 12. SBL weir discharging.

(usually, $\pm 2\%$) is prefixed, the flow through the depth corresponding to any prefixed permissible maximum error (Fig. 10b) is analogous to the flow through the base weir of any existing exact linear weir. This depth is generally called the 'base flow depth' or the 'threshold depth (d)' of the designed SBL weir. For all flows above this threshold depth, the theoretical h–q relationship (eqn(17)) can be replaced by the linear relationship given by eqn (27). Hence, for all flows in the range $d \leq h \leq \infty$, called the 'linearity range', the weir will pass discharges proportional to the head measured above a reference plane or datum located at 0.1187 above the weir crest (Fig. 10b). The validity of this linear relationship is further strengthened from the fact that the coefficient of discharge as obtained from experiments computed using the linear h–q relationship is constant to a high degree of accuracy.

4.4 *Experiments*

Experiments conducted on a typical SBL weir having a crest width of 40 cm in a rectangular channel measuring 18.5 m long, 1.2 m wide and 1.1 m deep and with its crest 20 cm above the bed of the channel show excellent agreement with the theory, giving a constant average coefficient of discharge of 0.61 (Fig. 11). Figure 12 shows a photograph of the SBL weir discharging.

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