J. Indian Inst. Sci., Sept.–Oct. 1995, 75, 537–558.

Formation and propagation of singularities on a nonlinear wavefront and a shockfront

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Received on February 24, 1995; Revised on March 8, 1995.

Abstract

Existence of two types of singularities, namely, cusps and kinks, on a nonlinear wavefront and a shock front, are postulated when these fronts are two-dimensional. The basic conservation laws governing the propagation of kink type of singularities are derived. The time of formation of a kink on a weakly nonlinear wavefront in the high-frequency approximation and on a weak shock front are derived. Propagation of kinks, obtained from numerical solution, is discussed in some cases. This paper raises many open questions regarding these singularities. The singularities discussed here appear frequently in gas dynamics and other physical phenomena. Physical motivation for the derivation of equations governing them have been discussed by Whitham and also by the author in earlier publications. Here an attempt is made to develop a theory in which first geometry (without any reference to physics) is used to derive as many kinematical results as possible and then physics is included in the form of transport equations.

Keywords: Singularities, cusps, kinks, nonlinear wavefront, shock front.

1. Introduction

Singularities on wavefronts are very common physical phenomena. They appear on the surface of tea or coffee in a cup when the cup is placed at a suitable place relative to a source of light and also when the rays of light pass through a gravitational lens in cosmological phenomena. In the linear theory of wave propagation in any medium governed by an arbitrary hyperbolic system of partial differential equations, the successive positions of a wavefront can be determined by the well-known Huygens' method of wavefront construction using the secondary spherical waves starting from the various points of the wavefront at any previous time[1]. Huygens' original method is equivalent to construction of the wavefront at any time, using the rays from the various points of the initial position of the wavefront. We note three fundamental properties of linear wavefront propagation.

1. *Self-propagation.* This means that a linear wavefront is determined by the information only on the wavefront at any previous time and is not influenced by the wavefronts which follow or precede it. For a linear wavefront, the information required is simply the position (and hence the geometry) of the wavefront and not even the amplitude of the wavefront.

2. *Local determinacy.* This means that the motion of an arbitrary small arc of a linear wavefront is independent of the neighbouring arcs.

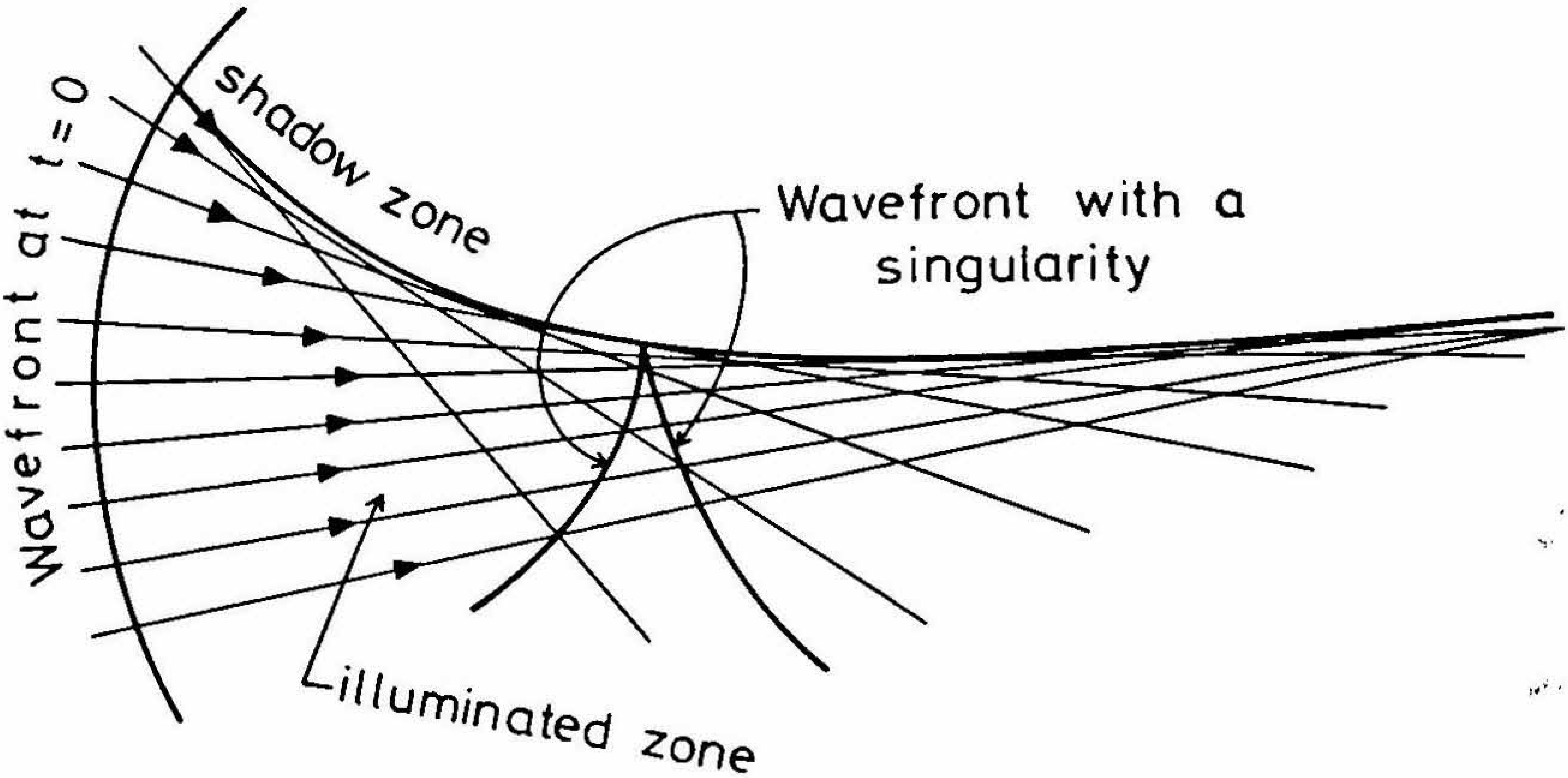

FIG. 1. An initially smooth wavefront folds with a cusp type of singularity on a caustic. Rays are taken to be straight lines.

3. *Reversibility in time.* According to linear propagation, where the amplitude of the wave has no influence, if a wavefront Ω_{t_1}, at time t_1 leads to a wavefront Ω_{t_2} at time t_2 ($>t_1$), then by reversing the direction of propagation velocity we can get Ω_{t_1} from Ω_{t_2}.

We shall point out later in this article the modification, if any, which nonlinearity produces on the above three properties of a linear wavefront.

Continuing the discussion of Huygens' method of linear wavefront construction with the help of rays, we note that in many cases the rays envelope a caustic surface separating an illuminated zone from a shadow zone. The wavefront folds itself on the caustic surface so as to remain in the illuminated zone and possesses a singularity which is a cusp (Fig. 1). Consider now a converging wavefront (for example, a wavefront which was initially a parabola); the caustic has two branches starting from an arête. In this case, the wavefront develops a pair of singularities as shown in Fig. 2. If all the rays converge to a point, the caustic degenerates into a focus. In the domain bounded by the two branches of the caustic, the linear wavefront crosses itself at a *node*. Study of very weak nonlinear effects on a node of a linear wavefront cannot be done in the single-mode high-frequency approximation, which forms the basis of our study in this article. It also appears that nodes are rare when (genuine) nonlinearity is present. Nodes are excluded from the study of singularities in this article.

It is interesting to view the geometry of the successive positions of a plane wavefront after it is reflected from the interior surface of a circular cylinder (Fig. 3). The illuminated part of the caustic is the domain bounded by the reflecting surface and the caustic surface. Soon after the reflection, say at the time t_1, the wavefront develops a pair of cusps both of which approach the arête at a critical time t_c after which the singularities

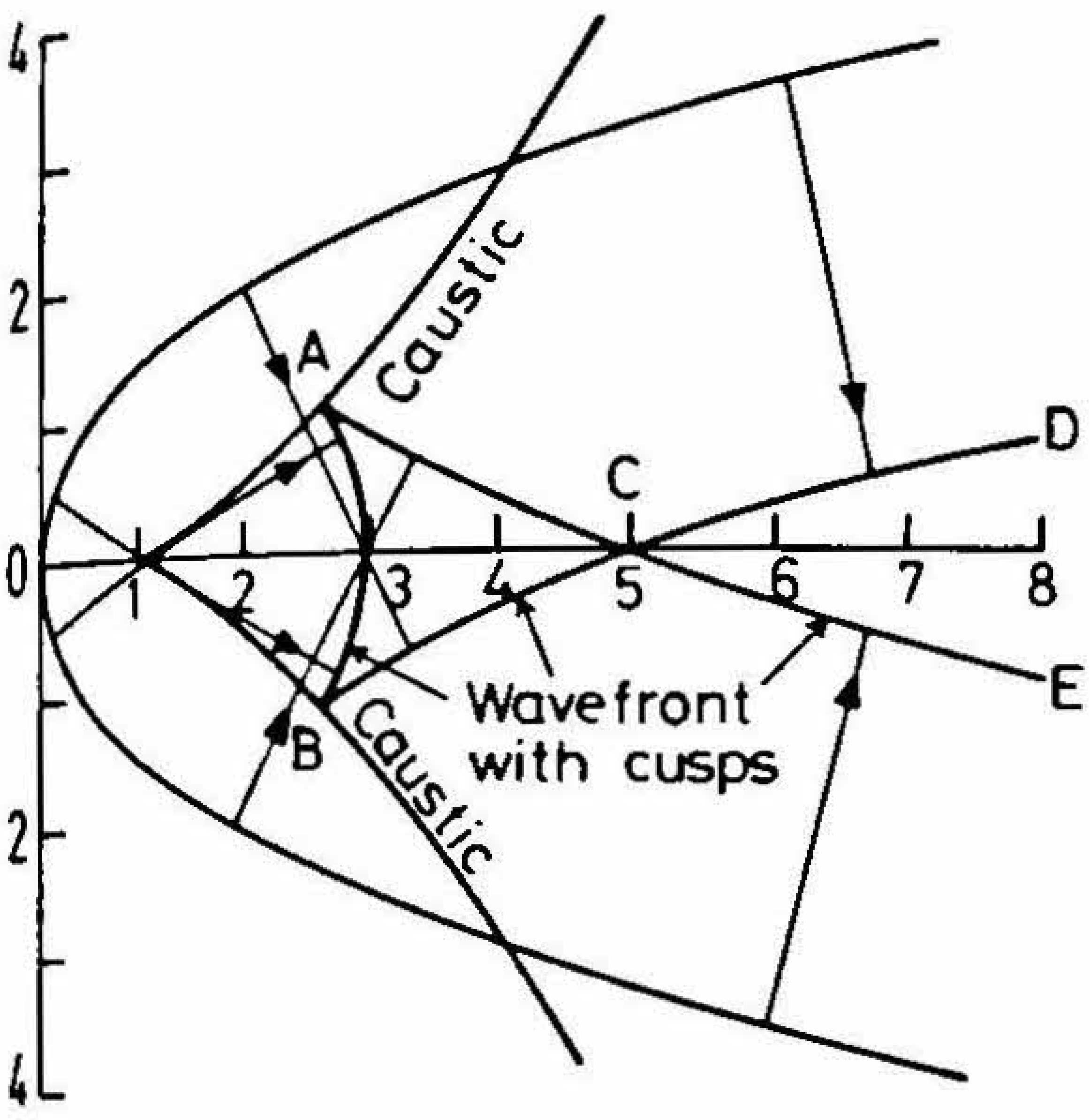

FIG. 2. A caustic generally starts from an arête, which itself is a cusp of the caustic.

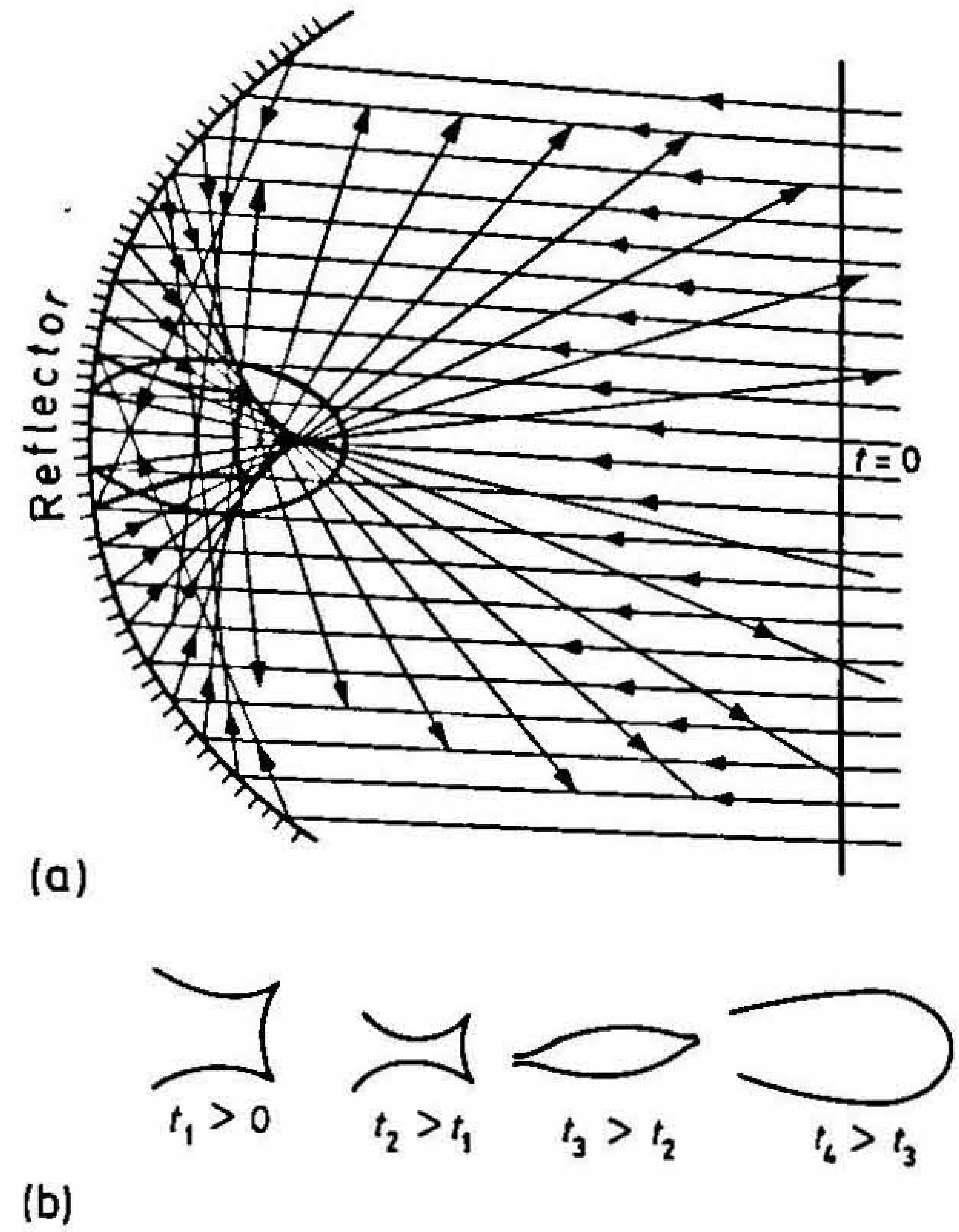

FIG. 3a. Caustic on the surface of tea in a cup, neglecting multiple reflections of the rays. b. Four typical shapes of the wavefront after a plane wavefront is reflected.

disappear and the wavefront becomes smooth as shown at time $t_4 > t_c$. This represents a phenomenon in which the transition at t_c takes place in reverse order of that depicted in Fig. 2 when the wavefront just meets the arête. This is an important observation and shows that the propagation of a linear wavefront is a reversible process, which we shall show not to be correct in general for a nonlinear wavefront.

Singularities do appear also on nonlinear fronts. Apart from the cusp singularities (we postulate that they appear also on a nonlinear front), we encounter a new type of singularity on a nonlinear front and we call this singularity a *kink*, across which both the direction of the normal and the amplitude (or intensity) of the discontinuity suffer finite jumps. The existence of a kink was first shown theoretically by Whitham[2] in 1957 by an approximate theory called shock dynamics. He named a kink on a shock front as a shock-shock. The first experimental results showing kinks on a shock front in gaseous media were obtained by Sturtevant and Kulkarni[3], and even today it is a great challenge to verify various observed properties of the flow field containing these kinks, especially the transition from the shock fronts containing kinks to the linear acoustic fronts containing only cusps.

Singularities of wavefronts were first discovered by Huygens in 1654. In the solution of linear hyperbolic partial differential equations, singularities have been studied in great detail[4] but these are related to the characteristics of multiplicity higher than one.

We are interested in problems in which the multiplicity of the characteristics is uniformly equal to one. Recently, Arnold[5] has also studied singularities of caustics and wavefronts. However, our approach is different and it is not clear whether the time evolution of the singularities can also be studied by the approach followed by Arnold.

Formation of singularities on a wavefront can be studied from the differential forms of suitable equations since, till the singularities are formed, the solution is either smooth or has discontinuities in higher-order derivatives. But as soon as a singularity appears on a wavefront, a discontinuity either in the amplitude or in the normal direction or in both comes into play. Hence propagation of singularities can be studied only from the conservation forms of the governing equations. In the next section we shall derive the conservation forms in the most general situation. Most of the material presented in this article is new. However, the research problems and the spirit of investigation of these is quite old and has been going on in our group for the last 20 years. Details of the derivation of many results here and related concepts can be found in the book by Prasad[6].

2. Conservation forms of the kinematical equations of a propagating curve

Let Ω_t be a curve representing a wavefront which occupies different positions at different times. For discussion in this section, wavefront refers also to a shock front. Distinction between a wavefront, across which state variables are continuous, and a shock front needs to be taken into account when we consider the dynamical equations. Associated with a wavefront Ω_t, there exists at every point of it a ray velocity $\chi = (\chi_1, \chi_2)$. This gives a one-parameter family of curves, called rays, each one of which is traced by a point on Ω_t moving with velocity χ. Expression for χ can be obtained only from the properties of the medium in which Ω_t propagates and depends also on the unit normal $n = (n_1, n_2)$ of Ω_t. We write the equation of Ω_t in the form

$$\Omega_t: \quad x = x(\xi, t), \;\; y = y(\xi, t), \tag{1}$$

where constant values of t give the positions of the propagating curve Ω_t at different times and ξ = constant represents a ray. In the case of isotropic wave propagation, rays are orthogonal to the family of wavefronts Ω_t. When the equation of Ω_t is represented in terms of a ray coordinate system (ξ, t) as in eqn (1) the ray velocity $\chi = (\chi_1, \chi_2)$ is given by

$$(x_t, y_t) = (\chi_1, \chi_2). \tag{2}$$

C, the speed of propagation of Ω_t, is

$$C = \langle n, \chi\rangle \equiv n_1\chi_1 + n_2\chi_2 \tag{3}$$

and the component T of the ray velocity in the direction of the tangent to Ω_t is

$$T = -n_2\chi_1 + n_1\chi_2. \tag{4}$$

Consider the curves Ω_t and Ω_{t+dt}. Let P' and Q' be the positions at time $t + dt$ of P and Q, respectively, on two rays at a distance $g d\xi$ on Ω_t (Fig. 4), where g is the associated metric. PN is the normal displacement of Ω_t in time dt. If the coordinates of Q' are $(x + dx, y + dy)$ then (dx, dy) is the displacement in the (x, y) plane corresponding to a displacement $(d\xi, dt)$ in the ray coordinate plane, so that

$$dx = (Cdt)n_1 - (gd\xi + Tdt)n_2,$$

$$dy = (Cdt)n_2 + (gd\xi + Tdt)n_1.$$

Let θ be the angle which the normal to Ω_t makes with the x axis ($n_1 = \cos\theta$, $n_2 = \sin\theta$). The above relation gives the Jacobian matrix of the transformation from (ξ, t) plane to (x, y) plane

$$\begin{pmatrix} x_\xi & x_t \\ y_\xi & y_t \end{pmatrix} = \begin{pmatrix} -g\sin\theta & C\cos\theta - T\sin\theta \\ g\cos\theta & C\sin\theta + T\cos\theta \end{pmatrix}, \tag{5a}$$

where

$$\frac{\partial}{g\partial\xi} = -n_2\frac{\partial}{\partial x} + n_1\frac{\partial}{\partial y}. \tag{5b}$$

The Jacobian, *i.e.*, the determinant of the Jacobian matrix, is $-gC$, which shows that the transformation between (ξ, t) and (x, y) is nonsingular as long as g and C are nonzero and finite.

Following Prasad *et al.*[7] we can derive a pair of kinematical relations in the conservation form by equating $x_{\xi t}$ to $x_{t\xi}$ and $y_{\xi t}$ to $y_{t\xi}$:

$$(g\sin\theta)_t + (C\cos\theta - T\sin\theta)_\xi = 0, \tag{6}$$

$$(g\cos\theta)_t - (C\sin\theta - T\cos\theta)_\xi = 0. \tag{7}$$

From these, we can deduce the following partial differential equations

$$g_t = C\theta_\xi + T_\xi, \tag{8}$$

$$\theta_t = -\frac{1}{g}C_\xi + \frac{1}{g}T\theta_\xi, \tag{9}$$

as kinematical relations for any propagating curve Ω_t. Equations (8) and (9) or their conservation forms (6) and (7) represent a system of two equations involving four quantities g, θ, C and T. For a wavefront which can be defined only in the high-frequency limit (high-frequency assumption implies that the wavelength of the wave is small compared to the radius of curvature of Ω_t and the length over which the state of the medium varies), the quantities C and T can be expressed in terms of an amplitude w of the wavefront Ω_t and its unit normal n. Therefore, to get a determined system of equations, we must add to these equations another evolution equation for w. Such an equation turns out to be a transport equation along a ray[8], which we shall discuss later in this article.

We shall now show the equivalence of equations (8) and (9) to the ray equations (or the bicharacteristic equations) of a hyperbolic system

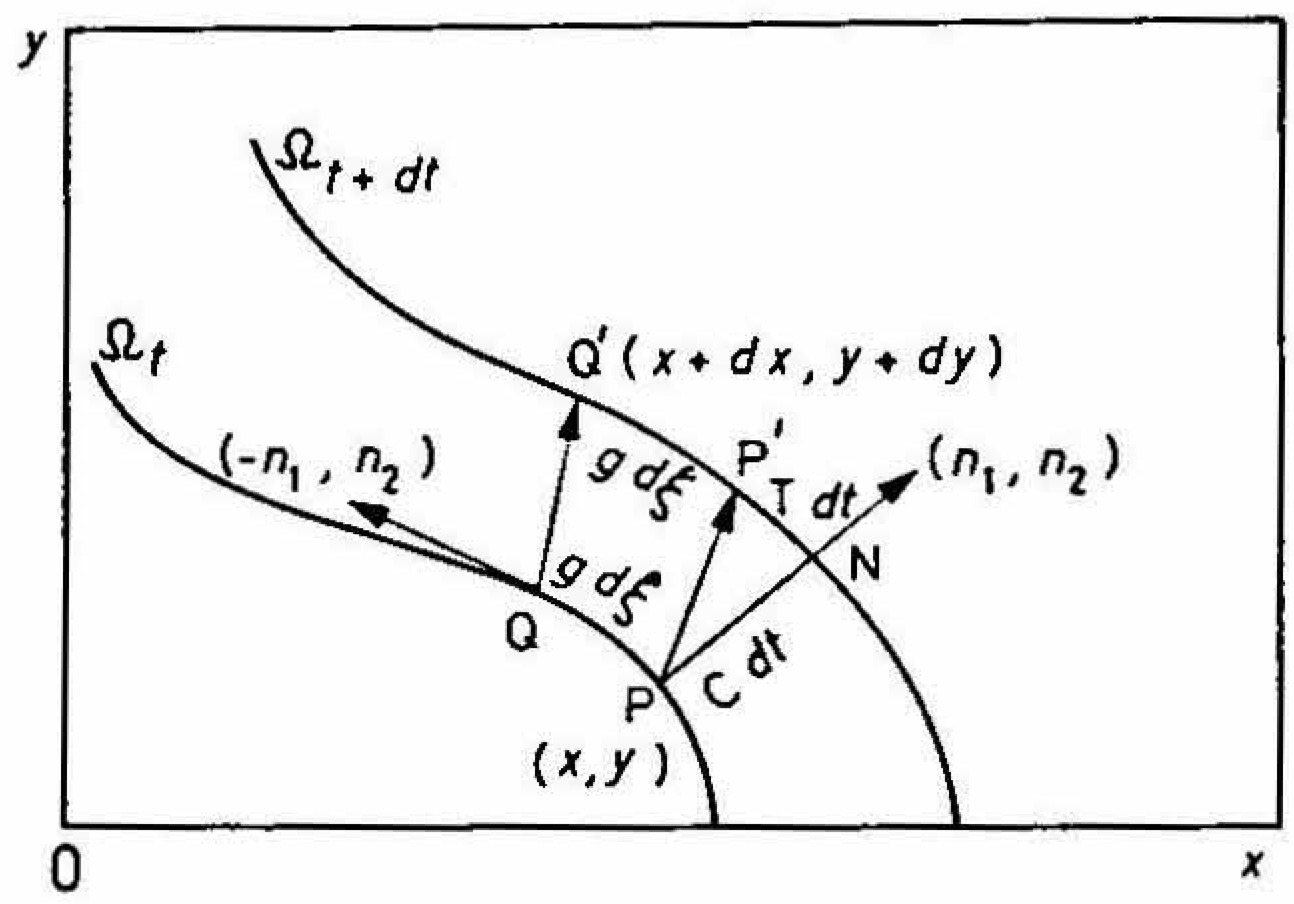

FIG. 4. P' and Q' are positions at time $t + dt$ of P and Q, respectively, on two rays at a distance of $g d\xi$ on Ω_t.

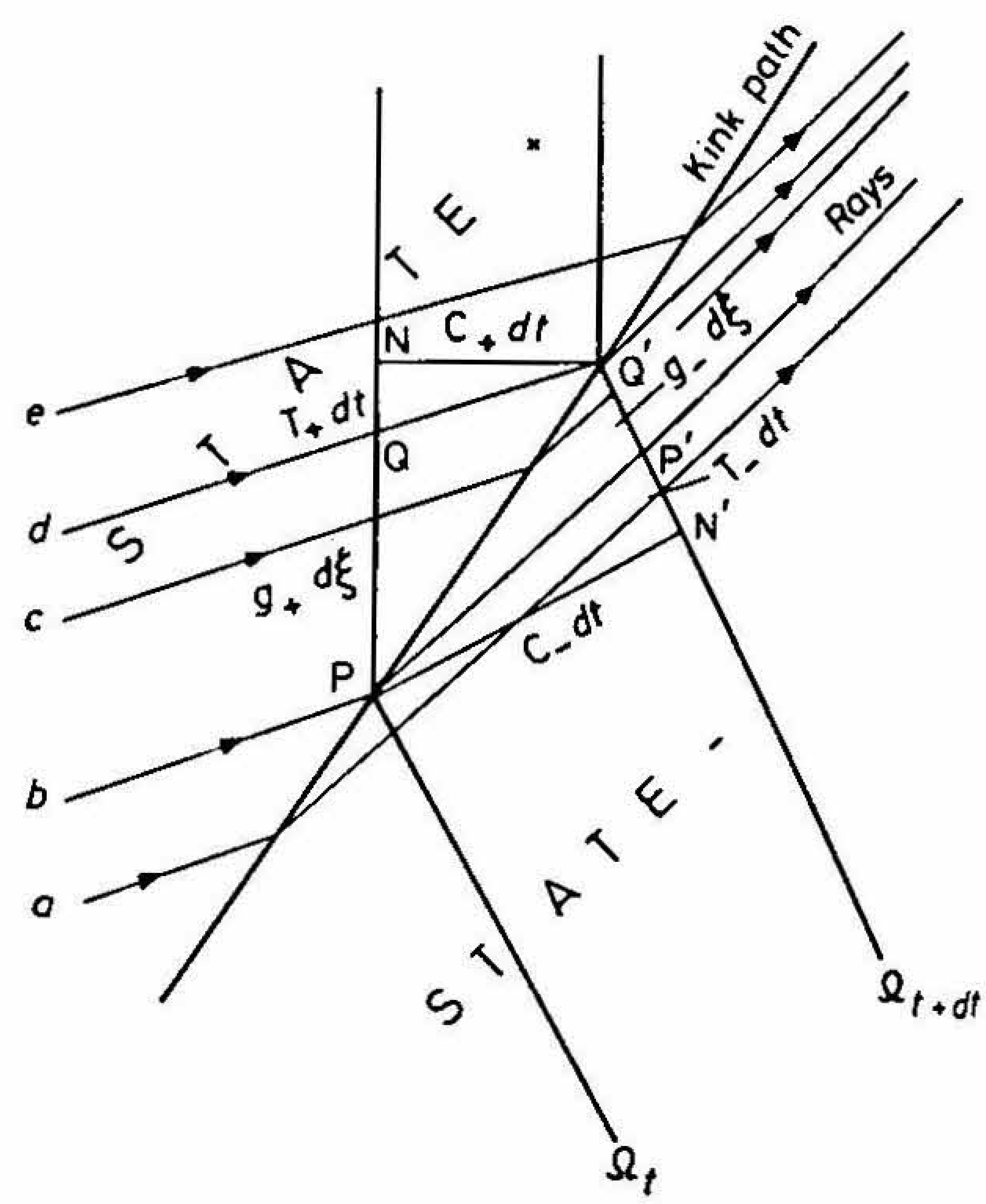

FIG. 5. Geometry of wavefronts and rays on the two sides of a kink.

$$Au_t + B^{(1)}u_x + B^{(2)}u_y + F = 0 \tag{10}$$

when Ω_t is taken to be the projection on the (x, y) plane of the section of the characteristic surface $\phi(x, y, t) = 0$ by $t =$ constant plane. Here u and $F(x, t, u)$ are n-dimensional column vectors and $A(x, t, u,)$, $B^{(1)}(x, t, u,)$ and $B^{(2)}(x, t, u,)$ are $n \times n$ matrices. In this case C is a characteristic velocity, $C = -\phi_t / |\nabla\phi|$ and $n = \nabla\phi / |\nabla\phi|$. The ray equations (equations (3.8) and (3.9) of Prasad[8]) become

$$\frac{dx}{dt} = \frac{lB^{(1)}r}{lAr} \equiv \chi_1, \quad \frac{dy}{dt} = \frac{lB^{(2)}r}{lAr} \equiv \chi_2 \tag{11}$$

and

$$\frac{d\theta}{dt} = \frac{1}{g(lAr)}\left(lC\frac{\partial A}{\partial \xi} - n_1\frac{\partial B^{(1)}}{\partial \xi} - n_2\frac{\partial B^{(2)}}{\partial \xi}\right)r, \tag{12}$$

where l and r are left and right null vectors of $A\phi_t + B^{(1)}\phi_x + B^{(2)}\phi_y$. Since

$$C(lAr) = l(n_1B^{(1)} + n_2B^{(2)})r, \quad n_1 = \cos\theta, \quad n_2 = \sin\theta$$

$$C_\xi = -\frac{1}{lAr}l(CA_\xi - n_1B_\xi^{(1)} - n_2B_\xi^{(2)})r + \frac{1}{(lAr)}l(-n_2B^{(1)} + n_1B^{(2)})r\theta_\xi. \tag{13}$$

Using eqns (4) and (14), and noting that d/dt in the (ξ, t) plane becomes the partial derivative $\partial/\partial t$, we find that eqn (12) reduces to eqn (9). To deduce eqn (8) from eqn (11), we differentiate $g^2 = x_\xi^2 + y_\xi^2$ with respect to t to get

$$gg_t = x_\xi x_{\xi t} + y_\xi y_{\xi t} = x_\xi x_{t\xi} + y_\xi y_{t\xi}.$$

Substituting in the above the expressions for x_t and y_t from eqn (11) and noting that $x_\xi/g = -n_2$, $y_\xi/g = n_1$, we get

$$\begin{aligned} g_t &= -n_2(\chi_1)_\xi + n_1(\chi_2)_\xi \\ &= (-n_2\chi_1 + n_1\chi_2)_\xi + (n_1\chi_1 + n_2\chi_2)\theta_\xi, \end{aligned}$$

which is the relation (8).

We can deduce a pair of kinematic equations in conservation form also in the (x, y) plane. The Jacobian of the transformation from (x, y) plane to (ξ, t) plane is given by

$$\begin{pmatrix} \xi_x & \xi_y \\ t_x & t_y \end{pmatrix} = \begin{pmatrix} -\frac{1}{gC}(C\sin\theta + T\cos\theta) & \frac{1}{gC}(C\cos\theta - T\sin\theta) \\ \frac{1}{C}\cos\theta & \frac{1}{C}\sin\theta \end{pmatrix}. \tag{14}$$

Equating the partial derivatives ξ_{xy} to ξ_{yx} and t_{xy} to t_{yx}, we get the conservation forms

$$\left(\frac{\cos\theta - (T/C)\sin\theta}{g}\right)_x + \left(\frac{\sin\theta + (T/C)\cos\theta}{g}\right)_y = 0, \tag{15}$$

$$\left(\frac{\sin\theta}{C}\right)_x - \left(\frac{\cos\theta}{C}\right)_y = 0. \tag{16}$$

The conservation forms (15) and (16) with $T = 0$ were obtained by Whitham[2]. These conservation laws are not suitable for studying the propagation of the singularities. However, we shall show later that the two sets (6), (7) and (15), (16) are equivalent in the sense that both lead to the same relation between the jumps in θ, C and T.

3. Two types of singularities and jump conditions across a kink

We start with an observation which we can take as a basic assumption: rays are neither lost nor created. Thus, the variable ξ introduced on a wavefront remains continuous even if a singularity appears on the wavefront, which may get folded or suddenly bent. Now two cases arise.

Case a. In the first case θ, g, C and T remain smooth functions satisfying the partial differential equations but the Jacobian $\partial(x, y)/\partial(\xi, t)$ given by the determinant of (5) vanishes in the (ξ, t) plane. This can happen at an isolated point in the (ξ, t) plane or along a curve m in the (ξ, t) plane. We consider only the situation when the Jacobian $= -gC$ vanishes along the curve m. The situation is again complicated. $-gC$ can vanish when C vanishes, an example of which is a sonic line in gas dynamics[9], where not only C but also T vanishes for wavefronts which are orthogonal to the stream lines. We further restrict our consideration to the case in which the Jacobian vanishes due to the vanishing of the metric g and $C \neq 0$. The image of m in the (x, y) plane is an edge. Consider now a point P on m. It can be proved[10] that at P there exists an exceptional direction in the (ξ, t) plane such that the image of any curve passing through P in the exceptional direction has a cusp on the edge in the (x, y) plane. This direction given by $(\dot{\xi}, \dot{t})$ is the right null vector of the Jacobian matrix at P. Since $g = 0$ at P, we get $\dot{\xi} \neq 0$, $\dot{t} = 0$, showing that the wavefront (t = constant) itself is in the exceptional direction at the points P of m. Thus, the wavefront at any t remains in a domain only on one side of the edge and has a cusp on it. A ray is not tangential to the wavefront in the (ξ, t) plane. Hence, the ray through P is not in the exceptional direction and is mapped into a curve which is tangential to the edge in the (x, y) plane. This shows that when $g = 0$ and $C \neq 0$, the edge is an envelope of the rays, *i.e.*, a caustic and a wavefront Ω_t has a cusp type of singularity on t. Such singularities appear frequently in linear wave propagation. *We postulate that cusps appear also on nonlinear wavefronts.*

Case b. We shall show in the next section that eqns (8) and (9) along with the transport equation for the amplitude in high-frequency approximation imply that the quantities θ, g, C and T, though continuous functions of ξ at $t = 0$, may become discontinuous at a point $K(\xi_p(t), t)$ on Ω_t. For a fixed t, limits of these quantities (denoted by subscripts − and +) as we approach K from lower and higher values of ξ to $\xi_p(t)$ are finite so that the jumps $[\theta] = \theta_+ - \theta_-$, $[g] = g_+ - g_-$, $[C] = C_+ - C_-$ and $[T] = T_+ - T_-$ are also finite. Since the jump $[\theta]$ is nonzero, the curve Ω_t suffers a sudden change in the tangent direction at a point P which is the image in the (x, y) plane of K in the (ξ, t) plane, and since $[\theta] \neq \pi$ (which can be seen from the results in a particular case in the next section), the point P is not a cusp of Ω_t but is a new type of singularity, which we call a *kink*. When we move on Ω_t, the Jacobian $-gC$ of the transformation also suffers a finite jump as we cross a kink and neither of the values $(-gC)_-$ and $(-gC)_+$ is zero. Figure 5 represents a kink phenomenon in the (x, y) plane. The kink at P on Ω_t occupies a position Q' on Ω_{t+dt}. The kink path separates two states + and −. PN' and $Q'N$ are the normals from P and Q' to the negative side of Ω_{t+dt} and positive side of Ω_t, respectively. $N'P' = T_- \, dt$ is the displacement of a ray along Ω_{t+dt} due to the tangential velocity T_- and $g_- \, d\xi$ is the distance along Ω_{t+dt} between P' and Q'. Corresponding quantities $g_+ \, d\xi$ and $T_+ \, dt$ on the positive side are PQ and QN.

Using the Pythagoras theorem we get

$$(g_+ \, d\xi + T_+ \, dt)^2 + (C_+ \, dt)^2 = (g_- \, d\xi + T_- \, dt)^2 + C_- \, dt)^2, \tag{17}$$

which shows that the kink velocity $K = d\xi/dt$ satisfies the quadratic equation

$$(g_-^2 - g_+^2)K^2 + 2(g_- T_- - g_+ T_+)K + (C_-^2 - C_+^2) + (T_-^2 - T_+^2) = 0, \tag{18}$$

which has real roots if

$$(g_- T_+ - g_+ T_-)^2 + (C_+^2 - C_-^2)(g_-^2 - g_+^2) > 0. \tag{19}$$

The kink velocity K in the (ξ, t) plane can also be deduced from the conservation forms (6) and (7). The jump relations obtained from these are

$$-K[g \sin\theta] + [C\cos\theta - T\sin\theta] = 0, \tag{20}$$

$$K[g\cos\theta] + [C\sin\theta + T\cos\theta] = 0. \tag{21}$$

Eliminating θ from these two we get eqn (18). This shows that the two conservation laws (6) and (7) represent conservation of distance in the (x, y) plane and hence are physically realistic conservation laws. Eliminating K from eqns (20) and (21), we get the following *Hugoniot curve*:

$$\cos(\theta_- - \theta_+) = \frac{C_- g_- + C_+ g_+}{C_- g_+ + C_+ g_-} + \frac{g_+ T_- - g_- T_+}{g_+ C_- + g_- C_+} \sin(\theta_- - \theta_+). \tag{22}$$

Let S be the slope of the path of the kink in (x, y) plane, *i.e.*, $S = (\mathrm{d}y/\mathrm{d}x)_{\text{kink}}$. The jump relation derived from conservation laws (15) and (16) give the following expressions for S:

$$S = \frac{C_- \cos\theta_+ - C_+ \cos\theta_-}{C_+ \sin\theta_- - C_- \sin\theta_+}, \tag{23a}$$

$$S = \frac{g_+ \sin\theta_- - g_- \sin\theta_+ + (T_- g_+ / C_-)\cos\theta_- - (T_+ g_- / C_+)\cos\theta_+}{g_+ \cos\theta_- - g_- \cos\theta_+ + (T_+ g_- / C_+)\sin\theta_+ - (T_- g_+ / C_-)\sin\theta_-} \tag{23b}$$

respectively. Equating the two expressions for S, we get the relation (22). Thus, the two sets of conservation laws (6) and (7) are equivalent in the sense that both lead to the same Hugoniot relation across a kink.

It is possible to get one more interesting result using only the kinematical equations. Consider a wavefront with x axis as the line of symmetry (Fig. 6) with two kinks P_1 and P_2 joining a straight disk (with θ_-, C_-, T_-) of the wavefront and two wings (with θ_+, C_+, T_+). The question arises: 'Do the kinks move away from or tend to approach one an

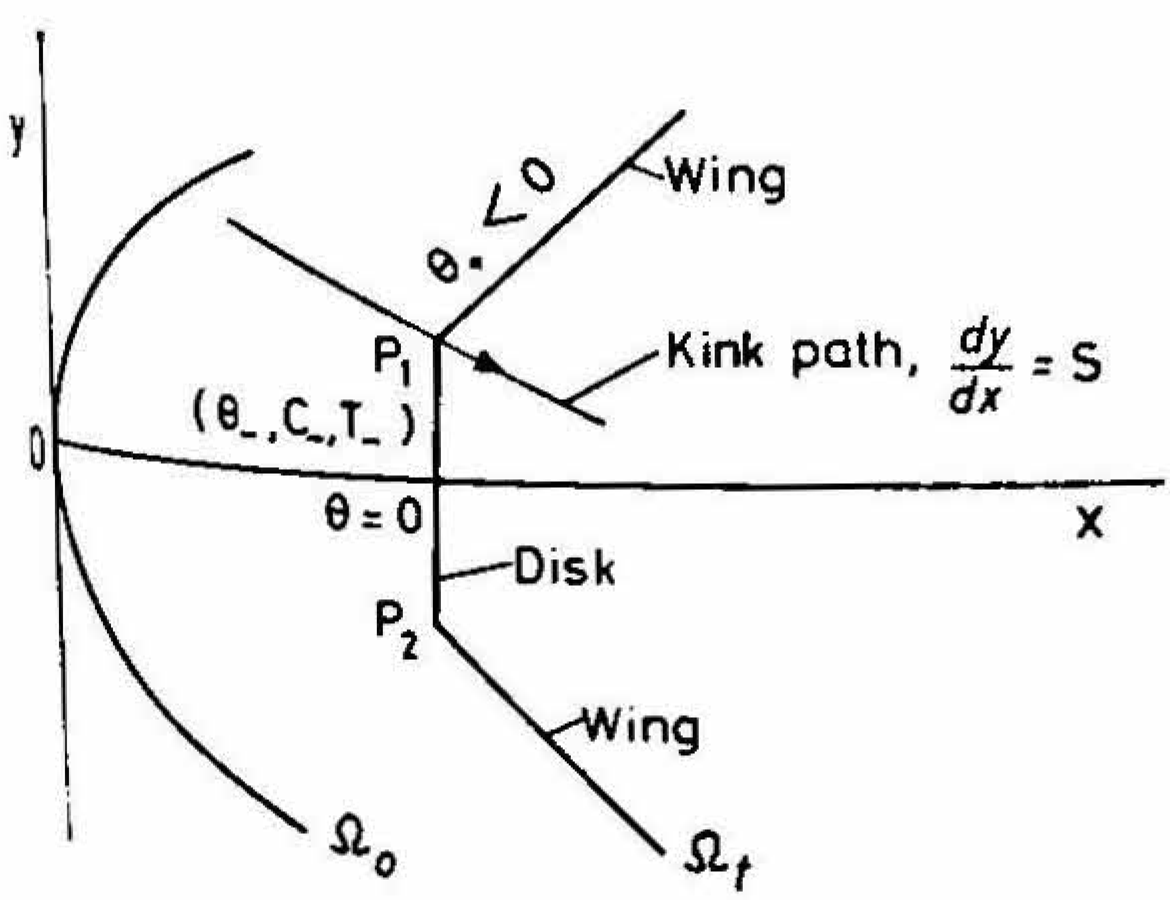

FIG. 6. The two kinks approach one another if $S < 0$.

other?'. To answer this, we note that the slope S of the upper kink path in (x, y) plane is negative if $S < 0$. We consider the whole configuration of the wavefront to be moving in the positive x direction, then $C_- > 0$ and since $\theta_- = 0$ and $\theta_+ < 0$, eqn (23) with $S < 0$ gives

$$C_- \cos \theta_+ < C_+. \tag{24}$$

Thus, a necessary and sufficient condition for the two kinks to approach each other is eqn (24). If a pair of kinks appears on an initially smooth converging wavefront symmetric with respect to the x axis, then $-\pi/2 < \theta_+ < 0$. In this case eqn (24) gives the sufficient condition $C_- < C_+$ for the two shocks to approach. This sufficient condition was derived by Kelvahan[11].

4. Formation of a kink

In order to study the formation of a kink on a wavefront, we need to add to the kinematical equations (2), (8) and (9) transport equations, for the amplitude of the wave and some other related variables along a ray. So far we have used the word *wavefront* only to represent a propagating curve Ω_t. In this section, we shall first clarify the meaning of wavefront and distinguish between two types of wavefronts. In general, the concept of a wavefront is an approximate one[12]. It involves three length scales: the first one denoted by H is the length over which the ambient medium (in which the wave propagates) varies significantly; the second one denoted by R is of the order of the radius of curvature of the wavefront; and the third one denoted by L is the length in the direction normal to the wavefront over which the amplitude of the wave varies significantly, and is called the wavelength of the wave whether the wave is periodic or not. A wavefront can be defined only in high-frequency or short-wavelength limit which implies that the wavelength is small compared to the other two, *i.e.*, $L/H \ll 1$ and $L/R \ll 1$. The concept of a wavefront becomes exact when a plane wavefront propagates in a uniform medium, *i.e.*, when $H = \infty$ and $R = \infty$, or when $L = 0$, *i.e.*, the wavefront carries either discontinuities in the normal derivatives of the amplitude (characteristic waves, more commonly known as weak discontinuities) or discontinuities in the amplitude itself for a system governed by first-order equations (shock waves or contact discontinuities). In any case, a high-frequency or short-wave assumption implies that all state variables of the system on a wavefront can be expressed in terms of the unit normal n of the wavefront and an amplitude variable w. This result is well known for a shock wave and has been shown to be true also for a high-frequency wave (either a continuous wave or a weak discontinuity[8,13,14]. In this article we are interested in the propagation of a shock front as well as that of a one-parameter high-frequency nonlinear wavefronts across which the amplitude of the wave changes continuously from one nonlinear wavefront to the neighbouring ones[8]. We study the formation of a kink on these two types of fronts separately. For details of the derivation of the equations, see Prasad[6].

4.1. *Nonlinear wavefront (NLWF) in a polytropic gas*

For a NLWF which is self-propagating, there appears only one transport equation along the ray, namely, the equation for the amplitude w on the wavefront. Consider the polytropic gas medium in which the density, particle velocity and pressure are denoted by ρ, $q = (q_1, q_2)$, and p, respectively. The ray velocity is given by

$$\chi = (q_1 + an_1,\ q_2 + an_2), \tag{25}$$

where a is the local velocity of sound. Then

$$C = a + n_1 q_1 + n_2 q_2, \tag{26}$$

$$T = -n_2 q_1 + n_1 q_2. \tag{27}$$

For a NLWF running into a medium in a uniform state at rest ($\rho = \rho_0$, $q = 0$, $p = p_0$, $a = a_0$) the perturbations due to a small-amplitude high-frequency wavefront are given by

$$q = nw, \quad p - p_0 = \rho_0 a_0 w, \quad \rho - \rho_0 = \frac{\rho_0}{a_0} w, \tag{28}$$

so that

$$C = a_0 + \frac{\gamma + 1}{2} w, \ T = 0, \tag{29}$$

where γ is the constant ratio of the specific heats. The transport equation along a nonlinear ray (associated with the NLWF) is

$$w_t = -\frac{1}{2g} a_0 w \theta_\xi. \tag{30}$$

We nondimensionalize the x and y coordinates by a typical length R in the problem (say, the radius of curvature of the NLWF at $t = 0$ at a suitable point), w by the sound velocity a_0 and time by R/a_0 and denote the nondimensional quantities also by x, y, t and w. We also introduce a Mach number m of the NLWF (we shall reserve the symbol M to denote the Mach number of a shock front) by

$$m = 1 + \frac{\gamma + 1}{2} w. \tag{31}$$

Equations (8), (9), (29) and (30) give

$$g = (m - 1)^{-2} e^{-2(m-1)}, \tag{32}$$

where we have removed the dependence of g on ξ by redefining ξ. Successive positions of a nonlinear wavefront and the distribution of w on t are finally given by

$$x_t = m \cos \theta, \qquad y_t = m \sin \theta, \tag{33}$$

$$\theta_t + \frac{1}{g} m_\xi = 0, \tag{34}$$

$$m_t + \frac{m-1}{2g} \theta_\xi = 0, \tag{35}$$

subject to the initial values

$$x(t, 0) = x_0(\xi), \qquad y(t, 0) = y_0(\xi) \tag{36}$$

for the NLWF position at $t = 0$, and initial distribution of amplitude

$$m(t, 0) = m_0(\xi). \tag{37}$$

Equation (36) should be consistent with the initial value of θ:

$$\theta(t, 0) = \theta_0(\xi). \tag{38}$$

The system of eqns (34) and (35) is elliptic for $m < 0$. For $m < 1$, *i.e.*, $w < 0$, eqn (28) shows that the nonlinear wave is an expansion wave. In this case, a singularity in the form of a kink cannot appear on the wavefront. We assume in the sequel that $m > 1$, *i.e.*, the wave is a compression wave. Then the characteristic velocities of the system (34) and (35) are

$$c_1,\ c_2 = \pm\sqrt{\frac{m-1}{2g^2}}, \tag{39}$$

which are real and distinct.

Simple wave solutions of eqns (34) and (35) can easily be obtained by choosing the initial data (37) and (38) such that the Riemann invariants $\theta \mp 2\sqrt{(\gamma+1)}w = r,\ s$ are constant. In a simple wave, we can relate m_ξ at any time on a characteristic to its initial value $m_0'(\xi)$. In the simple wave with r as constant, a discontinuity in m (as a function of ξ) appears if $m_0'(\xi) < 0$, in which case m_ξ tends to $-\infty$ at some point on the wavefront at a critical time given by

$$t_c = 2\sqrt{2} \min_{\xi_0 \in S} \{(m_0 - 1)^{3/2}\, e^{2(m_0-1)}\, |m_0'(\xi)|(5 + 4(m_0 - 1))\}^{-1}, \tag{40}$$

where S is the set of points on the initial line where $m_0'(\xi) < 0$. Thus, in this special simple wave solution of eqns (33)–(38), a kink on the wavefront appears at t_c.

The formation of a kink in a general solution of eqns (33)–(38) has also been studied recently by Monica and Prasad (unpublished), which we present below. Consider a discontinuity in the first derivatives of θ and m moving with the characteristic velocity c_1 in a continuous arbitrary unsteady solution. Then the jumps in the first derivatives are related by

$$[\theta_t] = -c_1[\theta_\xi], \qquad [m_t] = -c_1[m_\xi]. \tag{41}$$

Further, if r be the right vector of the characteristic matrix of eqns (34) and (35), *i.e.*, if

$$\begin{bmatrix} -c_1 & 1/g \\ (m-1)/2g & -c_1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = 0,$$

which leads to the choice

$$r_1 = 1, \qquad r_2 = gc_1, \tag{42}$$

then

$$[m_\xi] = gc_1[\theta_\xi]. \tag{43}$$

Differentiating with respect to t, the characteristic compatibility condition of eqns (34) and (35):

$$(m_t + c_1 m_\xi) + gc_1(\theta_t + c_1\theta_\xi) = 0, \tag{44}$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}m_t + \sqrt{\frac{m-1}{2}}\frac{\mathrm{d}\theta_t}{\mathrm{d}t} + \frac{1+4m}{2\sqrt{2(m-1)g}} - \left\{m_\xi m_t + \sqrt{\frac{m-1}{2}}m_t m_\xi\right\} + \frac{1}{2\sqrt{2(m-1)}}m_t(\theta_t + c_1\theta_\xi) = 0, \tag{45}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + c_1\frac{\partial}{\partial \xi}. \tag{46}$$

Since m itself is assumed to be continuous across the characteristic curve, the jump $[m_t]$ satisfies

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}m_t\right] = \frac{\mathrm{d}}{\mathrm{d}t}[m_t]. \tag{47}$$

Let the subscript + denote the value of the quantity in the unsteady state just ahead of the discontinuity. Taking the jump of eqn (45), writing the jump of the product terms like $[m_t\theta_\xi]$ as

$$[m_t\, \theta_\xi] = [m_t][\theta_\xi] + m_{t+}[\theta_\xi] + \theta_{\xi+}[m_t], \tag{48}$$

and using eqns (40) and (43), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}[m_t] - \frac{1+4m}{2(m-1)}[m_t]^2 + \frac{1}{2}\left\{\frac{1}{2\sqrt{2(m-1)}}\theta_{t+} + \frac{1+2m}{2g}\theta_{\xi+} - \frac{(1+3m)}{m-1}m_{t+} + \frac{1+4m}{2g\sqrt{2(m-1)}}m_{\xi+}\right\}[m_t] = 0. \tag{49}$$

We now note that the state (θ_+, m_+) on the positive (with respect to ξ increasing direction) satisfies the original eqns (34) and (35) so that we can express m_{t+} and θ_{t+} in terms of $\theta_{\xi+}$ and $m_{\xi+}$. The transport equation for the discontinuity in m_t reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}[m_t] + L_1(t)[m_t] + N_1(t)[m_t]^2 = 0, \tag{50}$$

$$L_1(t) = \frac{1}{4g}\left\{\frac{2m}{\sqrt{2(m-1)}} m_{\xi+} + (2+5m)\theta_{\xi+}\right\} \tag{51}$$

and

$$N_1(t) = -\frac{1+4m}{2(m-1)} < 0. \tag{52}$$

Once the solution on the positive side of the discontinuity is known (note that m and g being continuous, $m = m_+$, $g = g_+$), the coefficients in the transport eqn (50) are known functions of t along the characteristic curve and hence eqn (50) can be solved. A kink on the NLWF can possibly appear only when $[m_t]_{t=0} > 0$, which from eqn (40) implies that $[m_\xi]_{t=0} < 0$. The critical time t_c when a kink appears on the NLWF can be found by solving the eqn (50) numerically.

4.2. *Shock front in a polytropic gas*

A shock front is not self-propagating since the waves from behind catch up with the shock and influence its motion. This is mathematically represented by the existence of an infinite system of compatibility conditions[15] for the amplitude μ and the quantities μ_i, $i = 1, 2, \dots$. The first of these, namely, μ_1, is given by

$$\mu_1 = \left(N_1 \frac{\partial}{\partial x} + N_2 \frac{\partial}{\partial y}\right)\mu, \tag{53}$$

where (N_1, N_2) represent the unit normals to the shock front and μ is given in terms of the densities ρ and ρ_0 behind and ahead of the shock, respectively, by

$$\mu = \frac{\rho - \rho_0}{\rho_0}. \tag{54}$$

For a weak shock $0 < \mu \ll 1$ propagating into a medium in uniform state and at rest, the problem can be solved approximately by only two compatibility conditions[6], which can be written in terms of the Mach number M of the shock and its normal derivative N,

$$M = \left(1 + \frac{\gamma+1}{4}\mu\right)\Bigg|_{\text{shock}}, \qquad N = \left\{\frac{\gamma+1}{4}\left(N_1 \frac{\partial}{\partial x} + N_2 \frac{\partial}{\partial y}\right)\mu\right\}\Bigg|_{\text{shock}}, \tag{55}$$

where x and y are appropriately nondimensionalized here. The two compatibility conditions or the transport equations along a shock ray are

$$M_t + \frac{M-1}{2G}\Theta_\xi + (M-1)N = 0 \tag{56}$$

and

$$N_t + \frac{N}{2G}\Theta_\xi + 2N^2 = 0, \tag{57}$$

where G denotes the metric g for the shock front and θ for a shock front is denoted by Θ. We eliminate Θ_ξ from the above two equations and eqn (8) (with $T=0$, $g=G$ and $C=M$) and get two relations from which we can derive a homogeneous relation between M_t, N_t and G_t in the form

$$2NM_t + \frac{(M-1)N}{2GM} G_t - (M-1)N_t = 0. \tag{58}$$

For a weak shock, $0 < M-1 \ll 1$ we can replace $((M-1)/M)$ by $M-1$ and get an integral of eqn (58) in the form $G(M-1)^4 N^{-2} = h(\xi)$, where $h(\xi)$ can be obtained from the distribution of g, M and N on the initial position of the shock front. By changing ξ to another function of ξ we can choose $h(\xi) = 1$. In this case, we get[16]

$$G = N^2 (M-1)^{-4}. \tag{59}$$

The differential equations governing the successive positions of a shock front and the distribution of intensity on it can be obtained by solving the equations

$$x_t = M\cos\Theta, \qquad y_t = M\sin\Theta, \tag{60}$$

$$\Theta_t + \frac{(M-1)^4}{N^2} M_\xi = 0, \tag{61}$$

$$M_t + \frac{(M-1)^5}{2N^2}\Theta_\xi + (M-1)N = 0, \tag{62}$$

$$N_t + \frac{(M-1)^4}{2N}\Theta_\xi + 2N^2 = 0. \tag{63}$$

The system of three equations (61)–(63) is hyperbolic with characteristic velocities in the (ξ, t) plane: $0, \pm\sqrt{(M-1)/2G^2}$. Following the procedure used for the equations of the nonlinear ray theory, we can derive the transport equation for a jump in the first derivative across a characteristic curve moving with velocity $\sqrt{(M-1)/2G^2}$ in an arbitrary unsteady flow. The equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}[M_t] + L_2(t)[M_t] + N_2(t)[M_t]^2 = 0, \tag{64}$$

where the coefficients L_2 and N_2 are known functions of t along the characteristic curve from the known solution and the derivatives $M_{\xi+}$, $\Theta_{\xi+}$ and $N_{\xi+}$ ahead of the characteristic curve.

Equations (50) and (64) give the evolution of a discontinuity along a characteristic curve in a general unsteady solution of the equations of the NLRT or shock ray theory. Let us try to find their *steady* solutions (*i.e.*, solutions which depend only on ξ). The only such solution $(m = \bar{m}(\xi),\ \theta = \bar{\theta}(\xi))$ of eqns (34) and (35) of the NLRT is $\bar{m}$ = constant and $\bar{\theta}$ = constant, which represents a plane wavefront with a uniform distribution of the amplitude on it. For the shock ray equations (61)–(63) we note first that a constant solution ($N = 0$, Θ = constant, M = constant; for this we consider the equations in terms of the operator $\partial/\partial\lambda$ defined in eqn (5b) and not in terms of ξ; see comments in the next paragraph) representing a plane shock front moving with a uniform velocity is always a solution. For $N \neq 0$ and $M \neq 1$, a t-independent solution of eqns (56) and (57) implies an inconsistency, $N = 0$. Therefore, the only t-independent solution of the equation of the shock ray theory gives a shock of constant strength with $N = 0$, in which case the ray coordinate system is not valid. Therefore, the transport equation (64) is valid only for unsteady basic solution.

Before we pass on to a discussion of propagation of the kinks, we make an important observation regarding the role of the ray coordinate system. In the case of a linear wavefront, the right-hand sides of eqns (11) and (12) are independent of w and hence the rays are uniquely determined when the initial position of the wavefront is given. ξ can be chosen arbitrarily on the initial wavefront, which implies that the initial value of g can also be prescribed arbitrarily. Once this value of g is prescribed on the initial wavefront, its value at $t > 0$ is completely determined from eqn (8) written in the form

$$g_t = g\left(C\frac{\partial\theta}{\partial\lambda} + \frac{\partial T}{\partial\lambda}\right),$$

where $\partial/\partial\lambda$ is defined in eqn (5b). The metric g becomes singular at points where cusp type of singularities appear on the linear wavefront. The situation for a nonlinear wavefront and a shock front is different. Here the geometry of the wavefront and the intensity distribution on it mutually interact. The metric g tends to infinity as m or M tends to 1. Therefore, even though the equations in this section are written for a small-amplitude wave, these cannot be used to discuss the problem when the wave intensity tends to zero. One has to go back to the equations written in terms of the operator $\partial/\partial\lambda$. The problem of shock propagation becomes much more complex due to the fact that $G \to 0$ as $N \to 0$. One may then be tempted to use again the formulation in terms of the operator $\partial/\partial\lambda$, but this formulation is not found to be as good as the use of the ray coordinate system for numerical computation.

The above theory of shock propagation is a particular case of the new theory of shock dynamics (NTSD to distinguish it from Whitham's shock dynamics, 1957) for a weak shock, with only two compatibility conditions retained. We shall return to this theory in Section 6.

5. Propagation of a kink on a nonlinear wavefront

The nondimensional form of the two general conservation laws (6) and (7) for the propagation of a kink on a weakly nonlinear wavefront in terms of θ, m reduces to

$$(g \sin \theta)_t + (m \cos \theta)_\xi = 0 \tag{65}$$

and

$$(g \cos \theta)_t - (m \sin \theta)_\xi = 0, \tag{66}$$

where g as a function of m is given by eqn (32). The kink velocity K in the (ξ, t) plane satisfying eqn (18) with $C = m$ and $T = 0$ is given by

$$K^2 = \frac{m_-^2 - m_+^2}{g_+^2 - g_-^2}. \tag{67}$$

The pair of conservation laws (65) and (66) with relation (32) is sufficient to give the complete history of a kink in the ray coordinate system. The mapping to (x, y) plane obtained by integrating eqn (33) gives the successive positions of the nonlinear wavefront with kinks, whenever and wherever they appear. Here are three examples.

(a) A comparison of linear and nonlinear wavefronts starting from an initially parabolic shape is shown in Fig. 7. This solution was obtained numerically by Ramanathan[17]. This is a very interesting solution, which shows that the central nonlinear rays first tend to diverge and after crossing the arête start converging again. The nonlinear defocusing first prevents caustic formation but in the process the wave strength in the central region weakens considerably and, in fact, become less than the outer adjoining regions so that nonlinearity itself produces a focusing effect. The result of limited computation (in 1984, we did not have conservation forms and hence the numerical computation could not be carried out when the kinks appeared) shows that a pair of kinks have already appeared on the nonlinear wavefront at the points P and Q.

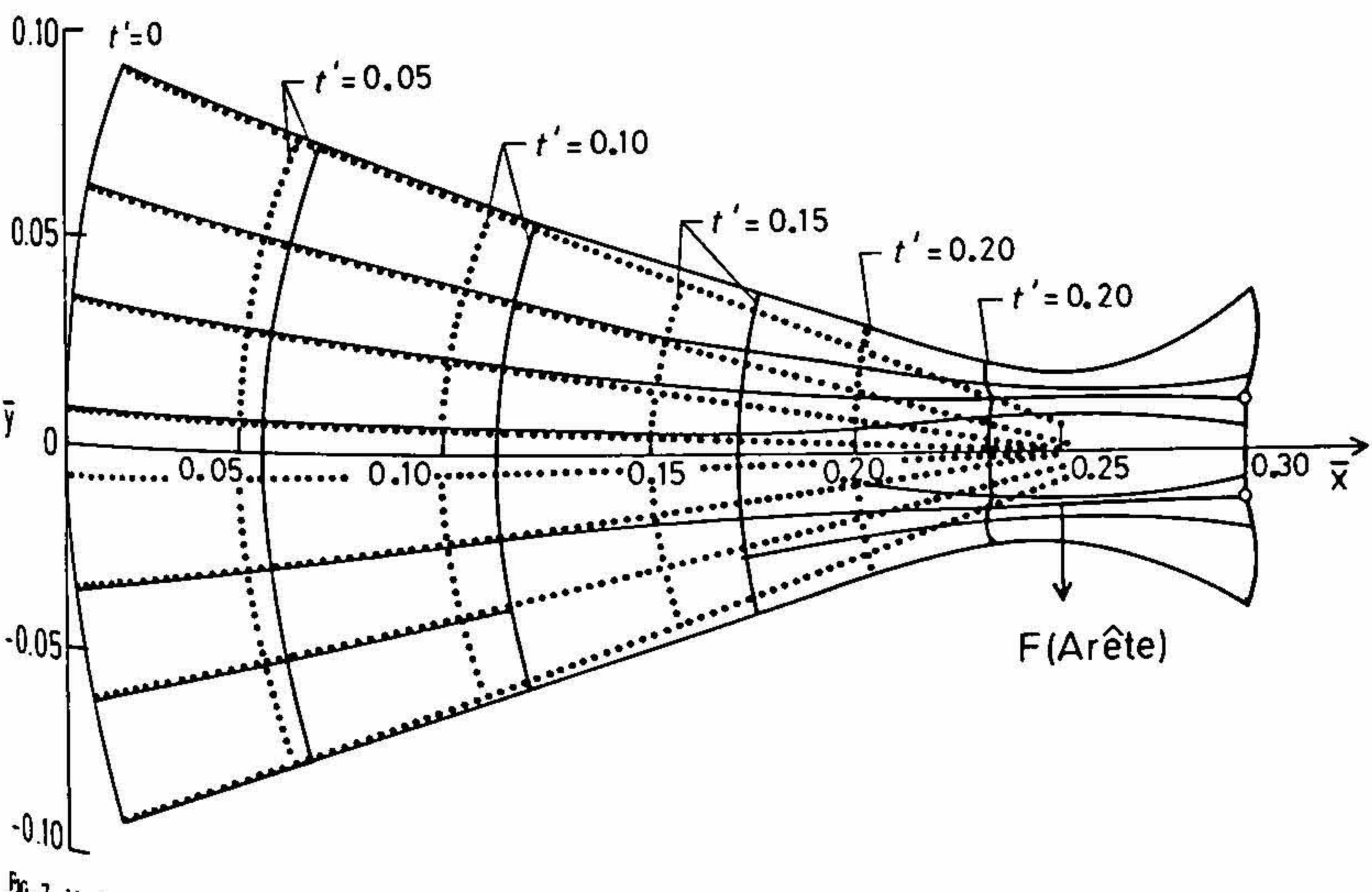

FIG. 7. Nonlinear wavefronts starting from an initially parabolic shape $y^2 = 0.5x$ and amplitude distribution $[(\gamma+1)/2](M_0 - 1) = 0.1 \exp(-|\theta_0|)$ with $\gamma = 1.4$: (...) linear wavefronts and rays; (——) NLWF and nonlinear rays.

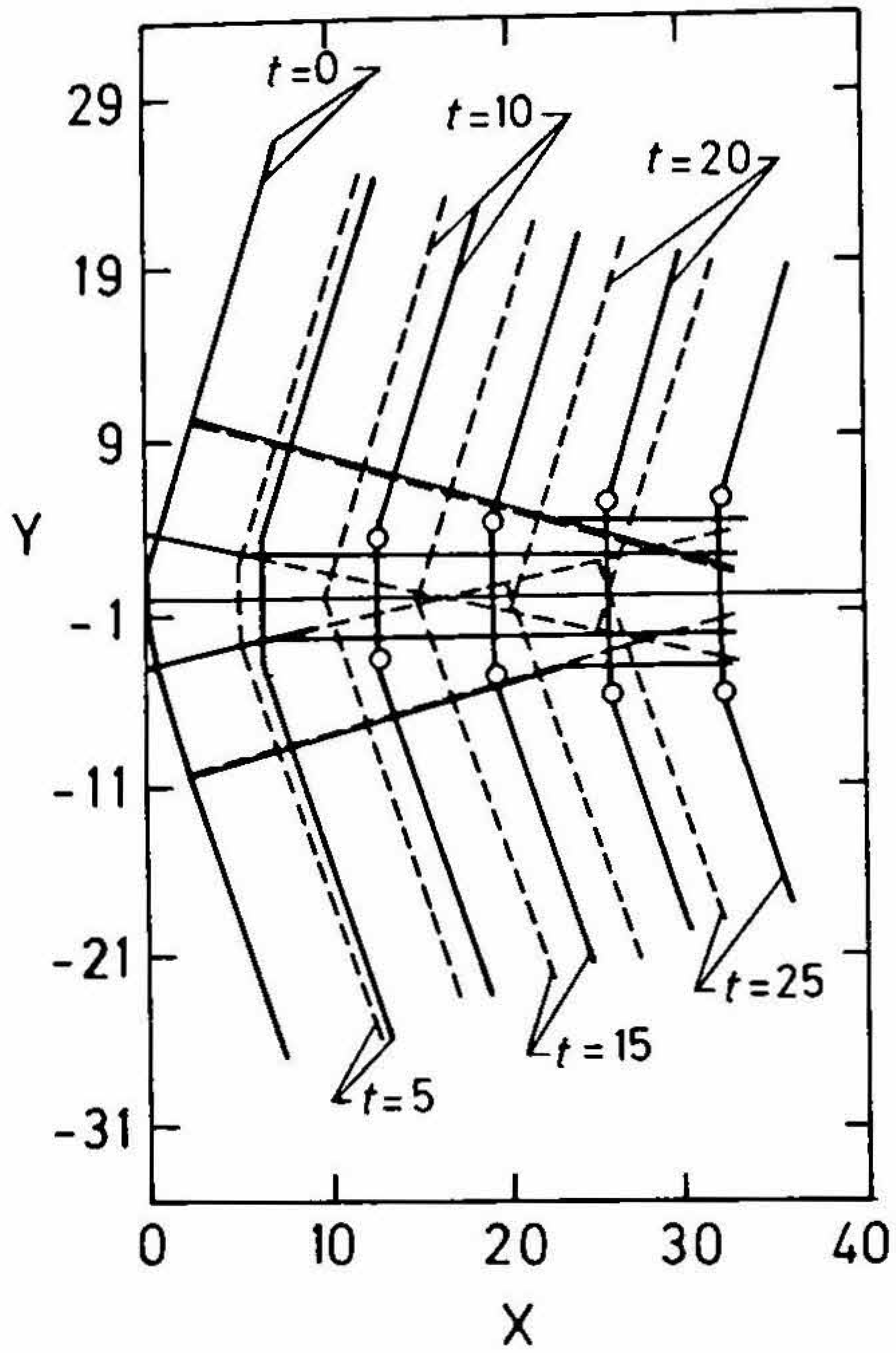

FIG. 8. Exact solution of the equations of nonlinear ray theory with a pair of kinks: (- - -) linear wavefronts and rays; (——) NLWF and nonlinear rays.

(b) Nonlinear wavefronts (with kinks) obtained from an exact solution of the equations of the nonlinear ray theory and linear wavefronts at the corresponding times are shown in Fig. 8. On the initial NLWF the distributions of $\theta(\xi, 0) = \theta_0(\xi)$ and $m(\xi, 0) = m_0(\xi)$ are given by

$$\theta_0(\xi_0) = \begin{cases} -0.3\sin(\pi\xi), & |\xi| \le \frac{1}{2}, \\ -0.3\xi/|\xi|, & |\xi| > \frac{1}{2}, \end{cases} \tag{68}$$

and

$$\left.\begin{aligned} \theta_0 + 2\sqrt{2(M_0-1)} &= 1.45, \quad \xi > 0, \\ \theta_0 - 2\sqrt{2(M_0-1)} &= -1.45, \quad \xi < 0. \end{aligned}\right\} \tag{69}$$

These initial data give rise to two simple wave solutions symmetrically situated with respect to the ξ axis and separated by a constant state region in which the rays are parallel to the x axis[7]. A pair of kinks appear in the solution and propagate on the NLWF.

6. Propagation of a kink on a shock front by NTSD

We rewrite the equations governing the propagation of a weak shock front with a kink. These consist of two equations in conservation form

$$(G \sin \Theta)_t + (M \cos \Theta)_\xi = 0, \qquad (70)$$

$$(G \cos \Theta)_t - (M \sin \Theta)_\xi = 0, \qquad (71)$$

and a partial differential equation

$$M_t + \frac{M-1}{2G}\Theta_\xi + (M-1)N = 0, \qquad (73)$$

and an algebraic relation between G, M and N, namely, eqn (59). This is a very special and new situation in which a discontinuous solution is to be considered of a system consisting of equations not all of which are in conservation form. Mathematical theory of such a system does not exist. However, we believe that the above system is sufficient to study any initial or boundary value problem in which a kink appears. Equations (70) and (71) give the kink velocity in the (ξ, t) plane, which can be used to fit the kink into two states: one ahead of it and another behind it. The two smooth states in the overlapping domains, each containing the kink path in the (ξ, t) plane, can be found at least in theory by solving the partial differential equations (61)–(63). This formulation, which has been just developed, is being tried in extensive numerical computation in the Department of Mathematics of the Institute. We present below the results of numerical computation of the differential equations of the NTSD and its extensions to include the sinusoidal state ahead of the shock to simulate the interaction of a shock front with turbulence) by Kevlahan[11].

(i) Figure 9 presents a comparison of the results from the NTSD with the experimental results of Sturtevant and Kulkarni[3]. M_{s0} represents the shock strength on an initially concave parabolic shock. We note that the shock at 1.14 in (b) with two kinks completely agrees with the experimental shock.

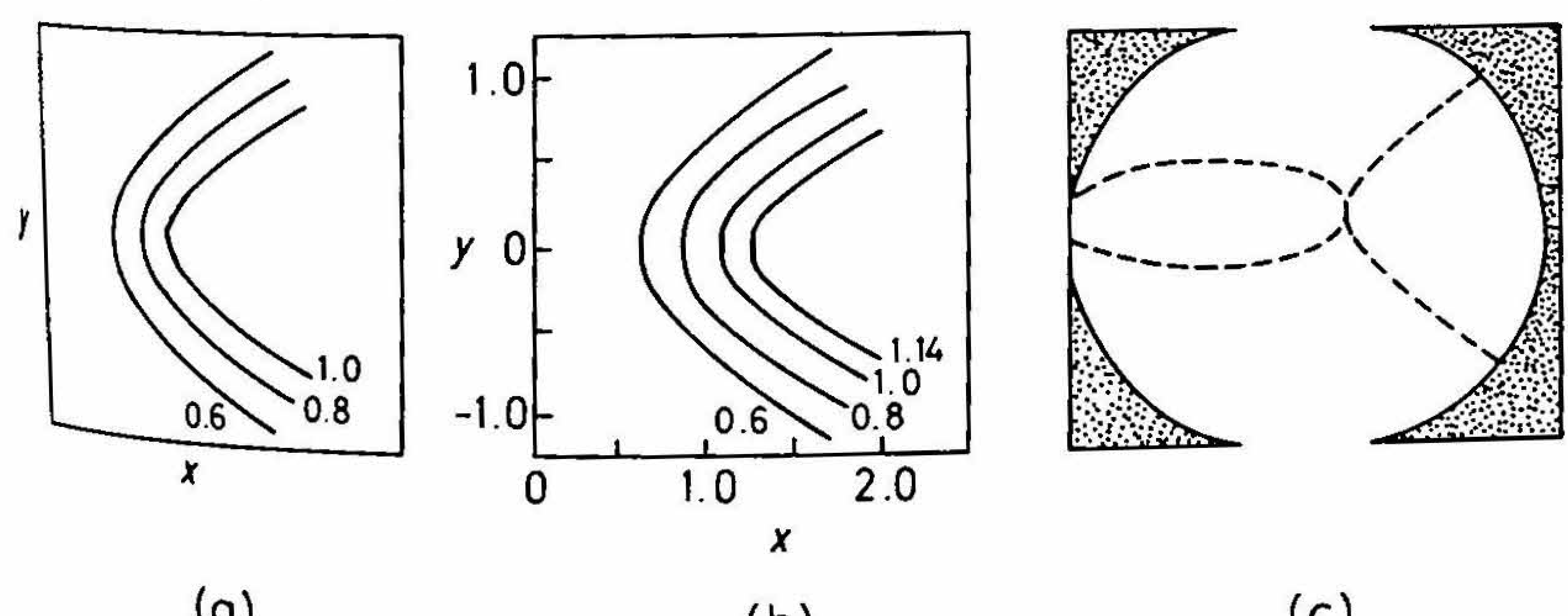

FIG. 9. Shock focusing (time is normalized suitably): a. $M_{s0} = 1$ (acoustic discontinuity at $t = 0.6$, 0.8 and 1.0 (focus)); b. $M_{s0} = 1.2$, $t = 0.6, 0.8, 1.0, 1.14$ (with a pair of kinks); c. Experimental shock $M_{s0} = 1.2$ at $t = 1.14$.

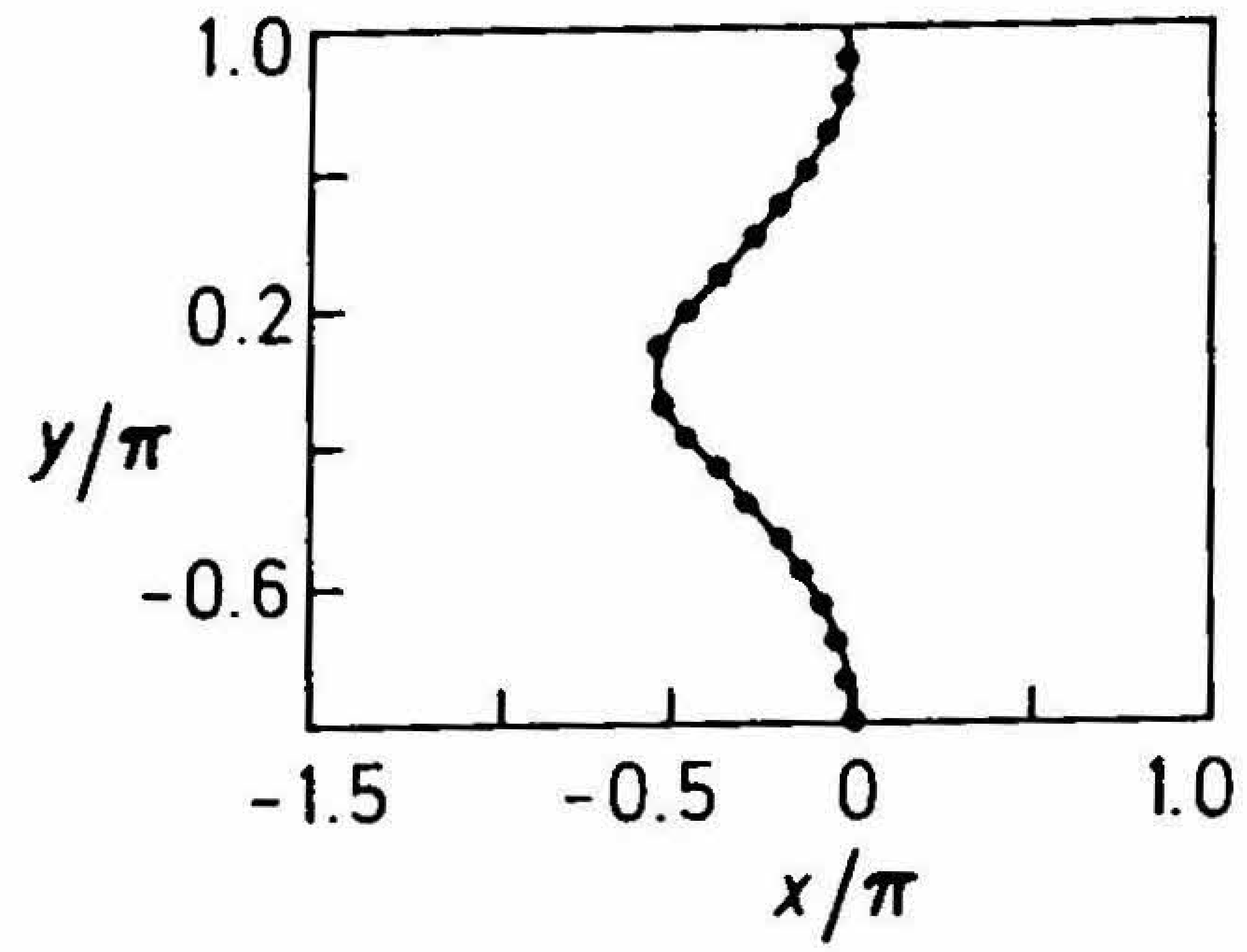

FIG. 10. Computation of the shock shape in a sinusoidal shear flow (at a time when kinks have clearly appeared) as predicted by NTSD (continuous line) and that by DNS (shown by circles) for $M_{s0} = 1.2$. The diameter of the circle is roughly the width of the DNS shock.

(ii) Figure 10 presents a comparison of the results of the NTSD with direct numerical simulation (DNS) of full gas-dynamic equations. Equations of the NTSD for an initially plane shock running into steady sinusoidal velocity field

$$\bar{u}_a(x, y) = -0.3\cos y, \qquad \bar{v}_a = 0 \tag{73}$$

were derived and solved numerically. It is found that numerical computation in NTSD takes only a fraction of time taken by DNS, depending upon the size of discretization. This is because in NTSD the number of space dimensions is reduced by one.

(iii) *Separation of the kinks on a shock front at a fixed time.* Consider an initially parabolic shock $x = \frac{1}{2}y^2$ with initial $N = 0$ and $M_{s0} =$ constant. A pair of kinks always appears and the shock strength in the central disk is uniform. Let ΔY_c be the kink separation at a particular time. Then Kevlahan[11] deduced the following empirical result:

$$\Delta Y_c \propto (M_{s0} - 1)^{0.8} \qquad \text{for } M_{s0} < 1.1. \tag{74}$$

For $M_{s0} > 1.1$, ΔY_c increases more slowly.

(iv) *Mean shock velocity.* For a curved shock with a periodic shape, the mean speed of the shock can be defined. It is found that an initially plane shock always travels faster in a nonuniform flow than in a uniform flow.

(v) *Shock turbulence interaction.* When a plane shock passes through a turbulent flow, Kevlahan[11] also tried to answer the question 'does the shock distort on the large (energetic) length scale or small (vortical) length scale?'. He found that the overall shape of the shock is determined by the energetic length scale but kinks on the shock front form at some extremum points of the vortical fluctuations. He also found that the vortical kinks form before energetic kinks.

In this section we have presented above only some of the results obtained numerically by NTSD. Kevlahan's thesis contains many more results.

7. Conclusion

It is clear from the above discussion that we have a very satisfactory theory for the formation and propagation of cusps and kinks on wavefronts. Kinks are nothing but shocks of the conservation laws in the ray coordinate system. Therefore, as discussed by Prasad[6], kinks represent irreversible phenomena and local determinacy breaks down. It has also been shown in this reference that an NLWF is self-propagating but a shock front is not. The system of equations of the NLRT or the shock ray theory have at least two distinct characteristic speeds in (ξ, t) plane and hence the range of influence of a point on the initial line increases with time (at least for a short time). Therefore, local determinacy breaks down for an NLWF and a shock front. Thus, we conclude that

(i) a linear wavefront has all the three properties of self-propagation, local determinacy and time reversibility;

(ii) an NLWF has the self-propagation and time reversibility properties as long as a kink does not appear but an NLWF with a kink has only one of the three properties, *i.e.*, that of self-propagation; and

(iii) a shock front has none of the above three properties.

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